Some Properties of Macroscopic Traffic Models

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Three "equations of state" are required to describe the traffic fluid. The first is volume = speed · density, and the second is the continuity of vehicles. There are at least four options for the third equation: (1) the deterministic speed-density model, (2) the equilibrium speed-density model, (3) the Payne model, and (4) the Ross model. Two restrictions on the space step *DX* and time step *DT* apply to numerical integrations of all four models. There is an additional restriction on *DT* that applies to the last three models and a special restriction on the "anticipation" term in the Payne model. The time required to perform numerical integrations of all four models is shown to be inversely proportional to the square of the length of the smallest feature represented. A general form for the "relaxation time" in the three non-deterministic models is derived. It is argued, on the basis of experience with the Ross model, that although a dependence upon speed is "correct," setting the relaxation time constant is adequate for most traffic purposes. The relationship between relaxation time and lost time at signals in the Ross model is shown to be linear.

This paper deals with five topics related to the macroscopic (speed, volume, density) representation of traffic. The topics are: (1) categorization of traffic formulations into four classes, (2) permissible step sizes when numerically integrating the traffic formulations, (3) execution time for numerical integrations, (4) form of dependence of the "relaxation time," a parameter in three of the formulations, upon average traffic speed, and (5) relationship between "lost time" and the relaxation time parameter in one of the models.

TYPES OF MACROSCOPIC TRAFFIC MODELS

In this paper, the term "traffic dynamics model" or "traffic formulation" means a complete set of relationships between traffic volume, average traffic speed, and traffic density. Such relationships may be looked upon as the "equations of state" of the traffic fluid.

Three relationships are required. The first relationship is inherent in the definitions of traffic volume, speed, and density:

$$Q = kv \tag{1}$$

where

$Q = Q(x,t)$ = traffic volume (veh/hr) at location x and time t,

$k = k(x,t)$ = vehicular density (veh/mi) at location x and time t, and

$v = v(x,t)$ = space-mean speed (mi/hr) at location x and time t. (Proof that this is the harmonic mean of the "spot" speeds is provided elsewhere (*1*).)

A second relationship, the continuity of vehicles, was pointed out by Lighthill and Whitham (*2*):

$$\partial k/\partial t + \partial Q/\partial x = S(x,t) \tag{2}$$

where

$\partial/\partial t$ and $\partial/\partial x$ indicate partial differentiation with respect to time t and with respect to location along the road x, respectively, and $S(x,t)$ = source strength of vehicles from ramps, parking lots, etc., which may be negative (veh/mi-hr).

Equations 1 and 2 are fundamental. All traffic models that deal with volume, speed, and density must incorporate them or equivalent relationships.

At least four possibilities for the third relationship have been proposed, as follows:

Deterministic Speed-Density Hypothesis

The deterministic speed-density traffic formulation states that the average traffic speed is a function of traffic density. [Greenshields (*3*) was the first to hypothesize that average traffic speed is a deterministic function of density, but innumerable investigators have followed his lead. A summary is provided elsewhere (*4*). The most recent authoritative work to adopt this approach is the Highway Capacity Manual (*5*), in its treatment of freeways.]

$$v = v(k) \tag{3}$$

The traffic-free speed is $v(0)$; $v(k_{jam}) = 0$, where k_{jam} is the so-called "jam density" of vehicles; and $\max[k \cdot v(k)]$ is the roadway capacity. The precise dependence of v upon k is not important to the arguments in this paper. $v(k)$ need not be single-valued except at $k = k_{jam}$ and $k = 0$.

Equilibrium Speed-Density Hypothesis

The equilibrium speed-density formulation states that there is an equilibrium speed, which is a function of density, to which actual speeds relax. [The author's experience is that the equilibrium speed-density hypothesis is accepted by traffic researchers but has never been specifically proposed in the

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literature. It is a logical implication of the work of Prigogine and Herman (*6*), who postulated that there is an equilibrium distribution of traffic speeds to which traffic relaxes.]

$$\partial v/\partial t + v\ \partial v/\partial x = [F(k) - v]/T \tag{4}$$

The terms on the left of the equal sign give the acceleration of traffic with respect to the moving traffic stream; $F(k)$ is the equilibrium speed, and T is the so-called relaxation time (the parameter which controls how quickly traffic returns to its equilibrium speed). Actual free speed, maximum flow, and density at $v = 0$ are variable depending upon the boundary conditions.

Payne's Hypothesis

To overcome the tendency to spontaneously lock up in the deterministic and equilibrium speed-density hypotheses, Payne added an anticipation term (*7, 8*):

$$\partial v/\partial t + v\ \partial v/\partial x = [F(k) - v]/T - (g/T)\partial k/\partial x \tag{5}$$

The anticipation constant is g. The net effect of Payne's addition is to cause traffic to accelerate when $\partial k/\partial t$ is negative—i.e., when the traffic anticipates lower density ahead. Otherwise, Payne's formulation is quite similar to the equilibrium speed-density hypothesis.

Ross's Hypothesis

In a recent paper (*9*), Ross claims that the three traffic formulations listed above are all grossly unrealistic. He proposes that the third defining equation is

$$\partial v/\partial t + v\ \partial v/\partial x = [F - v]/T \tag{6}$$

where F is the free speed on the roadway and is explicitly not dependent on k. Roadway capacity and jam density are accounted for by separate constraints; jam density flow is constrained to be incompressible.

INTEGRATION STEP SIZE

No matter which of the four traffic formulations is used, integration is frequently necessary. The simplest integration method is to step along the roadway using the chosen model to evaluate the volume, density, and speed at points a distance DX apart and to repeat that process every DT hours. The speed of the integration process is inversely proportional to the size of DX and DT; it is therefore important to make DX and DT as large as possible without compromising the accuracy of the integration. We investigate what restrictions there should be on DX and DT to ensure accurate integration.

Continuity of vehicles (equation 2) imposes some fundamental constraints on the integration of all four traffic formulations. Equation 2 can be rewritten in terms of finite differences:

$$Dk_i = S_i\, DT - (Q_i - Q_{i-1})\,(DT/DX) \tag{7}$$

where Dk_i is the change in density at location i between times DT apart, S_i is the traffic source strength at location i, and Q_i is the volume at location i.

The integration error will usually be small compared to k_i if Dk_i is small compared to k_i. Divide both sides of equation 7 by k_i:

$$Dk_i/k_i = (S_i/k_i)\, DT - (v_i - v_{i-1})(DT/DX) \tag{8}$$

where v_i is the volume at location i.

If both terms on the right are individually small, Dk_i/k_i will automatically be small. (Dk_i/k_i will also be small if the two right-side terms nearly cancel one another, but an integration scheme that relies on two large terms nearly cancelling one another would be impossible to implement.) Specifically:

$$DT << k_i/S_i \tag{9}$$

$$DX/DT >> F \tag{10}$$

where F is the free speed on the roadway.

Since the source/sink volume is $S_i{\cdot}DX$, not S_i, restriction 9 can be rewritten:

$$DX/DT >> \text{Source volume}/k_i \tag{11}$$

Inequalities 10 and 11 must both be satisfied if the integration is to give relatively accurate results. Inequality 10 means that the time step, DT, must be small enough and the space step, DX, large enough so that vehicles cannot cross an appreciable fraction the space step in one time step. Inequality 11 means that the time step must be small enough and the space step large enough that source/sink flows do not appreciably alter the number of vehicles in any space step during a time step. These conclusions are based entirely on equation 2 and, therefore, apply to all traffic formulations.

Since the third equation of the deterministic traffic formulation does not involve differentiation, it has no effect on the size of DX or DT in the deterministic model. Restrictions 10 and 11 are the only restrictions on DX and DT in the deterministic formulation.

In the equilibrium, Payne, and Ross formulations, the third equation of state does involve partial differentiation and, therefore, has an effect on the allowable sizes of DX and DT. The third equations in these formulations are similar enough to one another for the convergence properties under numerical integration of all three formulations to be analyzed at the same time. Consider the third equation of state from Payne's formulation converted to finite difference form:

$$\begin{aligned} Dv_i = {} & [F(k_i) - v_i](DT/T) - v_i(v_i - v_{i-1})\,(DT/DX) \\ & - (g/T)(k_i - k_{i-1})(DT/DX) \end{aligned} \tag{12}$$

There are no fixed values of DX and DT which guarantee that Dv_i will be small compared to v_i; we must settle for the weaker condition that Dv_i be small compared to the free speed, F. Again, each term individually must be small (compared to F).

The condition on the first term is

$$|\ [F(k_i) - v_i](DT/T)\ |\ << F \tag{13}$$

which is always satisfied if

$$DT << T \tag{14}$$

Since a term with similar convergence properties occurs in the equilibrium and Ross formulations, restriction 14 applies equally to the equilibrium, Payne, and Ross formulations.

The second term in equation 12 is automatically small com-

TABLE 1 CONDITIONS ON INTEGRATION STEP SIZES DX AND DT FOR THREE TRAFFIC SIMULATION PROGRAMS

Restriction	KRONOS (Deterministic)	FREFLO (Payne)	RFLO (Ross)
10. $DX/DT >> F$	68 mi/hr ?? 63 mi/hr	630 mi/hr >> 63 mi/hr	200 mi/hr ?> 63 mi/hr
11. $DX/DT >>$ source/k	68/mi/hr ?> 20 mi/hr	630 mi/hr >> 20 mi/hr	200 mi/hr >> 20 mi/hr
14. $DT << T$	Not applicable	0.000167 hr << 0.001875 hr	0.0005 hr << 0.0060 hr
16. $DX/DT >> g\ k_{jam}/T\ F$	Not applicable	630 mi/hr >> 68 mi/hr	Not applicable

NOTE: Restriction number refers to inequality number in the text. "??" means the restriction is not satisfied; "?>" means the restriction is marginally satisfied.

pared to F if restriction 10 is satisfied; no new restriction is needed.

The third term, the anticipation term, appears in Payne's formulation only.

$$| (g/T)(k_i - k_{i-1})(DT/DX) | << F \tag{15}$$

Condition (15) is guaranteed if

$$DX/DT >> g\ k_{jam}/T\ F \tag{16}$$

Restriction (16) applies to the Payne formulation only.

How well are these restrictions on DX and DT obeyed in practice? KRONOS (*10*) is the only recent simulation program to use the deterministic speed-density formulation. FREFLO (*11*) is the only simulation program to use the Payne formulation. RFLO, now under development at the Federal Highway Administration, uses the Ross formulation. (No example of a simulation using the equilibrium speed-density formulation could be found.)

It is assumed that the maximum average speed, F, is about 63 mi/hr. It is further assumed that typical source/sink flows ~ 100 veh/hr and worst case (smallest) traffic densities ~ 5 veh/mi, implying source flow/k ~ 20 mi/hr.

In the KRONOS program, the space step, DX, defaults to 100 feet and the time step, DT, defaults to 1 second, yielding DX/DT = 68 mi/hr. Table 1 shows how these values relate to restrictions 10, 11, 14, and 16.

In the FREFLO program, the space step, DX, is the link length; DX = 0.1 mi can be postulated. DT is one-tenth of the travel time on the shortest link (0.00017 hours, with our assumptions), automatically making DX/DT ten times F. The relaxation time, T, is proportional to DX and inversely proportional to the roadway capacity; for capacity = 2000 veh/lane-mi, T = 0.001875 hr. The anticipation constant, g, is proportional to DX and the roadway capacity; in these conditions, g = 0.028 mi^3/hr. Jam density is 143 veh/lane-mi—say, 286 veh/mi on a two-lane roadway. The term $g\ k_{jam}/TF$ evaluates to 68 mi/hr. (The similarity to DX/DT in the KRONOS program is coincidental.) Table 1 shows how these values relate to the applicable restrictions.

In the RFLO program, DX = 0.1 mi; DT = 0.0005 hr = 1.8 sec; (DX/DT = 200 mi/hr); T = 0.0060 hr. Table 1 relates these values to the applicable restrictions.

It is obvious from inspecting Table 1 that the KRONOS program does not satisfy restriction 10. This implies poor representation of density where it is changing rapidly. KRONOS is marginal with respect to restriction 11, implying that its representation of low-density traffic with comparatively large source/sink volumes is theoretically unsound. (Note, however, that the author's experience is that, although one wants the one-step change in any computed quantity to be less than 10% or so, integration steps that can, in theory, allow changes of 20 or 30% to work very well in practice. This is due to the fact that 20 or 30% changes only appear at such abrupt discontinuities that they rarely occur in real traffic.)

RFLO, with DX/DT less than four times the free traffic speed, is marginal with respect to restriction 10. Problems in the representation of density have not been detected in practice, but the possibility should be noted.

FREFLO satisfies all conditions well. In fact, the time step, DT, could probably be doubled or tripled in FREFLO without noticeable loss in accuracy.

MINIMUM EXECUTION TIME

Although the accuracy of the numerical integrations increases as DX becomes larger, precision increases as DX becomes smaller. If one wishes to simulate the effects of very small geometric features (such as an intersection wherein opposite streets are misaligned by, for example, 20 ft), one must make DX small (for example, 5 ft). Restrictions 10 and 11 both require that DT be made small proportionately as DX is made small. Since the number of space steps simulated is inversely proportional to DX and the number of time steps is inversely proportional to DT, the total computation time must be inversely proportional to the square of DX.

This conclusion—that minimum computation time is inversely proportional to the square of the length of the smallest feature simulated—applies to all four traffic formulations.

RELAXATION TIME: T

The three non-deterministic traffic formulations all use a quantity called the relaxation time, which has been symbolized by T here. The original formulation of the equilibrium speed-density hypothesis (equation 4) allows that T is probably a function of average traffic speed (v), but determining that functional dependence was beyond the scope of the original paper, which opted for the simplifying assumption that T is independent of v. The question now is, What is a good functional form for relaxation time, $T(v)$?

A small value for T means that the average traffic speed, v, relaxes to its equilibrium value quickly—that is, traffic acceleration is inversely proportional to T. Since a good deal is known about the acceleration of traffic as a function of speed, a functional form for $T(v)$ can be deduced.

The average power used to accelerate vehicles is

$$P = m{\cdot}v{\cdot}\text{acceleration} \tag{17}$$

where m is the average mass of vehicles. T is inversely proportional to the acceleration:

$$T = c\,(mv + b)/P \tag{18}$$

where c is a constant to be determined, and where b is a small number added to account for the fact that accelerations at $v = 0$ are not infinite (i.e., $T \neq 0$) but rather are limited by pavement friction and driver discomfort.

P is not the engine power used, but rather the power used to accelerate the vehicle after rolling and wind resistance have been overcome. The wind and rolling resistance are approximated by constant forces, so that the power used is linear in v. The dependence of T on v is therefore approximated by

$$T(v) = (c_1 + c_2\,v)/(1 - c_3\,v) \tag{19}$$

The three c's can be roughly estimated. We note that, when traffic is traveling at its maximum possible speed, it cannot accelerate further—i.e., $T \rightarrow \infty$ as $v \rightarrow v_{\max}$. Estimates are that the average maximum speed of the North American traffic stream is in the range 95 to 100 mi/hr and the average speed on level freeways is 63 mi/hr. A rough estimate of c_3 is therefore $0.67/F$, where F is the free, desired speed on the road.

The remaining two parameters, c_1 and c_2, can be chosen by noting that—in simulations with T held constant—T = 0.030 hr produces realistic traffic performance at speeds approaching the free speed (F), and T = 0.0053 hr produces signal discharge flows that represent about 2.1 seconds of lost time per green.

We conclude that the relaxation time, $T(v)$, can be approximately represented by

$$T = (0.0053 + 0.0047\,v/F)\ \mathrm{hr}/(1 - 0.67\,v/F) \tag{20}$$

At least three assumptions have been made in the above derivation:

1. The effects of traffic mix, roadway grade, and driving conditions on $T(v)$ are represented by making the constants which multiply v inversely proportional to F.
2. Deceleration behaves essentially the same as acceleration.
3. Traffic speed will never approach $F/0.67$.

These assumptions are mathematically convenient and not obviously wrong. The argument applies equally to the equilibrium, Payne, and Ross traffic formulations.

Extensive simulations with the Ross formulation indicate that there are no startling differences between the $T(v)$ variable as described above and T = constant—only subtle differences in the acceleration of traffic back to its desired speed upon leaving a bottleneck. The details of such accelerations have never been an important traffic issue. It is the author's conclusion that setting the relaxation time T = constant is adequate for most traffic purposes using the Ross formulation. This conclusion has not been tested for the equilibrium or Payne formulations.

"LOST TIME" AT SIGNALS

The relaxation time, T, affects all aspects of traffic behavior in the three formulations where it is used. Its most obvious effect is on "lost time" at signals. Because signal lost time is well known (*12*), this dependence can be used to estimate an appropriate value for T.

Consider a traffic signal with, for example, 0.01 hours of red alternating with 0.01 hours of green (cycle length 0.02 hr = 72 sec). It is straightforward to use any of the traffic formulations to simulate such a traffic condition. The results of one such simulation with the Ross model are shown in Figure 1.

When the simulation of Figure 1 is extended for a long time, a standing queue forms at the signal and the downstream flow stabilizes at 933 veh/hr. This implies an effective green time of 933/2000 = 0.466 hrs/hr or 33.6 sec/cycle. Lost time is: (36 sec of actual green per cycle) − (33.6 sec of effective green per cycle) = 2.4 sec/cycle.

Similar simulations can be repeated using different values of relaxation time and noting the resulting lost times. The relationship between relaxation time and lost time in the Ross formulation is shown in Figure 2. Figure 2 applies to the Ross formulation only.

SUMMARY

This paper has discussed five related topics. The conclusions are:

1. Three equations of state are required to describe the traffic fluid. The first is volume = speed · density, and the second is the continuity of vehicles. There are at least four options for the third equation: the deterministic speed-density model, the equilibrium speed-density model, the Payne model, and the Ross model.
2. Two restrictions on the space step DX and time step DT apply to numerical integrations of all four models. There is an additional restriction on DT that applies to the last three models and a special restriction on the anticipation term in the Payne model.
3. The time required to perform numerical integrations of all four models is inversely proportional to the square of the length of the smallest feature represented.
4. A general form for the relaxation time in the three nondeterministic models is derived. It appears, on the basis of experience with the Ross model, that although a dependence upon speed is "correct," setting the relaxation time constant is adequate for most traffic purposes.
5. The relationship between relaxation time and lost time at signals in the Ross model is linear.

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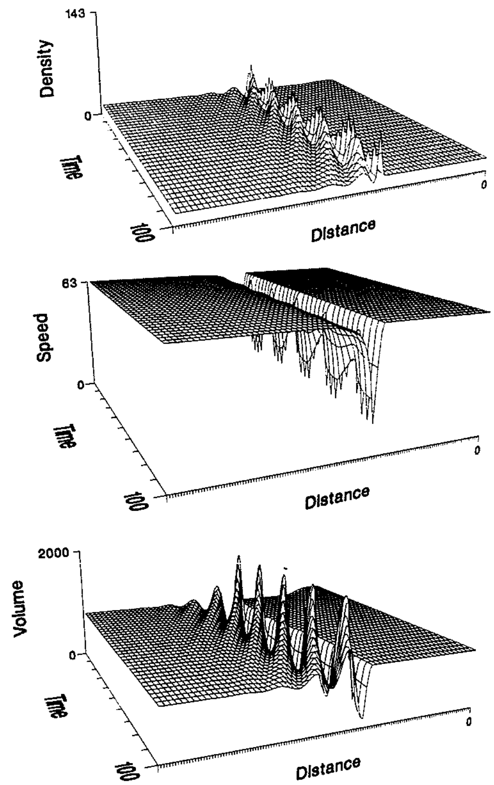

FIGURE 1 Density, speed, and volume as represented by the Ross traffic formulation at a traffic signal with actual green = 36 sec/cycle, actual red = 36 sec/cycle. Demand volume rises from 800 to 950 veh/hr. Traffic flows from right to left. Distance is in units of 0.1 mi (13 mi total). Time runs from 0.00 to 0.10 hr (five cycles), back to front. Jam density of vehicles is 143 veh/lane-mi. Free speed is 63 mi/hr.

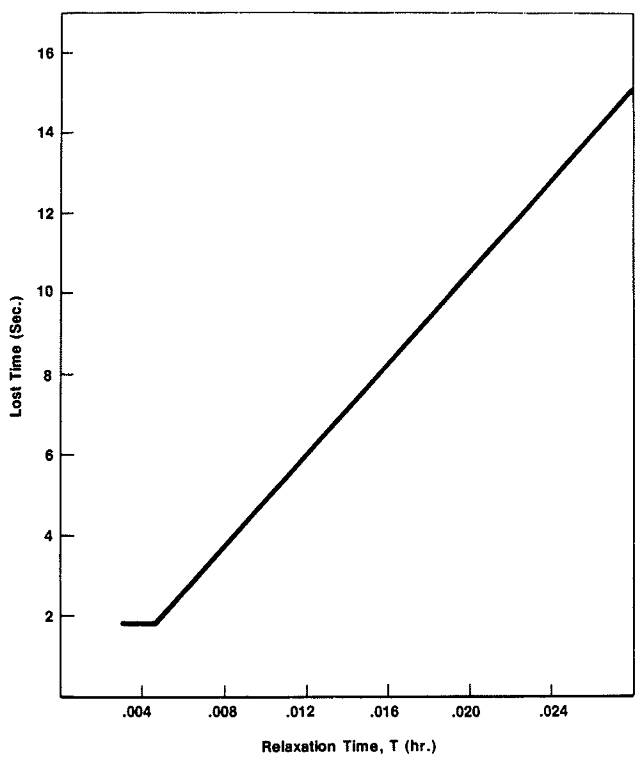

FIGURE 2 Signal lost time as a function of relaxation time, *T*, in the Ross model. Free speed = 63 mi/hr. Cycle length = 0.02 hr with 50% actual green. Lost times less than the integration step size (1.8 sec) are not observed.

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