Stability of Multilayer Systems Under Repeated Loads

LUTFI RAAD, DIETER WEICHERT, AND WAJIH NAJM

Multilayer systems such as pavements and railroads are subjected to repeated traffic loads of varying magnitude and number of repetitions. The accumulation of plastic strains in a given system may increase under repeated load applications, leading to a state of incremental collapse, or plastic strain may cease to increase with time, resulting in a stable response or shakedown condition. An improved numerical method for the application of shakedown theory to multilayer systems is presented. The proposed method is used to analyze two-layer pavements with varying geometry and material properties. Fatigue and shakedown analyses are also performed for two-layer systems consisting of a cement-treated layer and an asphalt concrete layer overlying a subgrade. Results of analyses are compared and discussed in relation to the design of such systems.

Multilayer systems such as pavements and railroads are subjected to repeated traffic loads of varying magnitude and number of repetitions. A major form of distress is the accumulation of plastic or permanent deformations associated with long-term repeated loading effects. The accumulation of plastic strains in a given system may increase under an additional number of load repetitions thereby leading to a state of incremental collapse or may cease to increase with time resulting in a stable response or a shakedown condition, as illustrated in Figure 1.

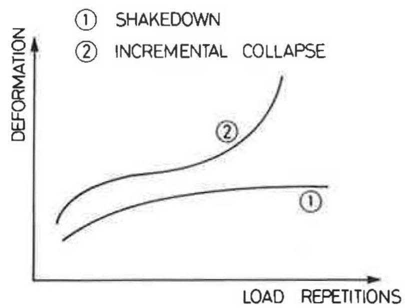


FIGURE 1 Shakedown and incremental collapse with load repetitions.

The shakedown theory was first presented by Melan (*1*). According to this theory, a system will shake down under repeated cyclic loads if a self-equilibrated residual stress field could be found such that equilibrium conditions, boundary conditions, and yield conditions are pointwise satisfied. It is assumed in this case that the material is elastic–perfectly plastic with convex yield surface and applicable normality condition and that viscous and inertial effects are negligible. The theory has been applied to discrete structures and plates (*2–5*). Although attempts have been made to apply the theory to general continua using various numerical algorithms (*6–9*), these methods fall short of satisfying all constraints associated with the shakedown theory, specifically, equilibrium conditions, boundary conditions, and yield criteria by an arbitrary chosen residual stress field in the system under consideration.

In this paper an improved numerical method for the application of shakedown theory to multilayer systems is presented. The proposed method is used to analyze two-layer pavements with varying geometry and material properties. Fatigue and shakedown performance for typical cement-treated layers and asphalt concrete layers overlying a clay subgrade is discussed.

PROPOSED NUMERICAL FORMULATION

The proposed numerical approach involves analysis of the discrete elements of the system using the finite-element method. Rectangular elements with four external primary nodes are used in this case. The displacement functions are complete to the second degree and satisfy compatibility conditions. A quasi-static analysis is implemented whereby inertial and viscous effects are assumed negligible. If stress states σ_o, σ_s, and σ_a correspond respectively to body forces P_o, statically applied load f_s, and repeated loads f_a, then according to shakedown theory the system will not collapse provided a stress increment $\Delta\sigma$ can be found such that equilibrium conditions, boundary conditions, and yield conditions are satisfied. In this respect, the following terms could be defined for a two-dimensional plane stress or plane strain analysis:

$(\sigma_{ij})_o$ = stresses due to body forces P_o at the center of a given element,
$(\sigma_{ij})_s$ = stresses due to statically applied forces f_s at the center of a given element,
$(\sigma_{ij})_a$ = stresses due to repeated loads f_a at the center of a given element,
$\Delta\sigma_{ij}$ = arbitrary stress increment applied at the center of each element,
S_{xi}, S_{yi} = result of forces in the x and y directions at a nodal point due to $\Delta\sigma_{ij}$ with respect to a global set of coordinates x-y,

L. Raad, Department of Civil Engineering, American University of Beirut, 850 Third Avenue, New York, N.Y. 10022. Current affiliation: Department of Civil Engineering, University of Alaska, 306 Tanana Dr., Fairbanks, Alaska 99775. D. Weichert, Institute of Mechanics, Ruhr University, Bochum, West Germany. W. Najm, American University of Beirut.

α = load multiplier associated with repeated loads f_a, and

f = yield function with yield occurring when $f \geq 0$.

In the analysis of a multilayer transportation support system, a Mohr-Coulomb failure criterion is proposed. The yield function (f) in this case will be given by

$$f = \sigma_1 - \sigma_3 \tan^2(45 + \phi/2) - 2C \tan(45 + \phi/2) \quad (1)$$

where

σ_1, σ_3 = major and minor principal stresses, respectively (compressive stresses are positive, tensile stresses are negative),

C = cohesion,

ϕ = angle of friction.

Determination of the shakedown load reduces mathematically to a minimizing function (Q) subject to a series of constraints stated as follows:

$$\text{Minimize } Q = -\alpha + \sum_{i=1}^{NP} (S_{xi})^2 + \sum_{i=1}^{NP} (S_{yi})^2 \quad (2)$$

Subject to the following constraints:

$$\alpha > 0 \quad (3)$$

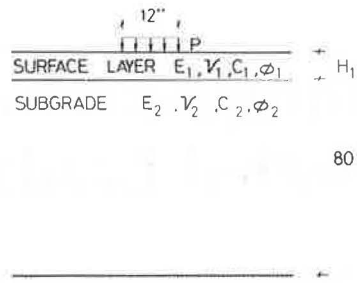
$$f[(\sigma_{ij})_o + (\sigma_{ij})_s + \alpha(\sigma_{ij})_a + \Delta\sigma_{ij}] \leq 0 \quad (4)$$

$$\sigma_3 \geq -2C \tan(45 - \phi/2) \quad (5)$$

where NP is the number of nodal points in the discretized system.

Minimizing Q, subject to the above constraints, would yield a maximum value of the load multiplier α which, when multiplied by f_a, would give the shakedown load for the system under consideration. It should be emphasized, however, that in this case, equilibrium conditions, boundary conditions, and yield criteria are satisfied in the "weak sense." This is a direct consequence of the numerical approach adopted.

Minimizing Q is based on a pattern search approach originally developed by Hooke and Jeeves (*10*). The minimization algorithm assumes a unimodal function. Therefore, if more than one minimum exists, several sets of starting solutions are recommended. However, in order to avoid this problem, the search is shifted to the vicinity of the region of interest by using a starting value of the load multiplier α_{st} such that the most critically stressed element in the system is on the verge of yielding. The search starts by determining Q for α_{st} and a set of $\Delta\sigma_{ij}$ that satisfy constraint conditions 4 and 5. During a given exploratory sequence, the α variable is allowed one disturbance in the direction of decreasing Q. Each of the stress variables is allowed as many disturbances, each equal to its step size and in the same direction, as long as the objective function Q decreases and the imposed constraints are satisfied. Otherwise, the exploratory sequence is rated a failure, and a new search is initiated about the last base point using smaller step sizes. The algorithm terminates when the values of the step sizes have been reduced to a certain preassigned value.



Where :

E_1 = Modulus of surface layer
E_2 = Modulus of subgrade layer
ν_1 = Poisson's ratio of surface layer
ν_2 = Poisson's ratio of subgrade
C_1 = Cohesion of surface layer
C_2 = Cohesion of subgrade layer
ϕ_1 = Angle of friction of surface layer
ϕ_2 = Angle of friction of subgrade layer
H_1 = Thickness of surface layer
P = Applied surface pressure

FIGURE 2 Representation of two-layer system consisting of a surface layer overlying a clay subgrade.

APPLICATIONS

The proposed numerical approach was applied to investigate the shakedown behavior of a two-layer system consisting of a surface layer overlying a clay subgrade (Fig. 2). The influence of layer stiffness, shear strength, and geometry on long-term stability under repeated loads was considered (Table 1). Fatigue and shakedown performance for cement-treated and asphalt concrete surface layers are compared in Table 2. Results of analyses are summarized below.

1. In the case where the surface layer overlies a soft subgrade, the shakedown load P_s increases with increasing surface layer modulus E_1 but decreases for higher values of E_1, as illustrated in Figure 3 (Cases 1A, 2A) and Figure 4 (Cases 5A, 6A). For lower values of E_1 the subgrade seems to be carrying a large proportion of the applied load. As E_1 increases, the subgrade will carry less, thereby resulting in a larger shakedown load. However, with further increase in E_1 the surface layer will carry most of the applied load in flexure, and the shakedown load will decrease for a given top-layer shear strength. For a stiff subgrade condition, no increase in P_s is observed with increasing E_1 for the range of values considered, which indicates that most of the load is carried through flexure of the surface layer as shown in Figure 3 (Cases 3A, 4A) and Figure 4 (Cases 7A, 8A).
2. The shakedown load increases with increase in surface layer thickness and shear strength, and subgrade stiffness and shear strength (Figs. 3 and 4).
3. Results of analyses indicate that shakedown of a system would still occur although a number of elements in the system exhibit plastic yield. In this case the plastic zone is "contained" and would not propagate further under repeated loads to induce incremental collapse. The influence of material properties on failure zones that develop when the shakedown load is applied is illustrated in Figure 5. The failure zone seems

TABLE 1 MATERIAL PROPERTIES USED IN TWO-LAYER SYSTEMS

	Surface Layer					Subgrade			
	E_1 (psi)	H_1 in	υ_1	C_1 (psi)	ϕ_1	E_2 (psi)	υ_2	C_2 (psi)	ϕ_2
Case 1A	0.5×10^6 1.0×10^6 1.5×10^6 3.0×10^6	9	0.25	100	35°	3000	0.47	3	0°
Case 2A	0.5×10^6 1.0×10^6 1.5×10^6 3.0×10^6	9	0.25	500	35°	3000	0.47	3	0°
Case 3A	0.5×10^6 1.0×10^6 1.5×10^6 3.0×10^6	9	0.25	100	35°	20000	0.47	20	0°
Case 4A	0.5×10^6 1.0×10^6 1.5×10^6 3.0×10^6	9	0.25	500	35°	20000	0.47	20	0°
Case 5A	0.5×10^6 1.0×10^6 1.5×10^6 3.0×10^6	15	0.25	100	35°	3000	0.47	3	0°
Case 6A	0.5×10^6 1.0×10^6 1.5×10^6 3.0×10^6	15	0.25	500	35°	3000	0.47	3	0°
Case 7A	0.5×10^6 1.0×10^6 1.5×10^6 3.0×10^6	15	0.25	100	35°	20000	0.47	20	0°
Case 8A	0.5×10^6 1.0×10^6 1.5×10^6 3.0×10^6	15	0.45	500	35°	20000	0.27	20	0°

to extend more into the subgrade with increasing cohesion of the top layer (Fig. 5a and b). This could be a result of increasing tensile strength of the top layer, resulting in a larger shakedown load and hence larger stresses in the subgrade layer. An increase in the number of failed elements is also obtained when the thickness of the top layer is reduced. In this case, a system with a 4-in.-thick top layer (Fig. 5c) would exhibit a much larger failure zone under applied shakedown load than a system with a 15-in.-thick top layer whose failure zone is similar to that shown in Figure 5a.

4. In a cement-treated surface layer (Cases 1B, 2B) or an asphalt concrete surface layer (Cases 3B, 4B), the variation of shakedown load with layer thickness is shown in Figure 6. An increase in layer thickness and subgrade stiffness would yield a larger shakedown load. The variation represented in Figure 6 reflects similar trends for all the cases considered. Specifically, an inflection point appears in all such representations. This would probably indicate more subgrade contribution to the shakedown load for systems with smaller surface-layer thickness.

5. A comparison between fatigue and shakedown performance for a cement-treated layer overlying a soft subgrade (Case 1B) is shown in Figure 7. The variation of the load required to initiate cracking on the underside of the stabilized

TABLE 2 MATERIAL PROPERTIES FOR SYSTEMS WITH CEMENT-TREATED AND ASPHALT CONCRETE LAYERS

	Surface Layer					Subgrade			
	E_1 (psi)	υ_1	H_1 (in)	C_1 (psi)	ϕ_1	E_2 (psi)	υ_2	C_2 (psi)	ϕ_2
Case 1B	1.5×10^6	0.25	4,6,9 15	200	35^o	3000	0.47	3	0^o
Case 2B	1.5×10^6	0.25	4,6,9 15	200	35^o	20000	0.47	20	0^o
Case 3B	1.5×10^6	0.25	4,6,9 15	500	35^o	3000	0.47	3	0^o
Case 4B	1.5×10^6	0.25	4,6,9 15	500	35^o	20000	0.47	20	0^o

layer with layer thickness for a given number of load repetitions is illustrated. Similar plots are shown for crack propagation to the surface of the cement-treated layer. The fatigue failure criterion used is presented elsewhere (*11*). Results indicate that for a given thickness of surface layer, crack initiation always occurs at a smaller load than the shakedown load. However, the load associated with fatigue crack propagation to the surface of the cement-treated layer could be greater or smaller than the shakedown load depending essentially on the thickness of the surface layer, interface conditions, and number of load repetitions under consideration. A similar illustration of fatigue failure for an asphalt concrete layer overlying a soft subgrade (Case 3B) is shown in Figure 8. The variation of applied load with layer thickness for a

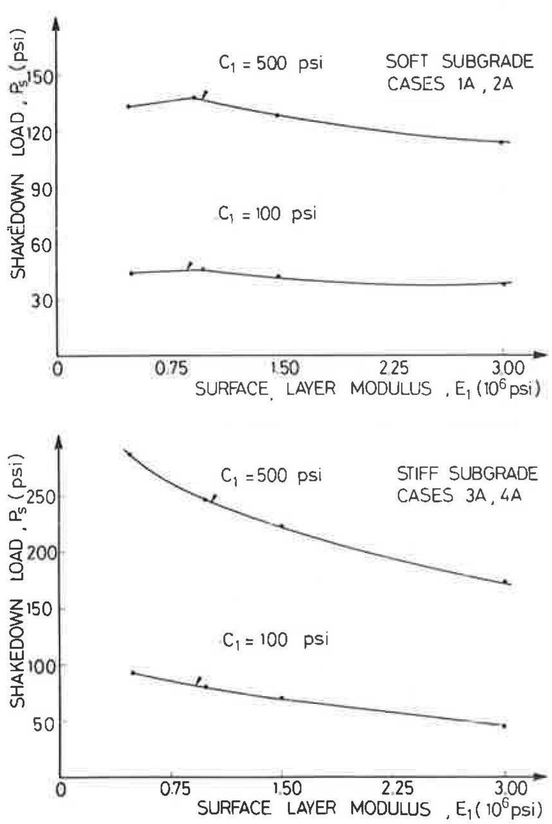


FIGURE 3 Influence of material properties on shakedown load for a 9-in. surface layer.

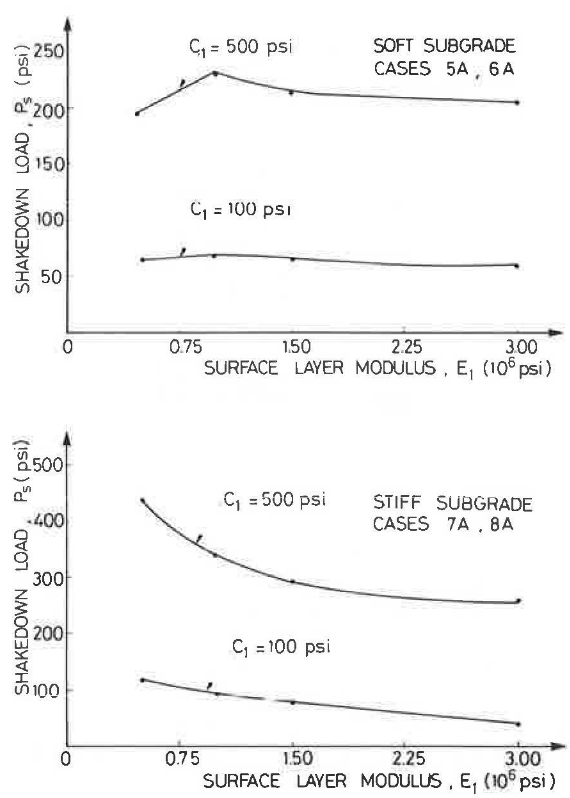


FIGURE 4 Influence of material properties on shakedown load for a 15-in. surface layer.

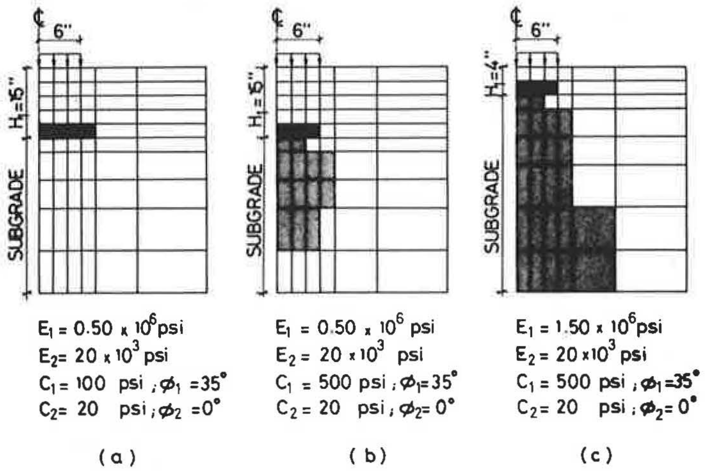


FIGURE 5 Failure zones under applied shakedown loads (scale 1:10).

given number of load repetitions required to cause fatigue in the asphalt concrete is determined using a fatigue failure criterion proposed by Brown and Pell (*12*). In this case the load required to induce fatigue in the asphalt concrete could be smaller or greater than the shakedown load, depending on layer thickness and number of load repetitions. It should be emphasized that in the case of the cement-treated layer and the asphalt concrete layer, thickness design represented by points above the shakedown curve could result in accelerated distress modes of fatigue and accumulated plastic deformations, whereas for points below the shakedown curve the system is relatively stable and could adapt to a longer service life after maintenance measures are implemented following signs of fatigue failure.

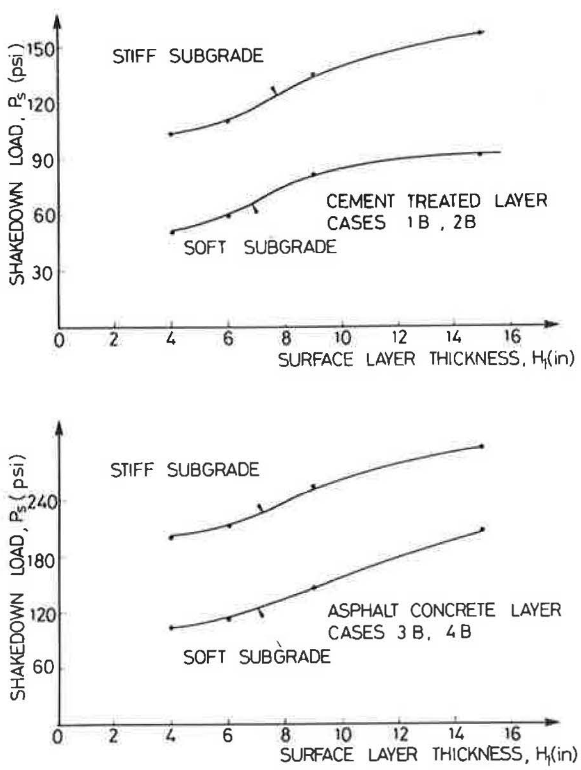


FIGURE 6 Influence of surface layer thickness on shakedown load.

SUMMARY AND CONCLUSIONS

An improved numerical method for the application of shakedown theory was proposed and applied in the analysis of two-layer systems. Results indicate that the shakedown load increases with increase in surface layer thickness and shear strength, and subgrade stiffness and shear strength. Shakedown and fatigue analyses were also conducted on two-layer systems consisting of a cement-treated layer and an asphalt concrete layer over a soft subgrade. Fatigue and shakedown behavior were compared, and results show that for a given thickness of stabilized layer the load associated with fatigue failure could be greater or smaller than the shakedown load, depending essentially on the number of load repetitions under consideration. It is concluded that design loads smaller than

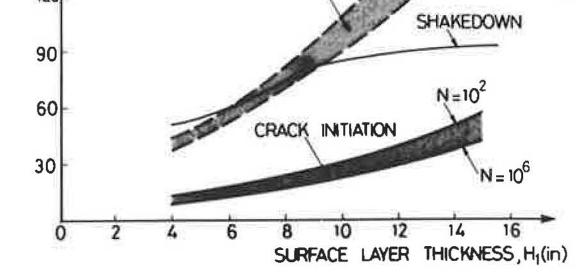


FIGURE 7 Fatigue and shakedown behavior for the cement-treated surface layer.

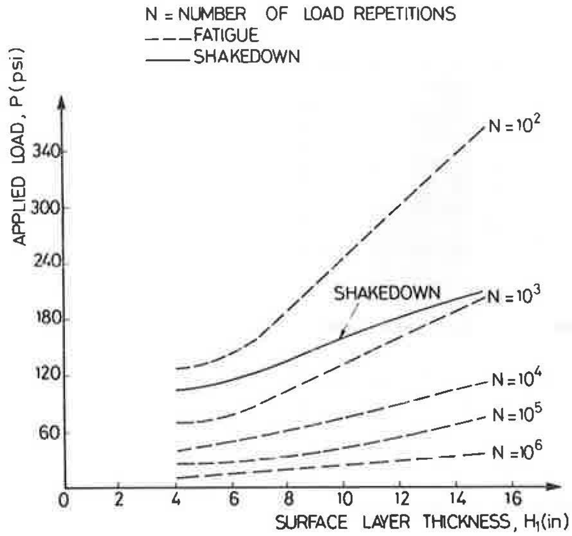


FIGURE 8 Fatigue and shakedown behavior for the asphalt concrete surface layer.

the shakedown load would result in a relatively stable response and a longer service life provided the system is periodically maintained following signs of fatigue in the surface layer.

Although the proposed numerical formulation is simple and yields a reasonable degree of convergence in the analysis of two-dimensional plane stress or plane strain multilayer systems, further research using higher order finite-element modeling and advanced optimization procedures is needed to improve shakedown predictions of pavement structures.

REFERENCES

1. E. Melan. *Theorie Statisch unbestimmter Systeme aus ideal-plastischen Baustoff.* Sitzungsberichte der Akademie der Wiissenschaften in Wien, Vol. IIa, 1936, pp. 145–195.
2. W. A. M. Alwis and P. Grundy. Shakedown of Plates under Moving Loads. *Proc., 7th Australian Conference on Mechanics of Structures and Materials.* 1980, pp. 191–196.
3. T. Belytschko. Plane Stress Shakedown Analysis by Finite Elements. *International Journal of Mechanical Sciences,* Vol. 14, 1972, pp. 619–625.
4. J. A. Konig. Shakedown Theory of Plates. *Archiwum Mechaniki Stosowanej,* Vol. 21, No. 5, 1969, pp. 623–637.
5. G. Maier and L. Corradi. Upper Bounds on Dynamic Deformations of Elastoplastic Continua. *Meccanica,* Vol. IX, March 1974, pp. 30–35.
6. B. L. Aboustit and D. V. Reddy. Finite Element Linear Programming Approach to Foundation Shakedown. *Proc., International Symposium on Soils Under Cyclic and Transient Loading.* Swansea, Wales, U.K., 1980, pp. 727–738.
7. G. N. Pande, W. S. Abdullah, and E. H. Davis. Shakedown of Elasto-Plastic Continua with Special Reference to Soil-Rock Structures. *Proc., International Symposium on Soils Under Cyclic and Transient Loading.* Swansea, Wales, U.K., 1980, pp. 739–746.
8. R. W. Sharp and J. R. Booker. Shakedown of Pavements Under Moving Surface Loads. *Journal of the Transportation Engineering Division, ASCE,* Vol. 110, No. TE1, 1984, pp. 1–14.
9. R. W. Sharp. Pavement Design Based on Shakedown Analysis. In *Transportation Research Record 1022,* TRB, National Research Council, Washington, D.C., 1985, pp. 99–107.
10. R. Hooke and T. A. Jeeves. Direct Search Solution of Numerical and Statistical Problems. *Journal of the Association for Computing Machinery,* Vol. 8, 1961, pp. 212–229.
11. L. Raad, C. L. Monismith, and J. K. Mitchell. Crack Propagation in Soil-Cement Bases Subjected to Repeated Wheel Loads. In *Transportation Research Record 690,* TRB, National Research Council, Washington, D.C., 1978, pp. 1–5.
12. S. F. Brown and P. S. Pell. A Fundamental Structural Design Procedure for Flexible Pavements. *Proc., 3rd International Conference on the Structural Design of Asphalt Pavements.* London, England, 1972, pp. 369–381.

Publication of this paper sponsored by Committee on Flexible Pavement Design.