Schedule Delay and Departure Time Decisions with Heterogeneous Commuters

RICHARD ARNOTT, ANDRE DE PALMA, AND ROBIN LINDSEY

The dynamics of morning rush-hour traffic congestion have been studied extensively in recent years. In most theoretical work, however, commuters are assumed to have identical travel cost functions and to face the same arrival-time constraints at work. In this paper, we allow commuters to differ in their travel time costs, their starting time at work, and the costs incurred from early and late arrival. Early in the rush hour, the departure rate exceeds road capacity, causing a queue to develop. Commuters order themselves systematically in the departure sequence to minimize their individual travel costs. The order in which different groups depart is not necessarily efficient. A time-varying congestion toll can be constructed to eliminate queueing and induce the optimal order of departure. Travel-cost savings from such congestion tolls and from road capacity investments are computed. Estimated benefits are generally biased if computed using average travel-cost parameters and average work start times rather than actual distributions. Savings tend to be overestimated if commuters differ primarily in travel costs, but can be underestimated for capacity investments when commuters differ primarily in starting times at work.

The trade-off between time spent commuting in the morning rush hour and the cost of arriving early or late at work has been studied extensively in recent years. With few exceptions, however, the theoretical work has been highly aggregative. Commuters are usually assumed to incur the same costs from travel and to face the same arrival-time constraints at work. For at least two reasons, the analysis needs to be generalized to allow for differences in commuters.

First, empirical studies by Ott et al. (*1*), Small (*2*), and Moore et al. (*3*) have found that different socioeconomic and age groups differ in their commuting behavior. White-collar workers as a group display greater variability in arrival time at work, and are more likely to arrive at the end of the rush hour than is the general population. Workers with children tend to arrive earlier and exhibit less variability in arrival time. Older workers and carpoolers also tend to arrive earlier, while the results for individuals with long commutes are mixed.

R. Arnott, Queen's University, Kingston, Ontario K7L 3N6, Canada. A. de Palma, Northwestern University, Evanston, Ill. 60208. R. Lindsey, University of Alberta, Edmonton, Alberta T6G 2H4, Canada.

Systematic differences between groups in commuting behavior can be attributed to differences in employer policies toward work hours, to family scheduling constraints, and to differences in the real or perceived cost of travel time. Professional and self-employed workers, for example, with high values of time but relatively flexible work hours, tend to travel on the tails of the rush hour to avoid the worst congestion. Carpoolers may depart earlier than single drivers to ensure that members with the earliest starting times arrive on schedule.

To explain the distribution of rush-hour travel times and to predict the response of traffic to changes in the system, it is necessary to account for differences in the commuting population.

A second reason for disaggregation is to provide a framework that will allow more accurate cost-benefit analysis of congestion tolls, road investments, and other transport policies. Most studies have employed single values for the cost of travel time, the desired arrival time at work, and the costs of arrival either earlier or later than desired. But since commuters of different types tend to order themselves systematically in the departure sequence, calculations based on single representative values will generally be biased. A tolling scheme, for example, may pass a cost-benefit testing using an aggregate model, but fail it after disaggregation, or vice versa.

Despite the apparent need to treat heterogeneity of the commuting population, only two authors have done so in the theoretical literature. Henderson (*4*, ch. 8) considers two groups of commuters, with identical costs of travel and waiting time, but different values of noncommuting time. On the assumption that no one arrives late for work, Henderson shows that the group with the lower value of time departs first. Travel time for this group is lower, but waiting time is higher than for the other group. In a later work, Henderson (*5*) again considers two groups of commuters, this time differing in schedule delay costs and with equal unit costs for early and late arrival. The group with higher costs travels at the peak of the rush hour, the other group on the tails.

Recently, Newell (*6*) provided a detailed analysis of the pattern of morning rush-hour departures when traffic flow is constrained by a bottleneck of constant capacity. Commuters are assumed to differ in their costs of travel time, their work start times, and their schedule delay costs (costs of early and late arrival). In this paper, developed independently of Newell's, the bottleneck framework is also adopted, and the focus is on differences between commuters in travel time and schedule delay parameters. The paper differs from Newell's in three respects.

First, Newell treats continuous distributions of commuters differing in both work start time and commuting costs, whereas we assume discrete distributions differing either in work start time or in commuting costs, but not both simultaneously. Our approach is thus less general than Newell's, but allows us to derive explicitly the equilibrium distribution of departure times. Second, we derive the socially optimal departure sequence of commuter groups and indicate how it differs from that in user equilibrium. And third, we calculate the benefits from optimal time-varying congestion tolls and road capacity investments. We also determine the direction of bias when benefits are computed using average population parameters rather than their actual distributions.

The queueing model on which our analysis is based was introduced by Vickrey (*7*) and extended by Hendrickson and Kocur (*8*), Fargier (*9*), and Arnott et al. (*10*). In the next section we review the main results of the model for identical commuters. In later sections, commuters are allowed to differ in their costs of travel time and schedule delay, differences in the relative costs of early and late arrival are considered, the assumption of homogeneous cost parameters is restored and focus is on differences in starting times at work, and conclusions are drawn.

REVIEW OF THE MODEL WITH IDENTICAL COMMUTERS

The precise assumptions and notation employed here follow our earlier model (*10*). N identical commuters travel each morning from home in the suburbs to work downtown. There is a single road along which each individual commutes in his own car. Travel is uncongested except at a single bottleneck (a bridge, tunnel, intersection, etc.) which at most s cars can traverse per unit of time. If the arrival rate at the bottleneck exceeds s, a queue develops. Travel time is

$$T(t) = T^f + T^v(t) \tag{1}$$

where T^f is travel time in the absence of a queue, $T^v(t)$ is waiting time at the bottleneck, and t is departure time from home. Without loss of generality, we set $T^f = 0$, so that an individual reaches the queue at the bottleneck as soon as he leaves home, and arrives at work upon exiting the bottleneck. The length of the queue, $D(t)$, is

$$D(t) = \int_{\hat{t}}^{t} r(\tau)\, d\tau - s(t - \hat{t}) \tag{2}$$

where $\hat{t}$ denotes the time at which the queue was last zero and $r(t)$ the departure rate. Waiting time at the bottleneck is

$$T^v(t) = D(t)/s \tag{3}$$

Individuals are assumed to have a common starting time at work t^*. Their travel cost is given by the linear function

$$C(t) = \alpha T^v(t) + \beta(\text{time early}) + \gamma(\text{time late}) \tag{4}$$

where for individuals who arrive early (before t^*), time late = 0, and for those who arrive late (after t^*), time early = 0. The parameter α measures the (vehicle operating and opportunity of time) costs of time spent in transit. β measures the cost of arriving an extra minute early at work and γ the cost of arriving an extra minute late. (These costs may be nonlinear, or exhibit discontinuities; we follow common practice in assuming linearity.) To assure that the equilibrium departure rate is finite we assume $\alpha > \beta$. For convenience, we define $\eta \equiv \gamma/\beta$ to be the relative unit cost of late arrival to early arrival.

Finally, t_n is defined to be the departure time for arrival at t^*, determined implicitly by the condition

$$t_n + T^v(t_n) = t^* \tag{5}$$

Henceforth, we take "depart early" to mean arrive early and "depart late" to mean arrive late.

User Equilibrium

In choosing when to leave home, individuals face a tradeoff between travel time and schedule delay. Individuals are assumed to have full information about the departure time distribution. Equilibrium obtains when no one can reduce costs by altering departure time. With identical individuals this means that costs are constant over the rush hour.

The equilibrium is depicted in Figure 1. Queue length is measured by the vertical distance between the cumulative departures and cumulative arrivals schedules. Travel time is measured by the horizontal distance. From the beginning of the rush hour at t_q until t_n, the queue builds up at a constant rate. Once past t_n the queue dissipates, again at a constant rate, reaching zero at the end of the rush hour at $t_{q'}$.

Over the interval $[t_q, t_n]$, the equal cost condition implies from equation 4 that

$$C(t) = \alpha T^v(t) + \beta[t^* - t - T^v(t)] \tag{6}$$

is constant. Differentiating equation 6 and using equations 2 and 3, the departure rate for individuals departing early is found to be

$$r(t) = r_E^e = s + \frac{\beta s}{\alpha - \beta} = \frac{\alpha s}{\alpha - \beta} \qquad t \in [t_q, t_n] \tag{7}$$

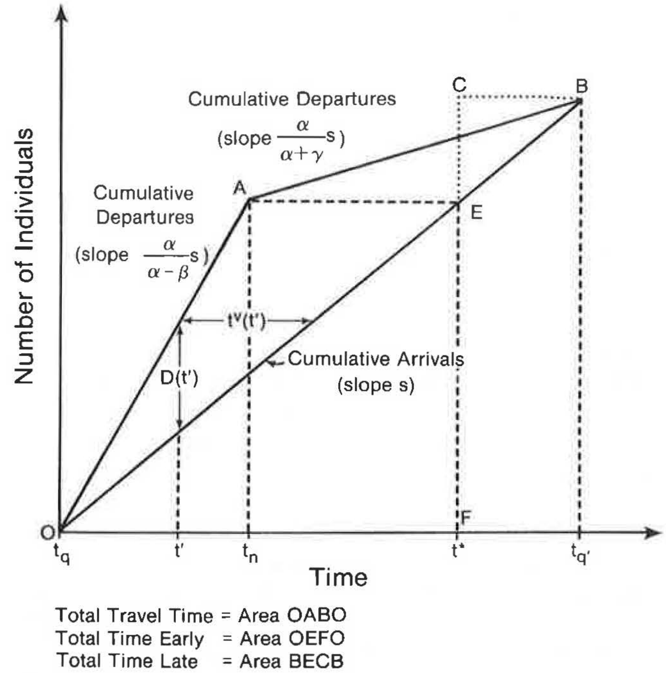

FIGURE 1 Cumulative arrivals and departures, queue length, total travel time, total time early, and total time late in user equilibrium.

where the superscript e denotes user equilibrium and the subscript E early arrivals. By similar reasoning, the late departure rate is

$$r(t) = r_L^e = s - \frac{\gamma s}{\alpha + \gamma} = \frac{\alpha s}{\alpha + \gamma} \qquad t \in [t_n, t_{q'}] \tag{8}$$

where the subscript L denotes late arrivals. Equating the costs of the first and the late individuals to depart,

$$\beta(t^* - t_q) = \gamma(t_{q'} - t^*) \tag{9}$$

Since the bottleneck operates at capacity throughout the rush hour,

$$t_{q'} = t_q + N/s \tag{10}$$

Combining equations 9 and 10, one obtains the following:

$$t_q = t^* - \frac{\eta}{1 + \eta}\frac{N}{s} \qquad t_q' = t^* + \frac{1}{1 + \eta}\frac{N}{s} \tag{11a, 11b}$$

Finally, using equations 5 and 7 and defining $\delta \equiv \beta\gamma/(\beta + \gamma)$ one obtains

$$t_n = t^* - \frac{\delta}{\alpha}\frac{N}{s} \tag{12}$$

Aggregate travel time and schedule delay are identified in Figure 1. Aggregate travel time costs (TTC), schedule delay costs (SDC), and travel costs (TC) are

$$TTC^e = SDC^e = \frac{\delta}{2}\frac{N^2}{s} \tag{13}$$

$$TC^e = TTC^e + SDC^e = \delta\frac{N^2}{s} \tag{14}$$

It is noteworthy that neither the timing of departures nor aggregate costs depend on the unit cost of travel time, α.

The Social Optimum

The social optimum is determined by minimizing the sum of travel time and schedule delay costs. To eliminate queueing while minimizing schedule delay costs, the departure rate is maintained at s throughout the rush hour. The time of first departure is chosen so that the first and last commuters incur equal costs, since otherwise costs could be reduced by moving an individual from the beginning of the rush hour to the end, or vice versa. Since this condition is also true of user equilibrium, the timing of the rush hour and the arrival distribution are the same as in equilibrium.

Denoting variables corresponding to the social optimum with a superscript o, aggregate costs are given by

$$TTC^o = 0 \tag{15}$$

$$SDC^o = TC^o = \frac{\delta}{2}\frac{N^2}{s} \tag{16}$$

Total costs are half their value in user equilibrium.

INDIVIDUALS WITH DIFFERENT α AND β, BUT THE SAME γ/β

Characterization of User Equilibrium and Social Optimum

In this section, we allow the unit costs of travel time and schedule delay to differ across commuters. We assume that commuters have the same relative cost of late arrival and desired arrival time, $\eta = \gamma/\beta$.

Let there be G groups of commuters indexed in order of decreasing relative cost of travel time, α/β, so that

$$\alpha_1/\beta_1 \geq \alpha_2/\beta_2 \geq \ldots \geq \alpha_G/\beta_G \tag{17}$$

Let N_i be the number of individuals in group i and N the number in all groups.

User Equilibrium

The equilibrium departure rate is described by the following theorem.

Theorem 1. In user equilibrium, a fraction $\eta/(1 + \eta)$ of each group departs early, with the remainder departing late. Group 1 is the first to depart early, then group 2 and so on to group G. Members of group G who do not depart early are the first to depart late, followed by the remainder of group $G - 1$ and so on until everyone has departed.

Theorem 1 is proved in Arnott et al. (*11*). (Proofs and derivations of the major results in this paper are contained in the earlier paper, which is available upon request.) Since the fraction of commuters in the homogeneous case who depart early is also $\eta/(1 + \eta)$, the rush hour begins and ends at the same time as with identical commuters. Individuals with the highest cost of travel time relative to schedule delay travel furthest out on the tails of the rush hour, as in Henderson's (*5*) model.

An example of equilibrium with four groups is shown in Figure 2. Group 1, with the lowest relative cost of travel time, travels at the beginning and end of the rush hour. Group 2 travels on adjacent time intervals, and so on. Group j follows group i in the departure sequence at time t_{ij}. Individuals arrive on time (at t^*) along the locus Ot^* with slope $-s$. To the left of Ot^*, individuals arrive early; to the right, they arrive late. The equilibrium travel costs incurred by each group are shown by the equal-cost contours labeled $C_1 \ldots C_4$. To the left of Ot^*, the slope of C_i is $\beta_i s/(\alpha_i - \beta_i)$; to the right, it is $-\gamma_i s/(\alpha_i + \gamma_i)$. The upper envelope of the equilibrium cost curves describes what can be called the travel equilibrium frontier (TEF). Commuters in each group select departure times that minimize their costs on the TEF. Group 2, for example, departs between t_{12} and t_{23} and again between t_{32} and t_{21}.

The Social Optimum

As is the case with identical individuals, the social optimum involves no queueing. To minimize schedule delay costs, the group with the largest value of α travels closest to t^*, with a fraction $\eta/(1 + \eta)$ departing early and the rest late. The group with the second highest value of α departs on adjacent time intervals with the same proportions early and late, and so on for the remaining groups.

The rush hour thus begins and ends at the same time as with identical individuals, and the same time as in user equilibrium. However, the order in which groups depart differs from equilibrium unless the ranking of groups according to α is the reverse of the ranking according to α/β, that is, $\beta_1 < \beta_2 < \ldots < \beta_G$. This is true if $\alpha_1 = \alpha_2 = \ldots \alpha_G$, but is not true in general. Thus, schedule delay costs are not necessarily minimized in equilibrium.

The social optimum can be brought about with a time-varying toll. A toll with four groups is drawn in Figure 3, where it is assumed that $\beta_2 < \beta_1 < \beta_3 < \beta_4$, so that groups 1 and 2 are reversed in the departure sequence relative to equilibrium. The equal-cost contour of group i rises with slope β_i for departures before t^* and falls with slope $-\gamma_i$ for departures after t^*. In equilibrium, group i departs early while the toll is increasing at rate β_i and late while the toll is decreasing at rate γ_i. The toll changes at a rate that just offsets any incentive to queue. Since commuters in a given group prefer their existing departure times to departure at any other times, the toll induces commuters to self-select into the socially efficient departure time intervals.

Cost-Benefit Analysis and Aggregation Bias

Since further calculation in the general case is tedious we shall focus, with little sacrifice of insight, on two groups. Suppose there are N_1 members of group 1 and N_2 of group 2, with $N = N_1 + N_2$ and $\alpha_2/\beta_2 < \alpha_1/\beta_1$. Let $f_2 \equiv N_2/N$ denote the fraction of commuters who are in group 2, and define $\alpha_r \equiv \alpha_2/\alpha_1$, $\beta_r \equiv \beta_2/\beta_1$, and $\delta_i \equiv \beta_i\gamma_i/(\beta_i + \gamma_i)$. To fix ideas it may be helpful to think of white-collar workers comprising group 1 and blue-collar workers group 2.

Total costs in equilibrium are as follows:

$$SDC^e = \frac{\delta_1}{2}\frac{N^2}{s}[1 + (\beta_r - 1)f_2^2] \tag{18}$$

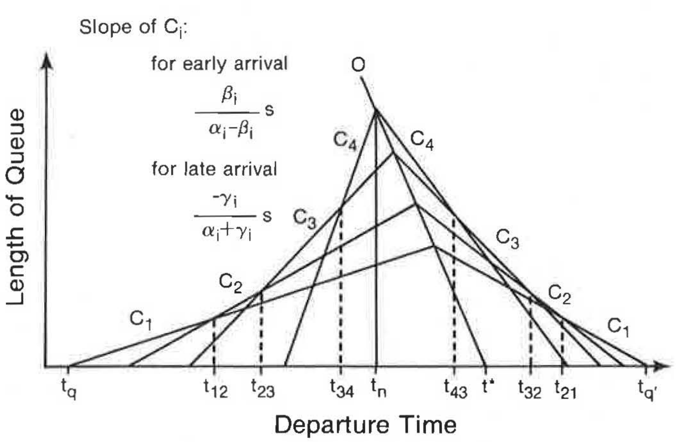

FIGURE 2 Equilibrium with four groups of commuters with different α and β and the same γ/β.

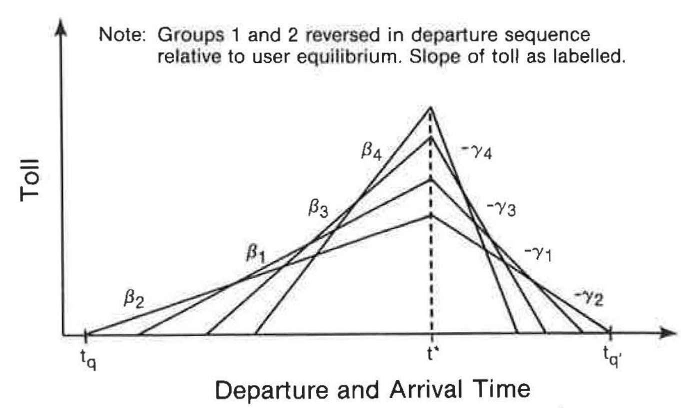

FIGURE 3 Optimal time-varying toll for groups of commuters with different α and β and the same γ/β.

$$TTC^e = \frac{\delta_1}{2}\frac{N^2}{s}[1 - 2(1 - \alpha_r)f_2 + (1 + \beta_r - 2\alpha_r)f_2^2] \quad (19)$$

$$TC^e = \delta_1 \frac{N^2}{s}[1 + (\alpha_r - 1)f_2 + (\beta_r - \alpha_r)f_2^2] \quad (20)$$

Note that total travel time costs depend on the relative unit costs of travel time for the two groups, β_r, whereas with homogeneous commuters TTC^e is independent of unit travel time costs, β.

In the social optimum there is no queueing. If $\beta_r > 1$, group 2 travels in the middle of the rush hour, just as in equilibrium, and with the same arrival time distribution. If $\beta_r < 1$, the optimal order of departure is reversed, and group 2 travels on the tails. Total costs in the social optimum are

$$SDC^o = TC^o = \begin{cases} \frac{\delta_1}{2}\frac{N^2}{s}[1 + (\beta_r - 1)f_2^2] \quad \text{if } \beta_r \geq 1 & (21a) \\ \frac{\delta_1}{2}\frac{N^2}{s}[1 - 2(1 - \beta_r)f_2 + (1 - \beta_r)f_2^2] \quad \text{if } \beta_r \leq 1 & (21b) \end{cases}$$

Expression 21a for $\beta_r \geq 1$ is identical to equation 18, while expression 21b for $\beta_r \leq 1$ obtains by interchanging subscripts in 21a.

To measure the bias introduced by treating commuters as identical, we use for the aggregate specification population-weighted average values of α and β:

$$\hat{\alpha} = (N_1\alpha_1 + N_2\alpha_2)/N = \alpha_1[1 + (\alpha_r - 1)f_2] \quad (22)$$

$$\hat{\beta} = (N_1\beta_1 + N_2\beta_2)/N = \beta_1[1 + (\beta_r - 1)f_2] \quad (23)$$

(Although this is not the only way that aggregate parameters might be determined, it is probably the most natural.) Using equations 13 and 14, the aggregate model yields the following values:

$$\hat{SDC}^e = \hat{TTC}^e = \hat{SDC}^o = \hat{TC}^o = \frac{\delta_1}{2}\frac{N^2}{s}[1 - (1 - \beta_r)f_2] \quad (24)$$

$$\hat{TC}^e = \delta_1 \frac{N^2}{s}[1 - (1 - \beta_r)f_2] \quad (25)$$

$$\hat{TTC}^o = 0 \quad (26)$$

Tolls

To determine the direction of aggregation bias in estimating the benefits from tolls, there are two cases to consider: $\beta_r \geq 1$ and $\beta_r < 1$. When $\beta_r \geq 1$, the optimal and equilibrium departure sequences coincide, so that toll benefits are simply travel time costs in equilibrium. Aggregation bias is computed by subtracting equation 19 from equation 24:

$$\hat{TTC}^e - TTC^e \overset{s}{=} (1 + \beta_r)/2 - \alpha_r \quad (27)$$

where $\overset{s}{=}$ means identical in sign. Combining equation 27 with the restriction $\alpha_r < \beta_r$ one obtains Figure 4. Aggregation leads to underestimation of toll benefits when $\alpha_r > (1 + \beta_r)/2$, and overestimation otherwise. To see why, note that since group 2 travels at the peak of the rush hour it bears a disproportionate fraction of travel time costs. If $\alpha_2 > \alpha_1$, the unit costs of travel time are underestimated in the aggregate model. This bias may be sufficient to outweigh the fact that total travel time is nevertheless overestimated with the aggregate model.

In the second case with $\alpha_r < 1$, it is clear from Figure 4 that travel-time cost savings from tolls are always overestimated with the aggregate model. However, tolls yield additional benefits by reducing schedule delay costs, so that a priori the direction of bias is unclear.

Road Investment

Much of the literature in urban transportation is concerned with the optimal capacity of the road network. Benefits are usually calculated on the assumption that road users are identical. The question arises whether aggregation creates a bias toward over- or underinvestment in capacity. Now since total costs in both the equilibrium and social optimum are inversely proportional to capacity, the marginal benefit from capacity expansion is proportional to total costs. Aggregation thus leads to overinvestment if calculated total costs exceed actual costs, and vice versa.

Subtracting equation 20 from equation 25, one finds for the user equilibrium that

$$\hat{TC}^e - TC^e \overset{s}{=} \beta_r - \alpha_r \geq 0 \quad (28)$$

Since $\beta_r \geq \alpha_r$ by assumption, total costs are overestimated with the aggregate model. Similarly, by subtracting equation 21 from equation 24 one can show that in the social optimum, total costs are overestimated unless $\beta_r = 1$. Thus, aggregation introduces a bias towards excessive road investment.

Numerical Examples

Some idea of the magnitude of aggregation bias can be obtained from the numerical examples in Table 1. Five sets of values for (β_r, α_r) are listed, corresponding to the five points labeled in Figure 4. Only the relative magnitudes of the table entries are meaningful. For simplicity, it is assumed that $N_1 = N_2$.

From the percentage figure in column 8 of Table 1, one can see that toll benefits are neither consistently overestimated nor underestimated under aggregation. In example 1, $\beta_r < 1$, and it is optimal for group 1 to travel in the middle of the rush hour, rather than group 2 as occurs in user equilibrium. The reduction in schedule delay costs from imposing an optimal toll (columns 4–7) exceeds the overestimate of travel time costs (the percentage figure in column 5). Aggregation thus leads to underestimation of toll benefits. In example 2, $\beta_r < 1$, and there is no reordering of departure times, hence no reduction in schedule delay costs. But this time, travel time costs are underestimated, so that aggregation again leads to underestimation of toll benefits. In the remaining three examples, schedule delay costs are unaffected by tolls, whereas aggregation leads to overestimation of travel time costs. Toll benefits are thus also overestimated.

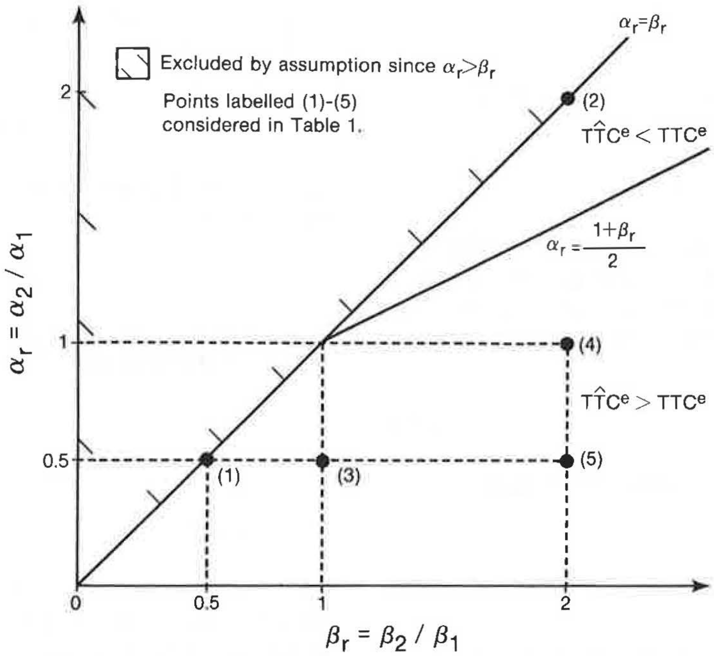

FIGURE 4 Aggregation bias in calculation of travel time costs with two groups of commuters with different α and β and the same γ/β.

TABLE 1 EXACT AND AGGREGATE CALCULATIONS OF SCHEDULE DELAY, TRAVEL TIME, AND TOTAL COSTS IN USER EQUILIBRIUM AND SOCIAL OPTIMUM WITH TWO COMMUTER GROUPS OF EQUAL SIZE DIFFERING IN α AND β, BUT WITH THE SAME γ/β

Example	β_r	α_r	Calculation	SDC^e	TTC^e	TC^e	$TC^o=SDC^o$	TC^e-TC^o
	1	2	3	4	5	6	7	8
1	0.5	0.5	Exact	0.4375	0.3125	0.7500	0.3125	0.4375
			Aggreg/Exact	85.7%	120%	100%	120%	85.7%
2	2.0	2.0	Exact	0.6250	0.8750	1.5000	0.6250	0.8750
			Aggreg/Exact	120%	85.7%	100%	120%	85.7%
3	1.0	0.5	Exact	0.5000	0.3750	0.8750	0.5000	0.3750
			Aggreg/Exact	100%	133%	114%	100%	133%
4	2.0	1.0	Exact	0.6250	0.6250	1.2500	0.6250	0.6250
			Aggreg/Exact	120%	120%	120%	120%	120%
5	2.0	0.5	Exact	0.6250	0.5000	1.1250	0.6250	0.5000
			Aggreg/Exact	120%	150%	133%	120%	150%

The aggregation bias in road investment can be determined from the values of total costs in equilibrium and the social optimum, shown respectively in columns 6 and 7. Consistent with the general results discussed above, total costs in the user equilibrium and the social optimum are never underestimated and are usually overestimated. In example 5, equilibrium costs are overestimated by 33 percent. With constant marginal construction costs, this would lead to overconstruction of road capacity by a factor $(1.33)^{1/2}$, or about 15 percent.

INDIVIDUALS DIFFERING WITH RESPECT TO γ/β

User Equilibrium and Social Optimum

In this section we allow the relative cost of late arrival $\eta \equiv \gamma/\beta$ to differ across commuters. For simplicity, α, β and t^* are assumed to be the same for everyone. Let there be G groups of commuters, indexed in order of decreasing η so that

$$\eta_1 > \eta_2 > \ldots \eta_G \qquad (29)$$

As before, N_i is the number of commuters in group i, and N the number in all groups.

User Equilibrium

User equilibrium is described by the following theorem.

Theorem 2. In user equilibrium, groups of commuters with the highest η depart late, in strict sequence of decreasing η, with group G the last to depart. Groups with lower η depart early. The order of departure of these groups is indeterminate. At most, one group departs both early and late.

Proof

Let C_G and C_{G-1} be cost curves for groups G and $G - 1$ respectively, as shown in Figure 5. Since $\eta G > \eta G - 1$, C_G has the same slope as C_{G-1} for early departure but is flatter for late departure. In equilibrium, the right-hand branch of C_G must intersect the abscissa to the right of C_{G-1}; otherwise members of group $G - 1$ could reduce their travel costs by switching to departure times chosen by group G. Since C_G cannot lie everywhere above C_{G-1}, the cost curves must intersect. Since cost curves for groups $G - 2$, $G - 3$, etc., are progressively steeper than C_{G-1} for late departure, group G must depart last, group $G - 1$ second last, and so on while individuals are departing late.

For individuals traveling early, the order of departure is indeterminate since all groups have the same cost curves. QED.

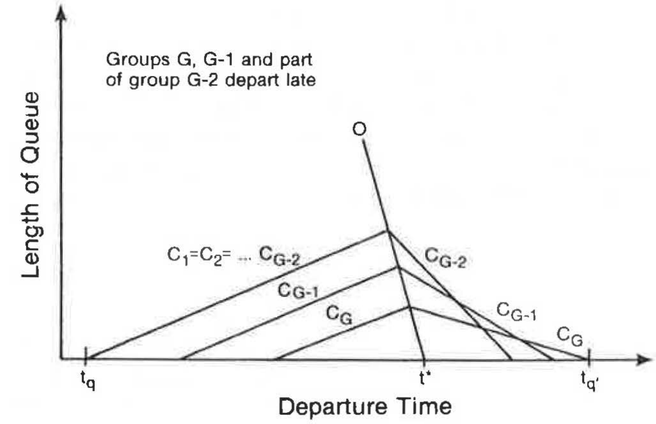

FIGURE 5 Equilibrium with groups of commuters with different γ/β.

According to theorem 2, individuals with the lowest relative cost of late arrival depart last, which is common sense.

The Social Optimum

The social optimum is intuitive. To prevent queueing, the departure rate is held at s throughout the rush hour. To minimize schedule delay costs, groups depart in strict sequence of decreasing η during late departure, just as in equilibrium.

The timing of the rush hour also turns out to be the same as in equilibrium, as was the case with commuters varying in α and β. Similarly, a time-varying toll can be constructed to decentralize the social optimum. Since schedule delay costs are minimized in the user equilibrium, toll benefits equal travel time saved.

Cost-Benefit Analysis and Aggregation Bias

At this point we again simplify by restricting attention to two groups. (The two-group case incorporates the solution for any number of groups when no more than two groups depart late.) Group 1 will largely comprise blue-collar workers, with relatively high penalties for late arrival at work. Group 2 will consist predominantly of white-collar workers. There are two cases to consider:

1. Group 2 is sufficiently small that all members depart late, as well as part of group 1,
2. Group 2 is sufficiently large that none of group 1 departs late.

Case 1

Case 1 applies when

$$f_2 < 1/(1 + \eta_2) \tag{30}$$

Aggregate costs are

$$SDC^e = TTC^e = SDC^o = TC^o = \frac{\delta_1}{2}\frac{N^2}{s}[1 - \mu_1] \tag{31}$$

$$TC^e = \delta_1 \frac{N^2}{s}[1 - \mu_1] \tag{32}$$

where

$$\delta_i \equiv \beta\gamma_i/(\beta + \gamma_i) \tag{33}$$

$$\mu_1 \equiv (1 - \eta_2/\eta_1)f_2\,[2 - (1 + \eta_2)f_2] > 0 \tag{34}$$

As is true of identical commuters, travel time costs and schedule delay costs are equal in equilibrium. Since $d\mu_1/df_2 > 0$ for values of f_2 satisfying equation 30, travel costs decrease monotonically with the proportion of commuters in group 2. Thus, in case 1, increasing the proportion of commuters with flexible work hours (i.e., converting group 1 workers to group 2) reduces total commuting costs.

Case 2

The solution for case 2 is the same as for identical commuters with $\eta = \eta_2$. Thus

$$SDC^e = TTC^e = SDC^o = TC^o = \frac{\delta_2}{2}\frac{N^2}{s} = \frac{\delta_1}{2}\frac{N^2}{s}[1 - \mu_2] \tag{35}$$

$$TC^e = \delta_2 \frac{N^2}{s} = \delta_1 \frac{N^2}{s}[1 - \mu_2] \tag{36}$$

where

$$\mu_2 \equiv \frac{1 - \eta_2/\eta_1}{1 + \eta_2} \tag{37}$$

To measure aggregation bias we again use a population-weighted average value:

$$\hat{\eta} = (N_1\eta_1 + N_2\eta_2)/N = \eta_1[1 + (\eta_2/\eta_1 - 1)f_2] \tag{38}$$

The aggregate specification yields

$$S\hat{D}C^e = T\hat{T}C^e = S\hat{D}C^o = T\hat{C}^o = \frac{\delta_1}{2}\frac{N^2}{s}[1 - \hat{\mu}] \tag{39}$$

$$T\hat{C}^e = \delta_1 \frac{N^2}{s}[1 - \hat{\mu}] \tag{40}$$

where

$$\hat{\mu} \equiv \frac{\delta_1}{2} N\left[1 - \frac{(1 - \eta_2/\eta_1)f_2}{1 + \eta_1 - f_2(\eta_1 - \eta_2)}\right] \tag{41}$$

Tolls

Because schedule delay costs at the social optimum are the same as in equilibrium, toll benefits equal travel time saved. Given equations 31, 35, and 39, the ratio of estimated to actual toll benefits in case 1 is

$$\frac{T\hat{T}C^e}{TTC^e} = \frac{1 - \hat{\mu}}{1 - \mu_i} \tag{42}$$

It is straightforward to show that $\hat{\mu} < \mu_1$ and $\hat{\mu} < \mu_2$, so that toll benefits are overestimated under aggregation.

Road Investment

The marginal benefit from capacity expansion is proportional to total costs. Given equations 32, 36, and 40, the ratio of estimated to actual total costs is

$$\frac{T\hat{C}^e}{TC^e} = \frac{1 - \hat{\mu}}{1 - \mu_i} > 1 \tag{43}$$

Hence, aggregation introduces a bias toward excessive road investment. This accords with the results shown earlier ("Individuals with Different α and β, but the Same γ/β").

INDIVIDUALS WITH DIFFERENT t^*

User Equilibrium and Social Optimum

Suppose now that commuters differ only with respect to t^*. One interpretation is that individuals differ in their starting time at work. An alternative interpretation is that individuals have the same work hours, but work at different distances from the bottleneck. Those with further to travel wish to pass through the bottleneck earlier.

Hendrickson and Kocur (*8*) address this problem under the assumption that the cumulative desired arrival time distribution, $W(t)$, crosses the cumulative arrivals schedule once, as shown in Figure 6. The equilibrium departure time distribution is the same as if all individuals desired to arrive at the crossing time $t^* = W^{-1}[\eta N/(1 + \eta)]$. Travel time costs are the same as with identical desired arrival times, while schedule delay costs are smaller. The arrival time distribution is socially optimal.

In this section, we extend Hendrickson and Kocur's analysis to discrete groups of commuters, allow for multiple crossings of the desired and actual arrival time distributions, and perform cost-benefit analysis. Groups can be thought of as teams of workers in industry or government departments with the same work hours.

Let N_i be the number of commuters with desired arrival time t_i^*, $i = 1 \ldots G$, with

$$t_1^* < t_2^* < \ldots t_G^* \quad (44)$$

Let $W(t)$ be the cumulative desired arrival time distribution:

$$W(t) = \sum_i N_i H(t - t_i^*) \quad (45)$$

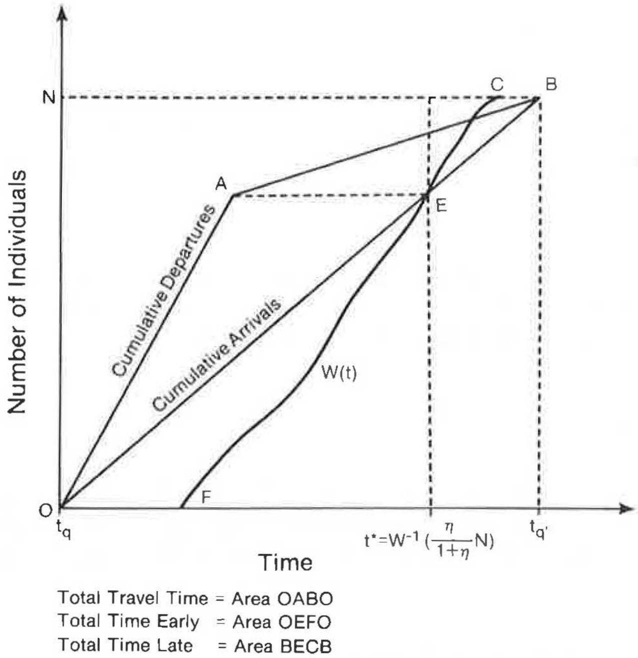

FIGURE 6 Equilibrium with a distribution of desired arrival times that crosses the cumulative arrivals schedule once.

where $H(x)$ is the Heaviside function: $H(x) = 1$ if $x \geq 0$, 0 otherwise. It is assumed that desired arrival times are sufficiently close that the bottleneck operates at capacity throughout the rush hour, implying

$$t_{q'} - t_q = N/s \quad (46)$$

User Equilibrium

The user equilibrium is described by the following theorem.

Theorem 3. Suppose that commuters differ only in desired arrival time. Let the cumulative desired arrival time distribution be $W(t)$. Then the equilibrium departure rate is given by

$$r(t) = \begin{cases} r_E^e \\ r_L^e \end{cases} \text{as } s(t - t_q) + D(t) \gtrless W\left[t + \frac{D(t)}{s}\right] \qquad t \in [t_q, t_{q'}] \quad (47)$$

where $D(t_q) = 0$, $D(t_{q'}) = 0$, and where r_E^e and r_L^e are defined in equations 7 and 8.

The intuition behind the theorem is that if at time t cumulative departures exceed cumulative desired arrivals, then the individual departing at t arrives early. If the individual and those departing just before and after him, who also arrive early, are content with their departure times then the departure rate must be r_E^e, just as with identical commuters. Similarly, when cumulative departures are less than cumulative desired arrivals, the departure rate must be r_L^e.

An example of equilibrium with three groups, in which the cumulative desired arrivals schedule $W(t)$ crosses the cumulative actual arrivals schedule three times, is shown in Figure 7. $t_{n,i}$ denotes the departure time for which an

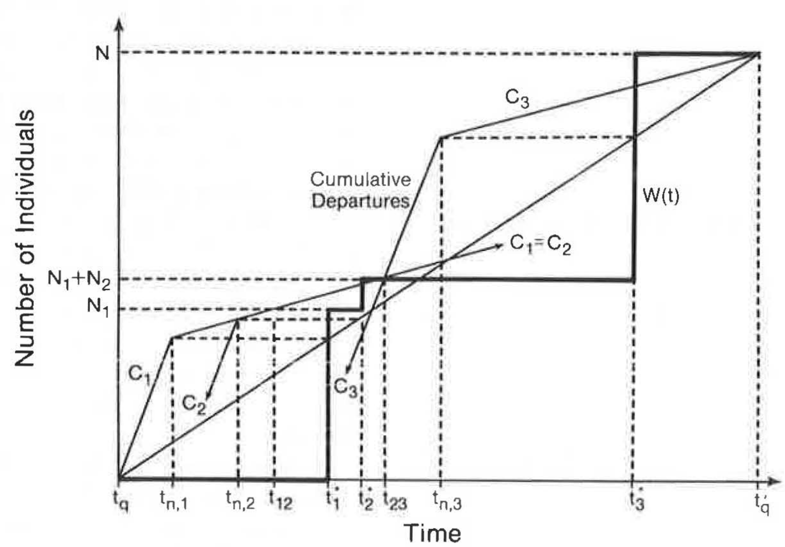

FIGURE 7 Equilibrium with three groups of commuters with different desired arrival times.

individual arrives at t_i^*. From t_q until $t_{n,1}$ cumulative arrivals exceed cumulative desired arrivals and the departure rate is r_E^e. Between $t_{n,1}$ and t_{23}, cumulative arrivals fall short of desired arrivals and the departure rate is r_e^L, and so on. All of group 2 arrives late, while some members of groups 1 and 3 are early and some late. The queue peaks at $t_{n,1}$ and $t_{n,3}$, and reaches a local minimum at t_{23}.

We have assumed that groups depart in strict sequence. If the equilibrium cost curves of two or more groups coincide, however, the order of departure is indeterminate. In Figure 7, for example, the equilibrium cost curves of groups 1 and 2, C_1 and C_2, coincide during the time interval $[t_{n,2}, t_{23}]$. Members of the two groups may depart at any time during this interval.

The Social Optimum

Since groups differ only in t^* it is clear that departure in strict sequence by group is optimal, although not necessarily the unique optimum. Both the timing of the rush hour and the indeterminacy in the order of departure are the same as in the user equilibrium. To see this, note that if two individuals in different groups are both arriving early or both late, they can be interchanged in the departure sequence without affecting total schedule delay costs. This is the same condition under which indeterminacy arises in equilibrium.

Two Groups

We now turn to a more thorough investigation of equilibrium with two groups. There are three cases to consider. In case 1, group 2 is sufficiently small that all members travel late and the timing of the rush hour is the same as if everyone belonged to group 1. Total travel time costs are the same as with identical commuters, but schedule delay costs are smaller.

In case 2, group 1 is sufficiently small that all members travel early and the timing of the rush hour is the same as if everyone belonged to group 2. Again, travel time costs are the same as with identical commuters while schedule delay costs are smaller.

In case 3, shown in Figure 8, the queue has a double peak, with group 1 traveling around the first peak and group 2 around the second. (As earlier it is assumed that t_1^* and t_2^* are sufficiently close that the rush hours for the two groups are connected.) The equilibrium cost curves of the two groups, C_1 and C_2, intersect only at t_{12}, so that all of group 1 departs before group 2. Both travel time costs and schedule delay costs are smaller than with homogeneous commuters.

Cost-Benefit Analysis and Aggregation Bias

From the preceding discussion it is clear that if t^* is assumed to be the same for everyone, then schedule delay costs are overestimated, whether the actual queue is single- or double-peaked. Travel time costs are calculated correctly if the queue is single-peaked, but overestimated if it is double-peaked.

Tolls

Since the timing of the rush hour and order of departure in user equilibrium are socially optimal, toll benefits equal travel time costs in equilibrium. Aggregation introduces

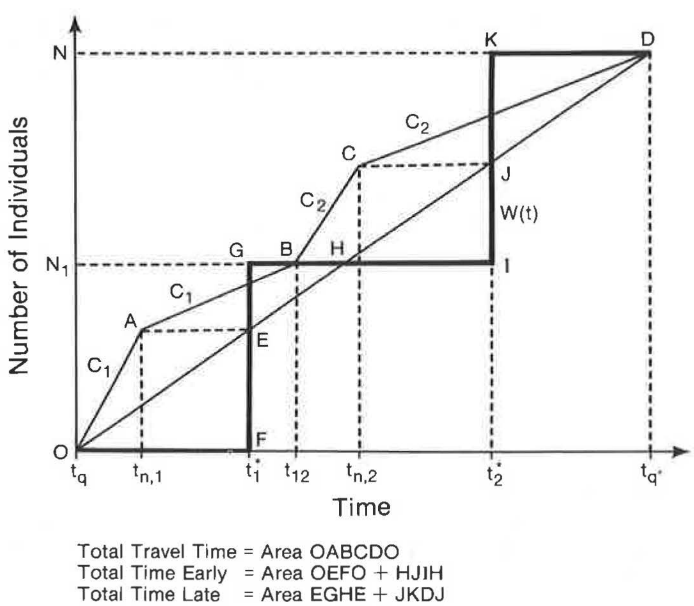

FIGURE 8 Equilibrium for two groups of commuters with different desired arrival times and a double-peaked queue.

no bias in calculated toll benefits when the queue is single-peaked because travel time costs are the same as with identical commuters. But toll benefits are overestimated when there is a double peak.

Road Investment

The marginal benefit from capacity expansion is found to be

$$|dTC^e/ds| = \delta(N/s)^2 \quad \text{with a single-peaked queue} \quad (48)$$

$$\left|\frac{dTC^e}{ds}\right| = \delta\left(\frac{N}{s}\right)^2 \left\{1 + \frac{[1 - (1 + \eta)f_2]^2}{2\eta}\right\} \quad \text{with a double-peaked queue} \quad (49)$$

Equation 48 for the single peak is the same as with identical commuters because the difference in schedule delay costs with identical and nonidentical commuters is independent of capacity.

When the rush hour is double-peaked, however, the marginal benefits of capacity expansion are greater than with identical commuters unless $f_2 = 1/(1 + \eta)$. The reason is that as capacity is expanded, the queue left by group 1 when group 2 begins to depart becomes smaller, leading to a more than proportionate decrease in overall travel time costs. This outweighs the fact that total travel costs are smaller with identical work start times, except if $f_2 = 1/(1 + \eta)$ when the two factors balance.

CONCLUDING REMARKS

We have analyzed the departure time decisions of morning commuters who differ in one of three respects: (1) travel time and schedule delay costs, (2) relative costs of early and late arrival, (3) desired arrival time. The principal results of the analysis are summarized in Table 2. In equilibrium, groups of commuters order themselves systematically in the departure sequence. Early in the rush hour the departure rate exceeds capacity, causing a queue to develop and increasing travel time costs. The timing of the rush hour is optimal in each of the above groups (1, 2, and 3) (although this is not true in general). Schedule delay costs need not be minimized in case 2, however, because

TABLE 2 SUMMARY OF RESULTS

Variable Parameters	Characteristics of User Equilibrium				Aggregation Bias[1] (Two Groups)	
	Order of Departure	Shape of Queue	Timing of Rush Hour	Schedule Delay Costs	Toll Benefits	Capacity Expansion Benefits
None (Homogeneous commuters)	N/A	Single-peaked	Optimal	Minimized	N/A	N/A
α and β (γ/β fixed)	High α/β groups travel on shoulders	Single-peaked with convex shoulders	Optimal	Generally not minimized	+ or −	+
γ/β (α and β fixed)	Smallest γ/β groups travel last	Single-peaked with convex right-hand shoulder	Optimal	Minimized	+	+
t*	In order of increasing t*, possibly indeterminate over limited periods	Single or multiple peaked	Optimal	Minimized	0 or +	0 or −

Aggregation bias refers to the error when a population of nonidentical commuters is treated as being homogeneous.
+ means the value calculated under aggregation is too large,
− means it is too small, and 0 means no bias

N/A: Not Applicable

the order in which commuters depart in user equilibrium is not necessarily optimal.

Cost-benefit studies of congestion tolls and of investments in road capacity often ignore heterogeneity of the commuting population. To measure the aggregation bias in computing benefits we used population-weighted average parameter values. We showed that aggregation usually leads to overestimation of benefits, although underestimation is possible. The direction and magnitude of bias depend on which parameters vary across commuters, and by how much.

There are several directions in which the analysis could be fruitfully extended.

Empirical Determination of Parameter Distributions

The results of this paper, as well as those of Newell (*6*), indicate that the qualitative characteristics of the departure time distribution are sensitive to the travel time and schedule delay costs of different commuter groups and to the proportion of commuters in each group. Further empirical work in the spirit of Small (*2, 12*) will be necessary before accurate cost-benefit calculations are possible, although the difficulty of obtaining adequate data may prove an obstacle.

Algorithms

For practical cost-benefit applications it will be necessary to divide the population into several groups and to allow all parameters to vary at once. Networks of roads and performance models more realistic than the idealized bottleneck should also be considered. To solve for user equilibrium in this more complex setup and to investigate the effects of smoothly varying or coarse (step-function) congestion tolls, an algorithmic approach will be necessary because analytical methods are intractable.

Welfare Considerations

Congestion tolls, road investments, and other transport or workplace policies affect the distribution of welfare. Because the cost of travel time varies with income and because socioeconomic groups differ in the flexibility of their work hours, the benefits of congestion relief tend to fall unequally. For example, both theoretical (*13, 14*) and empirical (*12, 15*) studies have concluded that the incidence of congestion tolls is probably regressive and that distributional effects can be significant compared to efficiency gains. In an earlier draft of this paper (*11*), we showed, using the bottleneck model with two commuter groups, that congestion tolls and road investments both tend to favor commuters with high values of travel time relative to schedule delay. Since such commuters are likely to be white-collar workers with above-average incomes, this finding is consistent with earlier findings that policies to reduce congestion tend to have regressive welfare effects. Clearly, the analysis needs to be done at a greater level of disaggregation and under more realistic assumptions.

REFERENCES

1. M. Ott, H. Slavin, and D. Ward. Behavioral Impacts of Flexible Working Hours. In *Transportation Research Record 767*, TRB, National Research Council, Washington, D.C., 1980, pp. 1-6.
2. K.A. Small. The Scheduling of Consumer Activities: Work Trips. *American Economic Review*, Vol. 72, No. 3, 1981, pp. 467–479.
3. A.J. Moore, P.P. Jovanis, and F.S. Koppelman. Modeling the Choice of Work Schedule with Flexible Work Hours. *Transportation Science*, Vol. 18, No. 2, 1984, pp. 141–164.
4. J.V. Henderson. *Economic Theory and the Cities*. Academic Press, New York, 1977.
5. J.V. Henderson. The Economics of Staggered Work Hours. *Journal of Urban Economics*, Vol. 9, 1981, pp. 349–364.
6. G. Newell. The Morning Commute for Nonidentical Travelers. *Transportation Science*, Vol. 21, No. 2, 1987, pp. 74–88.
7. W.S. Vickrey. Congestion Theory and Transport Investment. *American Economic Review*, Vol. 59 (Papers and Proceedings), 1969, pp. 251–261.
8. C. Hendrickson and G. Kocur. Schedule Delay and Departure Time Decisions in a Deterministic Model. *Transportation Science*, Vol. 15, No. 1, 1981, pp. 62–77.
9. P.H. Fargier. Effects of the Choice of Departure Time on Road Traffic Congestion: Theoretical Approach. In *Proceedings of the Eighth International Symposium on Transportation and Traffic Theory*, University of Toronto Press, 1983, pp. 22–263.
10. R. Arnott, A. de Palma, and R. Lindsey. Economics of a Bottleneck. *Journal of Urban Economics*, 1988.
11. R. Arnott, A. de Palma, and R. Lindsey. *Schedule Delay and Departure Time Decisions with Heterogeneous Commuters*. University of Alberta Department of Economics Working Paper No. 87–8, 1987.
12. K.A. Small. The Incidence of Congestion Tolls on Urban Highways. *Journal of Urban Economics*, Vol. 13, 1983, pp. 90–111.
13. R. Layard. The Distributional Effects of Congestion Taxes. *Economica*, Vol. 44, 1977, pp. 297–304.
14. A. Glazer. Congestion Tolls and Consumer Welfare. *Public Finance*, Vol. 36, No. 1, 1981, pp. 77–83.
15. D. Segal and T.L. Steinmeier, The Incidence of Congestion and Congestion Tolls. *Journal of Urban Economics*, Vol. 7, 1980, pp. 42–62.

Publication of this paper sponsored by Committee on Application of Economic Analysis to Transportation Problems.