Some Properties of Macroscopic Traffic Models

Paul Ross

Three “equations of state” are required to describe the traffic fluid. The first is volume = speed · density, and the second is the continuity of vehicles. There are at least four options for the third equation: (1) the deterministic speed-density model, (2) the equilibrium speed-density model, (3) the Payne model, and (4) the Ross model. Two restrictions on the space step $DX$ and time step $DT$ apply to numerical integrations of all four models. There is an additional restriction on $DT$ that applies to the last three models and a special restriction on the “anticipation” term in the Payne model. The time required to perform numerical integrations of all four models is shown to be inversely proportional to the square of the length of the smallest feature represented. A general form for the “relaxation time” in the three non-deterministic models is derived. It is argued, on the basis of experience with the Ross model, that although a dependence upon speed is “correct,” setting the relaxation time constant is adequate for most traffic purposes. The relationship between relaxation time and lost time at signals in the Ross model is shown to be linear.

This paper deals with five topics related to the macroscopic (speed, volume, density) representation of traffic. The topics are: (1) categorization of traffic formulations into four classes, (2) permissible step sizes when numerically integrating the traffic formulations, (3) execution time for numerical integrations, (4) form of dependence of the “relaxation time,” a parameter in three of the formulations, upon average traffic speed, and (5) relationship between “lost time” and the relaxation time parameter in one of the models.

**TYPES OF MACROSCOPIC TRAFFIC MODELS**

In this paper, the term “traffic dynamics model” or “traffic formulation” means a complete set of relationships between traffic volume, average traffic speed, and traffic density. Such relationships may be looked upon as the “equations of state” of the traffic fluid.

Three relationships are required. The first relationship is inherent in the definitions of traffic volume, speed, and density:

$$Q = kv$$  \hspace{1cm} (1)

where

$$Q = Q(x,t) = \text{traffic volume (veh/hr) at location } x \text{ and time } t,$$

$$k = k(x,t) = \text{vehicular density (veh/mi) at location } x \text{ and time } t,$$

$$\nu = \nu(x,t) = \text{space-mean speed (mi/hr) at location } x \text{ and time } t.$$ (Proof that this is the harmonic mean of the “spot” speeds is provided elsewhere (1).)

A second relationship, the continuity of vehicles, was pointed out by Lighthill and Whitham (2):

$$\partial k/\partial t + \partial Q/\partial x = S(x,t)$$  \hspace{1cm} (2)

where

$$\partial k/\partial t \text{ and } \partial Q/\partial x \text{ indicate partial differentiation with respect to time } t \text{ and with respect to location along the road } x,$$

$$S(x,t) \text{ = source strength of vehicles from ramps, parking lots, etc., which may be negative (veh/mi-hr).}$$

Equations 1 and 2 are fundamental. All traffic models that deal with volume, speed, and density must incorporate them or equivalent relationships.

At least four possibilities for the third relationship have been proposed, as follows:

**Deterministic Speed-Density Hypothesis**

The deterministic speed-density traffic formulation states that the average traffic speed is a function of traffic density. [Greenshields (3) was the first to hypothesize that average traffic speed is a deterministic function of density, but innumerable investigators have followed his lead. A summary is provided elsewhere (4). The most recent authoritative work to adopt this approach is the Highway Capacity Manual (5), in its treatment of freeways.]

$$\nu = \nu(k)$$  \hspace{1cm} (3)

The traffic-free speed is $\nu(0)$; $\nu(k_{\text{jam}}) = 0$, where $k_{\text{jam}}$ is the so-called “jam density” of vehicles; and $\max[k \cdot \nu(k)]$ is the roadway capacity. The precise dependence of $\nu$ upon $k$ is not important to the arguments in this paper. $\nu(k)$ need not be single-valued except at $k = k_{\text{jam}}$ and $k = 0$.

**Equilibrium Speed-Density Hypothesis**

The equilibrium speed-density formulation states that there is an equilibrium speed, which is a function of density, to which actual speeds relax. [The author’s experience is that the equilibrium speed-density hypothesis is accepted by traffic researchers but has never been specifically proposed in the...
The integration error will usually be small compared to \( k_i \) if \( \Delta k_i \) is small compared to \( k_i \). Divide both sides of equation 7 by \( k_i \):

\[
\Delta k_i = S_i DT - (Q_i - Q_{i-1})(DT/DX)
\]

where \( v_i \) is the volume at location \( i \).

If both terms on the right are individually small, \( \Delta k_i / k_i \) will automatically be small. \( \Delta k_i / k_i \) will also be small if the two right-side terms nearly cancel one another, but an integration scheme that relies on two large terms nearly cancelling one another would be impossible to implement. Specifically:

\[
DT << k_i/S_i
\]

\[
DX/DT >> F
\]

where \( F \) is the free speed on the roadway.

Since the source/sink volume is \( S_i DX \), not \( S_i \), restriction 9 can be rewritten:

\[
DX/DT >> Source volume/k_i
\]

Inequalities 10 and 11 must both be satisfied if the integration is to give relatively accurate results. Inequality 10 means that the time step, \( DT \), must be small enough and the space step, \( DX \), large enough so that vehicles cannot cross an appreciable fraction of the space step in one time step. Inequality 11 means that the time step must be small enough and the space step large enough that source/sink flows do not appreciably alter the number of vehicles in any space step during a time step. These conclusions are based entirely on equation 2 and, therefore, apply to all traffic formulations.

Since the third equation of the deterministic traffic formulation does not involve differentiation, it has no effect on the size of \( DX \) or \( DT \) in the deterministic model. Restrictions 10 and 11 are the only restrictions on \( DX \) and \( DT \) in the deterministic formulation.

In the equilibrium, Payne, and Ross formulations, the third equation of state does involve partial differentiation and, therefore, has an effect on the allowable sizes of \( DX \) and \( DT \). The third equations in these formulations are similar enough to one another for the convergence properties under numerical integration of all three formulations to be analyzed at the same time. Consider the third equation of state from Payne’s formulation converted to finite difference form:

\[
Dv_i = [F(k_i) - v_i](DT/T) - v_i(v_i - v_{i-1})(DT/DX)
\]

\[
- (g/T)(k_i - k_{i-1})(DT/DX)
\]

There are no fixed values of \( DX \) and \( DT \) which guarantee that \( Dv_i \) will be small compared to \( v_i \); we must settle for the weaker condition that \( Dv_i \) be small compared to the free speed, \( F \). Again, each term individually must be small (compared to \( F \)).

The condition on the first term is

\[
| [F(k_i) - v_i](DT/T) | << F
\]

which is always satisfied if

\[
DT << T
\]

Since a term with similar convergence properties occurs in the equilibrium and Ross formulations, restriction 14 applies equally to the equilibrium, Payne, and Ross formulations.

The second term in equation 12 is automatically small com-
TABLE 1 CONDITIONS ON INTEGRATION STEP SIZES \( DX \) AND \( DT \) FOR THREE TRAFFIC SIMULATION PROGRAMS

<table>
<thead>
<tr>
<th>Restriction</th>
<th>KRONOS (Deterministic)</th>
<th>FREFLO (Payne)</th>
<th>RFLO (Ross)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10. ( DX/DT &gt;&gt; F )</td>
<td>68 mi/hr ( ? &gt; 63 ) mi/hr</td>
<td>630 mi/hr ( ? &gt; 63 ) mi/hr</td>
<td>200 mi/hr ( ? &gt; 63 ) mi/hr</td>
</tr>
<tr>
<td>11. ( DX/DT &gt;&gt; ) source/k</td>
<td>68 mi/hr ( ? &gt; 20 ) mi/hr</td>
<td>630 mi/hr ( ? &gt; 20 ) mi/hr</td>
<td>200 mi/hr ( ? &gt; 20 ) mi/hr</td>
</tr>
<tr>
<td>14. ( DT &lt;&lt; T )</td>
<td>Not applicable</td>
<td>0.000167 hr ( ? &lt; 0.001875 ) hr</td>
<td>0.0005 hr ( ? &lt; 0.0060 ) hr</td>
</tr>
<tr>
<td>16. ( DX/DT &gt;&gt; g \ k_{\text{aju}}/T F )</td>
<td>Not applicable</td>
<td>630 mi/hr ( ? &gt; 68 ) mi/hr</td>
<td>Not applicable</td>
</tr>
</tbody>
</table>

Note: Restriction number refers to inequality number in the text. "?" means the restriction is not satisfied; "?=?" means the restriction is marginally satisfied.

pared to \( F \) if restriction 10 is satisfied; no new restriction is needed.

The third term, the anticipation term, appears in Payne’s formulation only.

\[ |(g/T)(k_i - k_{i-1})(DT/DX)| << F \]  

Condition (15) is guaranteed if

\[ DX/DT >> g k_{\text{aju}}/T F \]  

Restriction (16) applies to the Payne formulation only.

How well are these restrictions on \( DX \) and \( DT \) obeyed in practice? KRONOS (10) is the only recent simulation program to use the deterministic speed-density formulation. FREFLO (11) is the only simulation program to use the Payne formulation. RFLO, now under development at the Federal Highway Administration, uses the Ross formulation. (No example of a simulation using the equilibrium speed-density formulation could be found.)

It is assumed that the maximum average speed, \( F \), is about 63 mi/hr. It is further assumed that typical source/sink flows = 100 veh/hr and worst case (smallest) traffic densities \( \sim 5 \) veh/mi, implying source flow/k = 20 mi/hr.

In the KRONOS program, the space step, \( DX \), defaults to 100 feet and the time step, \( DT \), defaults to 1 second, yielding \( DX/DT = 68 \) mi/hr. Table 1 shows how these values relate to restrictions 10, 11, 14, and 16.

In the FREFLO program, the space step, \( DX \), is the link length; \( DX = 0.1 \) mi can be postulated. \( DX \) is one-tenth of the travel time on the shortest link (0.00017 hours, with our assumptions), automatically making \( DX/DT \) ten times \( F \). The relaxation time, \( T \), is proportional to \( DX \) and inversely proportional to the roadway capacity; for capacity = 2000 veh/lane-mi, \( T = 0.001875 \) hr. The anticipation constant, \( g \), is proportional to \( DX \) and the roadway capacity; in these conditions, \( g = 0.028 \) mi/hr. Jam density is 143 veh/lane-mi—say, 286 veh/mi on a two-lane roadway. The term \( g k_{\text{aju}}/T F \) evaluates to 68 mi/hr. (The similarity to \( DX/DT \) in the KRONOS program is coincidental.) Table 1 shows how these values relate to the applicable restrictions.

In the SFLO program, \( DX = 0.1 \) mi; \( DT = 0.0005 \) hr = 1.8 sec; \( DX/DT = 200 \) mi/hr; \( T = 0.0060 \) hr. Table 1 relates these values to the applicable restrictions.

It is obvious from inspecting Table 1 that the KRONOS program does not satisfy restriction 10. This implies poor representation of density where it is changing rapidly. KRONOS is marginal with respect to restriction 11, implying that its representation of low-density traffic with comparatively large source/sink volumes is theoretically unsound. (Note, however, that the author’s experience is that, although one wants

the one-step change in any computed quantity to be less than 10% or so, integration steps that can, in theory, allow changes of 20 or 30% to work very well in practice. This is due to the fact that 20 or 30% changes only appear at such abrupt discontinuities that they rarely occur in real traffic.)

FREFLO, with \( DX/DT \) less than four times the free traffic speed, is marginal with respect to restriction 10. Problems in the representation of density have not been detected in practice, but the possibility should be noted.

FREFLO satisfies all conditions well. In fact, the time step, \( DT \), could probably be doubled or tripled in FREFLO without noticeable loss in accuracy.

MINIMUM EXECUTION TIME

Although the accuracy of the numerical integrations increases as \( DX \) becomes larger, precision increases as \( DX \) becomes smaller. If one wishes to simulate the effects of very small geometric features (such as an intersection wherein opposite streets are misaligned by, for example, 20 ft), one must make \( DX \) small (for example, 5 ft). Restrictions 10 and 11 both require that \( DT \) be made small proportionately as \( DX \) is made small. Since the number of space steps simulated is inversely proportional to \( DX \) and the number of time steps is inversely proportional to \( DT \), the total computation time must be inversely proportional to the square of \( DX \).

This conclusion—that minimum computation time is inversely proportional to the square of the length of the smallest feature simulated—applies to all four traffic formulations.

RELAXATION TIME: \( T \)

The three non-deterministic traffic formulations all use a quantity called the relaxation time, which has been symbolized by \( T \) here. The original formulation of the equilibrium speed-density hypothesis (equation 4) allows that \( T \) is probably a function of average traffic speed \( v \), but determining that functional dependence was beyond the scope of the original paper, which opted for the simplifying assumption that \( T \) is independent of \( v \). The question now is, What is a good functional form for relaxation time, \( T(v) \)?

A small value for \( T \) means that the average traffic speed, \( v \), relaxes to its equilibrium value quickly—that is, traffic acceleration is inversely proportional to \( T \). Since a good deal is known about the acceleration of traffic as a function of speed, a functional form for \( T(v) \) can be deduced.

The average power used to accelerate vehicles is

\[ P = m \cdot v \cdot \text{acceleration} \]  

(17)
where \(m\) is the average mass of vehicles. \(T\) is inversely proportional to the acceleration:

\[
T = c (mv + b)/P
\]  

(18)

where \(c\) is a constant to be determined, and where \(b\) is a small number added to account for the fact that accelerations at \(v = 0\) are not infinite (i.e., \(T \neq 0\)) but rather are limited by pavement friction and driver discomfort.

\(P\) is not the engine power used, but rather the power used to accelerate the vehicle after rolling and wind resistance have been overcome. The wind and rolling resistance are approximated by constant forces, so that the power used is linear in \(v\). The dependence of \(T\) on \(v\) is therefore approximated by

\[
T(v) = (c_1 + c_2 v)/(1 - c_3 v)
\]  

(19)

The three \(c\)'s can be roughly estimated. We note that, when traffic is traveling at its maximum possible speed, it cannot accelerate further—i.e., \(T \rightarrow \infty\) as \(v \rightarrow v_{\text{max}}\). Estimates are that the average maximum speed of the North American traffic stream is in the range 95 to 100 mi/hr and the average speed on level freeways is 63 mi/hr. A rough estimate of \(c_3\) is therefore 0.67/F, where \(F\) is the free, desired speed on the road.

The remaining two parameters, \(c_1\) and \(c_2\), can be chosen by noting that—in simulations with \(T\) held constant—\(T = 0.030\) hr produces realistic traffic performance at speeds approaching the free speed (\(F\)), and \(T = 0.0053\) hr produces signal discharge flows that represent about 2.1 seconds of lost time per green.

We conclude that the relaxation time, \(T(v)\), can be approximately represented by

\[
T = (0.0053 + 0.0047 v/F) \text{ hr}/(1 - 0.67 v/F)
\]  

(20)

At least three assumptions have been made in the above derivation:

1. The effects of traffic mix, roadway grade, and driving conditions on \(T(v)\) are represented by making the constants which multiply \(v\) inversely proportional to \(F\).
2. Deceleration behaves essentially the same as acceleration.
3. Traffic speed will never approach \(F/0.67\).

These assumptions are mathematically convenient and not obviously wrong. The argument applies equally to the equilibrium, Payne, and Ross traffic formulations.

Extensive simulations with the Ross formulation indicate that there are no startling differences between the \(T(v)\) variable as described above and \(T = \text{constant}\)—only subtle differences in the acceleration of traffic back to its desired speed upon leaving a bottleneck. The details of such accelerations have never been an important traffic issue. It is the author's conclusion that setting the relaxation time \(T = \text{constant}\) is adequate for most traffic purposes using the Ross formulation. This conclusion has not been tested for the equilibrium or Payne formulations.

**“LOST TIME” AT SIGNALS**

The relaxation time, \(T\), affects all aspects of traffic behavior in the three formulations where it is used. Its most obvious effect is on “lost time” at signals. Because signal lost time is well known (12), this dependence can be used to estimate an appropriate value for \(T\).

Consider a traffic signal with, for example, 0.01 hours of red alternating with 0.01 hours of green (cycle length 0.02 hr = 72 sec). It is straightforward to use any of the traffic formulations to simulate such a traffic condition. The results of one such simulation with the Ross model are shown in Figure 1.

When the simulation of Figure 1 is extended for a long time, a standing queue forms at the signal and the downstream flow stabilizes at 933 veh/hr. This implies an effective green time of 933/2000 = 0.466 hrs/hr or 33.6 sec/cycle. Lost time is: (36 sec of actual green per cycle) - (33.6 sec of effective green per cycle) = 2.4 sec/cycle.

Similar simulations can be repeated using different values of relaxation time and noting the resulting lost times. The relationship between relaxation time and lost time in the Ross formulation is shown in Figure 2. Figure 2 applies to the Ross formulation only.

**SUMMARY**

This paper has discussed five related topics. The conclusions are:

1. Three equations of state are required to describe the traffic fluid. The first is volume = speed · density, and the second is the continuity of vehicles. There are at least four options for the third equation: the deterministic speed-density model, the equilibrium speed-density model, the Payne model, and the Ross model.
2. Two restrictions on the space step \(DX\) and time step \(DT\) apply to numerical integrations of all four models. There is an additional restriction on \(DT\) that applies to the last three models and a special restriction on the anticipation term in the Payne model.
3. The time required to perform numerical integrations of all four models is inversely proportional to the square of the length of the smallest feature represented.
4. A general form for the relaxation time in the three nondeterministic models is derived. It appears, on the basis of experience with the Ross model, that although a dependence upon speed is “correct,” setting the relaxation time constant is adequate for most traffic purposes.
5. The relationship between relaxation time and lost time at signals in the Ross model is linear.

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FIGURE 1  Density, speed, and volume as represented by the Ross traffic formulation at a traffic signal with actual green = 36 sec/cycle, actual red = 36 sec/cycle. Demand volume rises from 800 to 950 veh/hr. Traffic flows from right to left. Distance is in units of 0.1 mi (13 mi total). Time runs from 0.00 to 0.10 hr (five cycles), back to front. Jam density of vehicles is 143 veh/lane-mi. Free speed is 63 mi/hr.

FIGURE 2  Signal lost time as a function of relaxation time, \( T \), in the Ross model. Free speed = 63 mi/hr. Cycle length = 0.02 hr with 50% actual green. Lost times less than the integration step size (1.8 sec) are not observed.
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