Traffic Flow Theory and Chaotic Behavior

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Many commonly occurring natural systems are modeled with mathematical expressions and exhibit a certain stability. The inherent stability of these equations allows them to serve as the basis for engineering predictions. More complex models, such as those for modeling traffic flow, lack stability and thus require considerable care when used as a basis for predictions. In 1960, Gazis, Herman, and Rothery introduced their generalized car-follow equation for modeling traffic flow. Experience has shown that this equation may not be continuous for the entire range of input parameters. The discontinuous behavior and nonlinearity of the equation suggest chaotic solutions for certain ranges of input parameters. Understanding the chaotic tendencies of this equation allows engineers to improve the reliability of models and predictions based on those models. This paper describes chaotic behavior and briefly discusses the methodology of the algorithm used to detect its presence in the car-follow equation. Also discussed are two systems modeled with the equation and their associated chaotic properties.

Classical mathematical models for natural systems, most often linear, provide well-behaved results for a wide range of input parameters. These models, such as Greenshield's for traffic flow (*1*), are characterized as predictable, deterministic, and exhibiting a kind of stability:

$$u = u_f (1 - k/k_j) \quad (1)$$

where

u = speed,
u_f = free flow speed,
k = density, and
k_j = jam density.

Another physical example of a system with inherent stability is a pendulum displaced 5.001° from vertical and released; its motion will closely follow that of a pendulum displaced 5° from vertical. The point is that small changes in initial conditions should produce small changes in resulting motion. The same assumption is often made concerning models of more complex systems, and that assumption brings certain freedoms: if small changes produce small changes, then there is stability inherent in making predictions from a given mathematical model; if small changes in initial conditions produce large changes, care must be taken when predicting based on that mathematical model.

Other models consisting of mostly nonlinear relationships, often in the form of differential and iterative equations, provide exceptions to behavior patterns typical of the classical models. Two characteristics of mathematical models are their ability to be predictable and deterministic. Any nonlinear equation can possess both, one, or none of these characteristics. An example illustrating the difference between the predictable and deterministic would include the differential equation $dx/dt = Rx(1 - x)$, and the iterative equation $x_{n+1} = Rx_n(1 - x_n)$. The equation $dx/dt = Rx(1 - x)$, is classified as a predictable and deterministic equation—knowing $x(0)$, the value of x at any time t is $[x(0)exp(Rt)]/\{1 + [exp(Rt) - 1]\, x(0)\}$. The iterative equation $x_{n+1} = Rx_n(1 - x_n)$ is deterministic—knowing x_0 precisely gives x_1, but from some values of R it is not predictable, because the only way to find $x_{1,000,000}$ from x_0 is to iterate the equation 1 million times. This example illustrates another feature of some systems, "sensitive dependence on initial conditions." A small uncertainty in $x(0)$ will produce a small change in $x(t)$ for the differential equation, while a small change in x_0 for the iterative equation, for certain values of R, produces complete uncertainty. Specifically, if $R = 3.9$ and x_0 is between 0 and 1, every term x_n in sequence also lies between 0 and 1. Taking $x_0 = 0.4$ yields $x_{28} = 0.259$, while taking $x_0 = 0.4000001$ yields $x_{28} = 0.870$. This clearly demonstrates that this equation is sensitive to initial conditions and that small—0.0000001—changes in the input parameter can produce large changes in the results.

Unpredictability does not imply that any values for the variables can occur, and for some systems a subset of variables called an "attractor" exists to which the system evolves. Although constrained to lie on the attractor, the unpredictability arises from not knowing the long-term position on the attractor. Such behavior is often reflected in the complicated geometry of the attractor. An example of a system with a simple attractor would be (in polar coordinates) $dr/dt = r(1 - r)\ d\theta/dt = 1$. The attractor is the unit circle $r = 1$; a point $0 < r < 1$ spirals outward toward $r = 1$ and a point $r > 1$ spirals inward toward $r = 1$. Regardless of initial position, except $r = 0$, all paths are eventually arbitrarily close, traveling counterclockwise around $r = 1$ at a constant rate.

A fluid turbulence model, developed by Lorenz, was a system of three differential equations with three parameters:

$$dx/dt = -\sigma x + \sigma y$$

$$dy/dt = rx - y - xz$$

$$dz/dt = xy - bz$$

For parameter values $\sigma = 10$, $r = 28$, and $b = 2.7$, Figure 1 shows the attractor for the system. As intertracings between the two lobes show, the path does not lie on a two-dimensional surface, nor does it fill any three-dimensional region in space. This means that the attractor is a "fractal" and not a standard mathematical object. Another feature of

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fractals is that magnification of any portion of the attractor reveals increasingly finer structures, in direct contrast to standard geometrical shapes—for example, the circle, which on magnification becomes even simpler, more like a straight line.

A simple example of a fractal is the Koch curve, constructed by repeated applications of a certain geometrical process. This process involves subdividing a line segment into three equal lengths, erecting an equilateral triangle over the middle third, and removing the base of the triangle.

This process is repeated in the x-axis plane for the four segments of one-third length (Figure 2). The self-similarity of the Koch curve is apparent, and sufficiently magnifying any portion of the curve reproduces the entire shape.

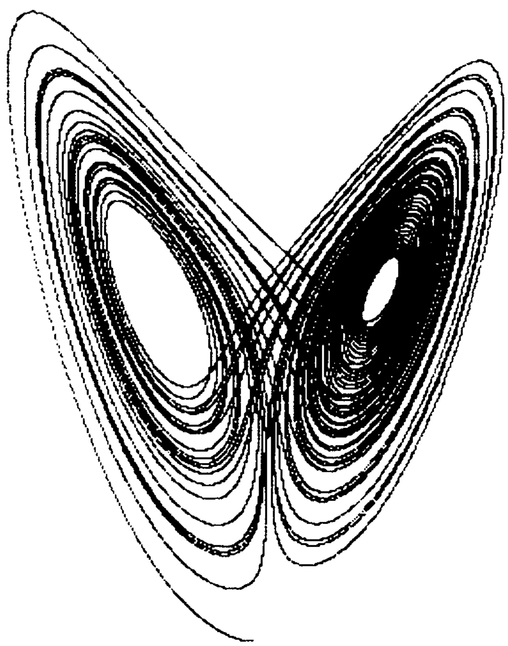

FIGURE 1 The Lorenz Attractor (x-axis is horizontal, y-axis is vertical).

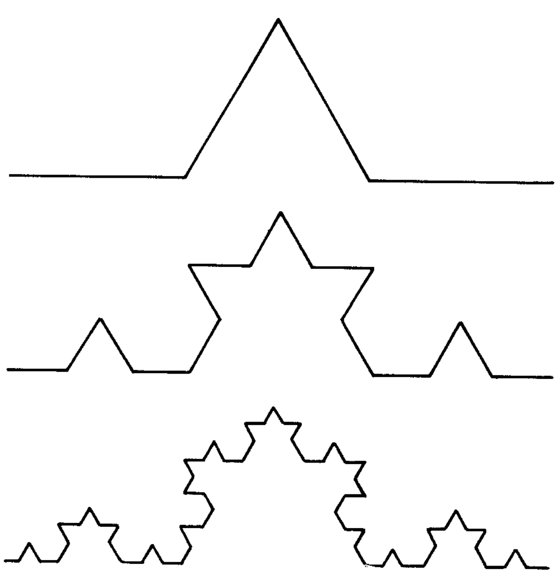

FIGURE 2 The first four stages of the Koch curve.

A "strange attractor" is one that is fractal, and chaotic dynamics are often a manifestation of a strange attractor. To determine whether a system is chaotic, strange attractors must be detected and quantified.

In the realm of dynamics, chaotic systems have three primary characteristics:

1. There is sensitive dependence on initial conditions,
2. The attractor cannot be decomposed into smaller attractors that do not interact, and
3. Any trajectory is arbitrarily close to a periodic trajectory.

Chaos, primarily associated with a state of disorder and generally considered detrimental to systems, has been discovered as a state of high order based on the geometry of the attractors. Unlike stochastic behavior, which arises from the statistical effects of treating large numbers of interacting particles representing a threshold of indeterminism, chaotic behavior is completely deterministic—but unpredictable—and occurs in systems involving as few as one variable. Predictability of chaotic systems is still limited to knowledge of long-term behavior of the associated attractor. New mathematical techniques allow attractors for nonlinear systems to be evaluated, identified, and quantified. Also, by altering input parameters, the shape of the associated attractor can be controlled and, thus, systems can be designed to produce reliable results even in chaotic states. The ability to identify and quantify attractors provides the initial steps in evaluating nonlinear chaotic systems.

Engineers have been applying the chaotic theory to thousands of systems, including thermodynamics, electrical systems, material engineering, and dynamical systems. In the field of civil engineering, chaos theory has been applied to structural vibrations and hydraulic systems. Traffic flow modeling, which contains many highly nonlinear differential equations, also offers applications for chaotic theory.

As early as 1935, engineers were developing models to describe traffic flow principles, consisting of mathematical expressions to describe basic as well as complex physical, human, and vehicular interactions. Early models for uninterrupted macroscopic traffic flow consisted of explaining functional relationships between speed, flow, and density, disregarding precise interactions between individual vehicles. Later models, called microscopic or car-following models, were developed to describe behavior of a traffic stream by the complex interrelationships involved as one vehicle follows another, and by behavior of pairs of vehicles.

In 1960, Gazis, Herman, and Rothery (GHR) developed a generalized car-following model (2) in which driver response is inversely proportional to the spacing between vehicles, as follows:

$$\ddot{X}_{n+1}(t + T) = \frac{\alpha\,[\dot{X}_{n+1}(t + T)]^m}{[X_n(t) - X_{n+1}(t)]^l}\,[\dot{X}_n(t) - \dot{X}_{n+1}(t)] \qquad (2)$$

where

= speed,
= accelerations,

X_n = position of the leading vehicle,
X_{n+1} = position of the following vehicle,
T = lag time, and
α, m, and l = constant parameters.

This model was well accepted and has since been reintroduced with various modifications. Experience with the equation has shown that it may not be continuous for the entire range of input parameters. The discontinuous behavior and nonlinearity of the GHR traffic-flow equation suggest chaotic solutions for certain ranges of input parameters. The chaotic realms represent areas where disturbances may not be dampened and predictability is limited. By identifying the range of chaotic solutions and the input parameters yielding such solutions, engineers can make greater use of these models. Also, engineers can control reliability of results in the chaotic realm by altering the shape of the associated attractor through modifications of input parameters—but this avenue must be reserved for future investigation.

This paper discusses application of chaotic theory to the GHR traffic-flow equation. It includes a brief discussion of the methodology used to detect chaos in the GHR equation—a more detailed description of the methodology can be obtained from the authors—and two examples of systems modeled using the GHR equation and their associated chaotic properties. A variety of input parameters are evaluated in a detailed system and the resulting chaotic properties are discussed.

DISCUSSION OF CHAOS

In recent years, "chaos," a new method of evaluating nonlinear dynamics, has arisen and received wide attention in journal articles on mathematics, physics, chemistry, biology, and engineering (*3,4*). Many nonlinear differential and difference equations with an adjustable parameter exhibit chaotic behavior for some ranges of that parameter. This section describes what constitutes chaotic behavior and the methods used to quantify chaos. In many examples, chaotic dynamics can be characterized by presence of a strange attractor in the state space of the system.

To quantify the complexity of strange attractors, an extension of the familiar notation of dimension is used. Consider a smooth curve C in three-dimensional space. An approximation of the length, L, of C can be obtained by finding the smallest number—$N_C(e)$—of cubes of side length, e, needed to cover C, and computing $N_C(e) \times e$. As e is taken smaller, this approximation improves and the limit $L = \lim_{e\to 0} N_C(e) \times e$. Similarly, for a smooth surface, S, in three-dimensional space, the area A is given by $A = \lim_{e\to 0} N_S(e) \times e^2$. The curve is one-dimensional, and the two-dimensional surface is exhibited by the exponent of e in the expression of length (the one-dimensional measure) or area (the two-dimensional measure).

Consider a simple example, where the curve is the line segment $C = [(x,0,0)\colon 0 \le x \le 1]$ and the surface in the square $S = [(x,y,0)\colon 0 \le x,y \le 1]$. Then for small e, $N_C(e) = 1/e$ and $N_S(e) = 1/e^2$, so $L = 1$ and $A = 1$. Notice that trying to measure the area of C yields

$$\lim_{e\to 0} N_C(e) \times e^2 = \lim_{e\to 0} (1/e) \times e^2 = 0$$

and trying to measure the length of S yields

$$\lim_{e\to 0} N_S(e) \times e = \lim_{e\to 0} (1/e^2) \times e = \infty$$

Considering just the curve C, observe that for any number $d < 1$, $\lim_{e\to 0} N_C(e) \times e^d = \infty$, and for any $d > 1$, $\lim_{e\to 0} N_C(e) \times e^d = 0$. Thus, the d-dimensional measure of curve C has the following properties: it is infinite for $d < 1$ but 0 for $d > 1$, and the length for $d = 1$. Similarly, the d-dimensional measure of surface S is infinite for $d < 2$, 0 for $d > 2$, and the area for $d = 2$.

For the Koch curve, the computation is more interesting. Taking $e = (1/3)^n$, it follows that $N(e) = 4^n$ and so the Koch curve has length

$$\lim_{n\to\infty} 4^n(1/3^n) = \infty$$

and has area

$$\lim_{n\to\infty} 4^n(1/3^n)^2 = 0$$

Thus the dimension of the Koch curve lies between 1 and 2. A straightforward calculation shows that the exponent d for which

$$0 < \lim_{e\to\infty} N(e) \times e^d < \infty$$

is given by

$$d = \lim_{e\to 0} \ln(N(e))/\ln(1/e)$$

This is the capacity dimension of the set and is closely related (and often equal) to the Hausdorff dimension. (All possible countable coverings of the set must be considered for the Hausdorff dimension, not simply those by cubes.) Observe that the Koch curve has a dimension of ln 4/ln 3.

If the dimension of a set is not an integer, then the set is a fractal, but some sets have integer dimensions that are fractals. The precise definition of fractal involves defining yet another dimension—the topological dimension—which is beyond the scope of this paper.

METHODOLOGY AND RESULTS

Methodology

This section describes the methodology used in developing a computer algorithm to test for presence of chaos in nonlinear systems. In measuring the capacity dimension of differential equation systems, counting boxes $N(e)$ can cost a lot in computer memory and time. These problems can be avoided by using Liapunov exponents. An infinitesimal sphere, centered about a point on a solution curve of the differential equation, evolves after a short time into an ellipsoid. The Liapunov exponents are natural logarithms of the ratios of the semimajor axes of the ellipsoid to the radius of the sphere, time-averaged over the trajectory.

A relationship between the capacity dimension and the Liapunov exponents is expressed in a conjecture of Kaplan and Yorke (*5*). They arrange the Liapunov exponents in nonincreasing order and allow k to be the largest integer for which the sum of the exponents is greater than 0. The Kaplan-Yorke conjecture is that

$$\sigma = k + ((\delta_1 + \delta_2 + \cdots + \delta_k)/\delta_{k+1})$$

Although there are counterexamples to this conjecture, it is often true and holds rigorously under very general conditions $\sigma < k + (\delta_1 + \cdots + \delta_k)/\delta_{k+1})$. Determining the Liapunov exponents requires some care. The authors use a method developed by Shimada and Nagashima (*6*), and also independently by Bennetin, Galgani, and Strelcyn (*7*). Together with the Kaplan-Yorke conjecture, this method gives computational access to the dimension of attractors of high-dimensional systems.

Computing the first Liapunov exponent is sufficient to test for the presence of chaos. A positive Liapunov exponent indicates stretching of nearby trajectories, thus guaranteeing the sensitive dependence on initial conditions that characterizes chaos.

As a test, this method (algorithm) was used to compute the dimension of the Lorenz attractor (Figure 1), and the accepted value of 2.06 was obtained. Because of the complexity of the calculations and the agreement to two decimal places, the algorithm used in this report was considered accurate.

Results

The GHR equation was solved by a four-point Runge-Kutta method, modified for a delay differential equation. Tangent vectors also were processed as an array, their evolutions being governed by the Jacobian of the GHR equation. To prevent focusing of the transported tangent vectors to the direction of that with the largest Liapunov exponent, the Gram-Schmidt method was applied to produce a new orthonormal basis (*7,8*). The Liapunov exponents are the natural logarithms of the lengths of the transported tangent vectors, time-averaged along the trajectory. The Kaplan-Yorke conjecture then is applied to determine the Hausdorff dimension. The initial traffic model, consisting of eight vehicles and no disturbances (i.e., intersections, signals, bottlenecks, etc.) was developed with the GHR traffic flow equation (Equation 2) and tested for the presence of chaotic behavior. The following parameter values were selected for the system:

Variable	*Description*	*Value*
n	Number of Vehicles	8
T	Lag Time	1 sec
k_j	Jam Density	260 vehicle/mile
u_f	Free Flow Speed	55 mph
u_o	Steady State Speed	40 mph
l	Constant Parameter	2

The value of l was selected, based on ranges previously used by Ceder and May (*9*). Values for two additional variables m and α were calculated, as a subroutine in the program, using equations derived from the GHR equation:

$$m = 1 - \frac{\ln[1 - k/k_j]l - 1}{\ln[u_o/u_f]}$$

and

$$\alpha = \frac{(l - 1) \times u_f (1 - m)}{(1 - m)\, k_j\, (l - 1)}$$

The step size selected was 0.01 sec, requiring the algorithm to generate matrices of 100 rows and 16 columns to compute and store values. The program was written in Pascal and designed to compute only the first Liapunov exponent, which is sufficient to detect chaotic behavior. The simplicity of this problem, as well as the cost of computer time, did not warrant calculation of the capacity dimension; that will be reserved for the next system to be discussed.

Calculation of the first Liapunov exponents for 5,000 sec required about 8 hr of CPU time on a VAX 11-785 computer. The resulting Liapunov exponents were positive, indicating sensitive dependence on initial conditions, and thus showing the presence of chaotic behavior in the GHR traffic flow equation for these parameters, even for a simple system.

Figure 3 shows change in the first Liapunov exponents for the first 500 sec. It shows oscillations that occur due to transient behavior or system noise, caused by numerical rounding.

Figure 4 illustrates change in Liapunov exponents over time for the first 5,000 sec. No oscillations are apparent because the graph scale does not allow for sufficient detail. The large positive value (about 375) of the first Liapunov exponent, resulting after the transients have died, indicates sensitive dependence on initial conditions. The magnitude of the first Liapunov exponent should not be used as an indicator of quantitative degree of chaos in the GHR equation. No mathematical evidence exists directly relating magnitude of the first Liapunov exponent to the degree of chaotic behavior present.

To further clarify this equation's sensitive dependence on initial conditions, a small sinusoidal perturbation (range between 0 and 0.1) was added to the velocity parameter of the lead vehicle. The graph of the first Liapunov exponent versus time for the sinusoidal perturbation (Figure 4) has a more pronounced peak in the curve and a lower resulting value for the first Liapunov exponent (about 355) after all transients have died out (near 5,000 sec). This indicates that the system with perturbation settles more quickly to an attractor than the undisturbed system. This system's sensitive dependence on initial conditions is clearly illustrated by a comparison of the two graphs, showing how a small change in the adjustable parameter significantly affects the shape of the solution curve for the first Liapunov exponent.

A second system, consisting of a coordinated signal network, was modeled with the GHR traffic-flow equation (Figure 5). The network had five signals spaced at intervals ranging from 500 to 1,500 ft. The network was coordinated with a 60-sec cycle, and offsets between consecutive signals were computed accordingly. It was loaded with eight vehicles at the design speed of 30 mph (44 ft/sec). Initial vehicle positions were selected so that no vehicle was located within an intersection or directly affected by a signal indication for the first second. This was necessary to allow the computer algorithm to initialize the matrices necessary to compute and store position and velocity values. Also, the network was designed so that the entrance and exit rates of vehicles were identical. This was accomplished by including 1,500 ft of additional roadway from Signal 5 to Signal 1 and simplified the modeling.

The program was modified for the network model so that each vehicle constantly looked at the light ahead of it. If the light was green, the acceleration term for that vehicle was not changed. If the light was yellow or red, a negative term was added to the acceleration, if necessary, to stop the vehicle at the light. For example, if, when the light turned yellow, the vehicle was close enough to the light to pass through the

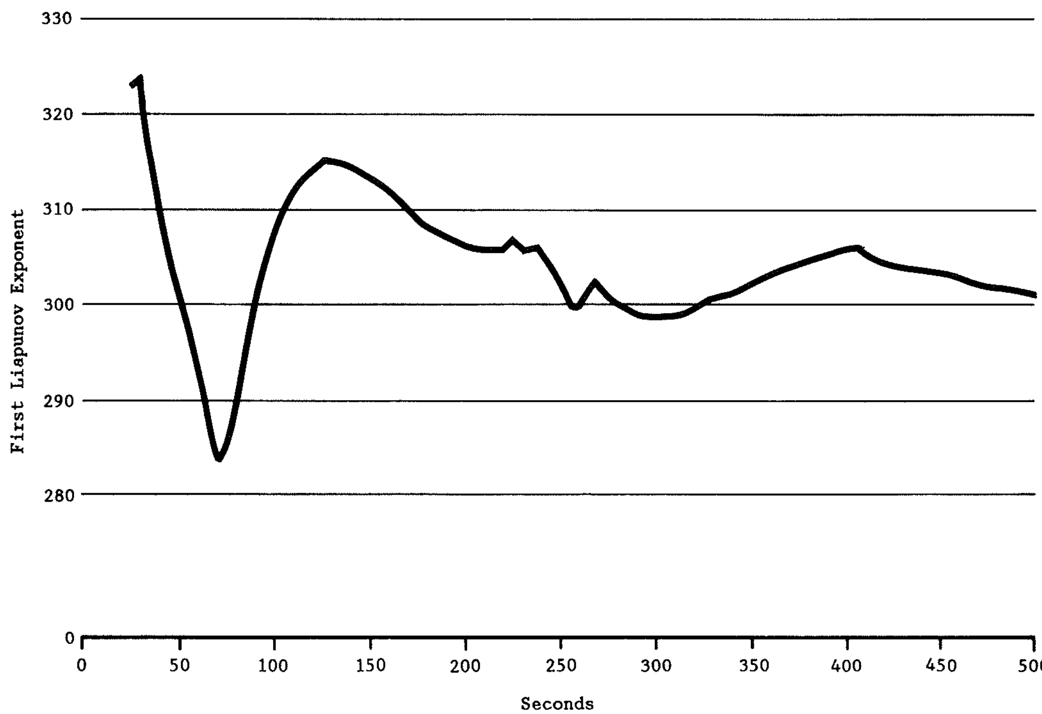

FIGURE 3 First Liapunov exponents versus time (500 seconds).

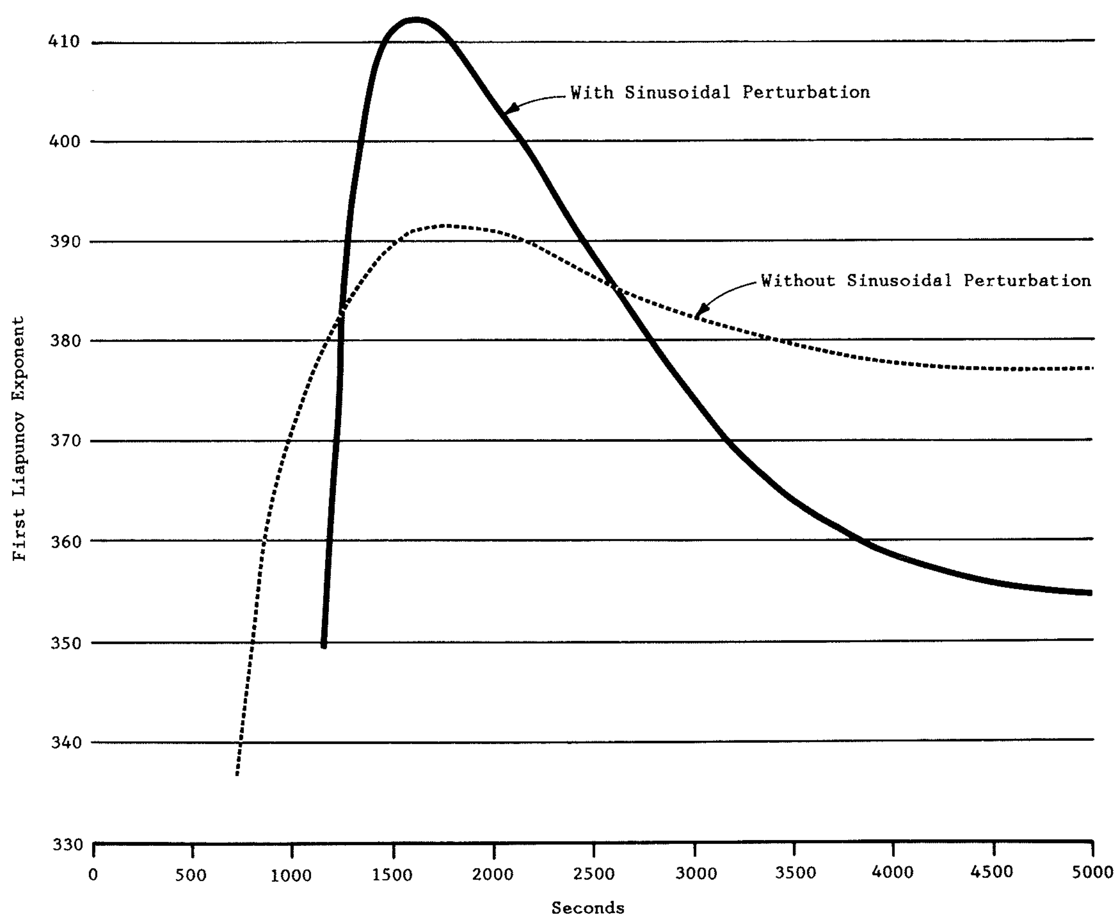

FIGURE 4 First Liapunov exponents versus time (5,000 seconds).

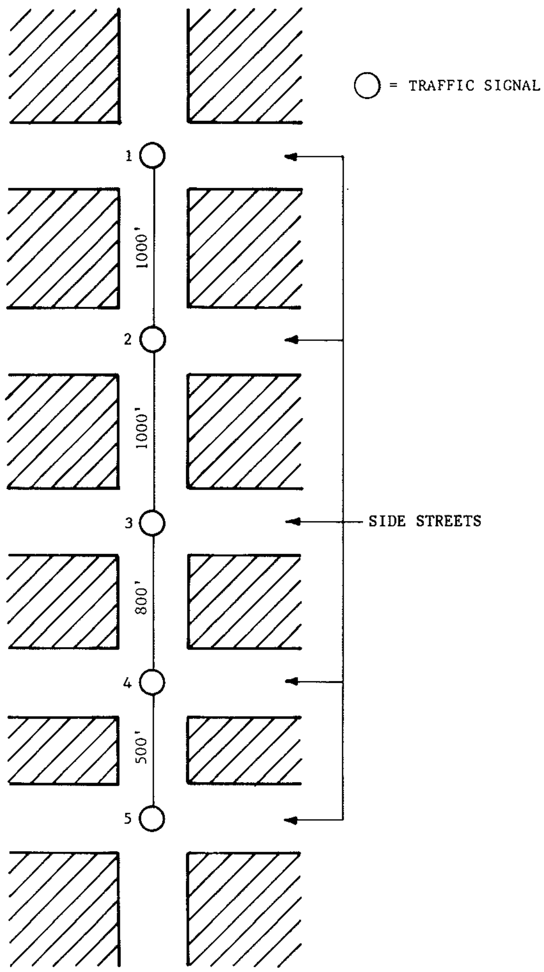

FIGURE 5 Traffic signal network.

intersection before the light turned red, then the acceleration term was not modified. If a vehicle was stopped at a red light, when the light turned green a positive acceleration term was added to bring the vehicle up to the speed limit, provided that this would not result in collision with another vehicle.

Capacity dimensions were calculated for each second for the traffic signal network, using initial speeds of 40, 44 and 50 ft/sec. Figure 6 illustrates the relationship between initial velocities and their resulting capacity dimensions. The capacity dimension for the design speed of 44 ft/sec was 14.25, indicating the presence of a strange attractor—an attractor that is fractal—to which the system can be reduced. This also shows that for an initial velocity of 44 ft/sec, 14 degrees of freedom (14 variables) are necessary to examine the system at any point in time. However, the resulting capacity dimension for initial speeds of 40 and 50 ft/sec is 16.0 (16 degrees of freedom), the maximum for this system. This further demonstrates the system-sensitive dependence on initial conditions and shows that the system modeled is inherently less complex at the design speed.

CONCLUSIONS

Chaotic behavior has been shown to exist in two relatively simple systems modeled with the GHR traffic flow equation (Equation 2). This was done by demonstrating the equation's sensitive dependence on initial conditions (positive first Liapunov exponents) and the presence of a strange attractor (indicated by noninteger capacity dimension). Two different capacity dimensions resulted from simulations using three different initial velocity parameters. The design speed of 44 ft/sec resulted in a capacity dimension of about 14, and speeds slightly higher and lower resulted in a dimension of 16. This

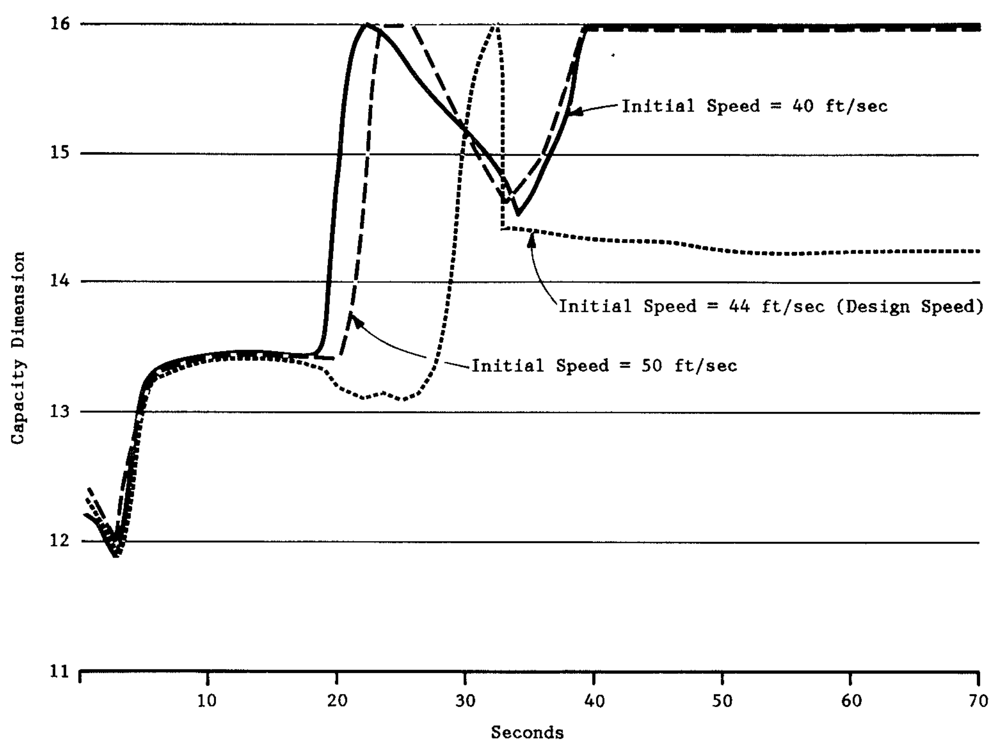

FIGURE 6 Capacity dimension versus time.

finding indicates that the degree of freedom and complexity of the system increase as speeds deviate from the design speed.

As work continues, more details regarding the attractor's geometric properties will be investigated. Knowing the geometric limitations of the attractor will improve predictions. Information on how the attractor changes shape with various input parameters will also be obtained, making more precise predictions possible for greater ranges of input parameters. Finally, attempts will be made to quantify the degree of robustness—effects caused by large changes of input parameters—further improving the reliability of predictions based on the GHR equation.

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