Maximum Entropy Spectral Analysis of Transverse Crack Spacing in Continuously Reinforced Concrete Pavements

JIAN LU, B. FRANK MCCULLOUGH, AND C. L. SARAF

The characteristics of cracks of continuously reinforced concrete pavements (CRCP) are generally evaluated by analyzing the distribution of transverse crack spacing in the pavement. Statistical analysis of data produces the mean and the standard deviation of crack spacing. However, these parameters are not always sufficient for characterizing the crack spacing of CRCP. Therefore, this paper proposes an alternate method for analyzing the transverse crack spacing data. This method, maximum entropy spectral analysis (MESA), analyzes the data in the frequency domain rather than in the space domain. In this paper, the uniformity and variability of crack spacing are also defined. By using MESA, the uniformity and variability of crack spacing can be observed in the frequency domain. The results of analyses using MESA indicate that this method can intuitively distinguish the characteristics of transverse crack spacing distribution in CRCP containing different types of coarse aggregates. The limitations of space domain analysis can be made up by MESA.

The quality of continuously reinforced concrete pavements (CRCP) is sometimes related to the transverse crack spacing characteristics. It is known that the crack spacing is distributed randomly, and therefore, statistical methods are usually used to determine the crack spacing characteristics. Intuitively, researchers analyze and evaluate the characteristics of crack spacing in the space domain (i.e., probability distribution analysis, and variance analysis). However, because of the limitations of space domain analysis, it is sometimes difficult to determine the characteristics of crack spacing of CRCP. In such cases, the alternative is to analyze the crack spacing characteristics in the frequency domain. Maximum entropy spectral analysis (MESA), which is presented here, can be used to analyze the characteristics of transverse crack spacing in CRCP in the frequency domain.

This paper introduces the MESA method for evaluating transverse crack spacing characteristics in CRCP and describes the procedure for comparing the characteristics of transverse crack spacing between two kinds of CRCP, one built with mixes that contained limestone (LS) aggregates, and the second built with mixes that contained siliceous river gravel (SRG) aggregates. Using the MESA method, it was found that the crack spacing characteristics of pavements containing LS aggregates are significantly different from those of pavements containing SRG aggregates.

Center for Transportation Research, The University of Texas at Austin, Austin, Tex. 78705-2650.

MESA is a method of analysis in the frequency domain. It can be used in other analyses, such as roughness analysis of pavement surface profiles, vehicle vertical vibration analysis, degree of passenger comfort analysis, and determination of the skid characteristic of pavement surfaces.

LIMITATIONS OF PROBABILITY ANALYSIS

It is known that if the CRCP contains N cracks, and the ith crack spacing is denoted as x_i, then the crack spacing sequence is defined as

$$\{x_i\} = \{x_1, x_2, \ldots, x_N\} \tag{1}$$

Figures 1a and 1b show crack spacing sequences for two test sections in Texas. These spacings can be represented as

$$\{x_i\} = \{6.2, 4.5, \ldots, 3, 7.4\} \text{ (ft) (50051N)}$$

and

$$\{x_j\} = \{12.4, 8.5, \ldots, 6.7, 6.9\} \text{ (ft) (130154E)}$$

These sequences are called discrete space domain sequences, because they can be measured by observations.

From Figures 1a and 1b, one cannot decide which test section has better crack spacing characteristics. Usually, the probability density function (PDF) of the crack spacing sequence can be used to characterize the pavement crack spacing sequence statistically. Our experience with the analysis of the data indicates that the crack spacing is approximately log normally distributed. The PDF of crack spacing can be represented by

$$f(x) = \frac{1}{(2\pi\, bX)^{1/2}}\, e^{\frac{-(\ln X - a)^2}{2b^2}} \tag{2}$$

where the mean value (μ) and variance (σ^2) can be expressed as follows:

$$\mu = e^{a + b^2/2}$$

and

$$\sigma^2 = e^{2a + b^2/2}\,(e^{b^2} - 1)$$

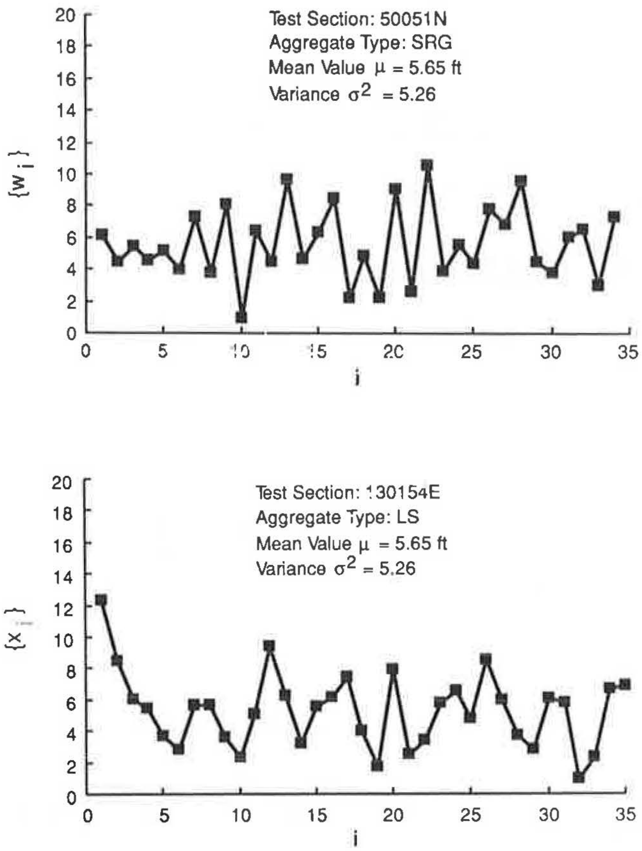

FIGURE 1 Crack spacing sequence: *Top*, test section 50051N; *bottom*, test section 130154E.

where

$$a = \frac{1}{2}\ln\left(\frac{\mu^4}{\mu^2 + \sigma^2}\right)$$

$$b^2 = \ln\left(1 + \frac{\sigma^2}{\mu^2}\right) \tag{3}$$

In Equation 2, only two parameters must be estimated, i.e., μ and σ^2. The PDF of the crack spacing can be easily obtained with these parameters. However, two parameters are not enough to characterize the crack spacing sequence, because it is possible to obtain approximately the same PDF from two different crack spacing sequences although their statistical characteristics may be different. For example, the PDFs of the crack spacing sequences of the two test sections shown in Figures 1a and 1b are almost the same, as indicated below:

$\mu = 5.65$ ft; $\sigma^2 = 5.26$ (test section: 50051N)

$\mu = 5.33$ ft; $\sigma^2 = 5.59$ (test section: 130154E)

However, their maximum entropy spectral density functions are different, as shown in Figure 2 (these two spectral density functions are normalized at frequency $\omega_k = 0$). This means that, although two crack spacing sequences present the same distribution characteristics in the space domain, such as PDF, if the sequences are transformed into a spectral density function in the frequency domain, the spectral density function of these two sequences will probably show different characteristics.

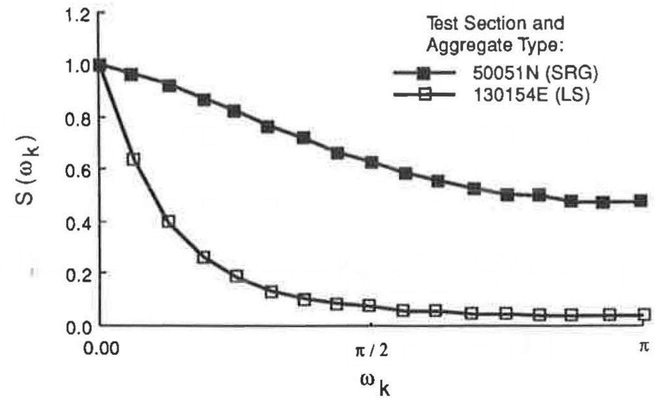

FIGURE 2 Normalized spectral density functions.

In this paper, the MESA of transverse crack spacing was used to analyze the effect of aggregate type on the transverse crack spacings in CRCP. This method provided another way to analyze transverse crack spacing characteristics in CRCP.

MAXIMUM ENTROPY SPECTRAL ANALYSIS

MESA can be as described as follows.

Discrete Spectral Transformation

From Equation 1, the transverse crack spacing sequence in CRCP can be considered as a discrete sequence, or

$\{x_i\} = \{x_1, x_2, \ldots, x_N\}$

Consider the inverse discrete Fourier transformation (*1*):

$$x_i = \frac{1}{N}\sum_{k=0}^{N-1} H_k\, e^{jik2\pi/N} \quad (i = 1, 2, \ldots, N) \tag{4}$$

where

N = length of the sequence,
x_i = the ith crack spacing,
H_k = the weights ($k = 0, 1, \ldots, N - 1$), and
$j = (-1)^{1/2}$.

Equation 4 shows that x_i can be considered as the weighted summation of sine function $e^{jik2\pi/N}$. Now define

$\omega_k = k2\pi/N \; (k = 0, 1, \ldots, N - 1)$

and then

$$x_i = \frac{1}{N}\sum_{k=0}^{N-1} H_k\, e^{ji\omega_k} \tag{5}$$

and

$e^{ji\omega_k} = \sin \omega_k i + j \cos \omega_k i$

Usually, variable ω_k is called frequency. Obviously, ω_k is within the range of 0, $2\pi(N-1)/N$. According to Equation 5, the bigger the H_k, the more sine function components with frequency ω_k the crack spacing sequence $\{x_i\}$ will contain. Mathematically, it can be proven that

$$H_k = H(\omega_k) = \sum_{i=-\infty}^{+\infty} x_i e^{ji\omega_k} \times \{\omega_k = 0, 2\pi/N, 4\pi/N, \ldots, 2\pi(N-1)/N\} \tag{6}$$

In other words, $H(\omega_k)$ is the discrete Fourier transformation of $\{x_i\}$ and is the function of frequency ω_k. Equation 6 implies that space domain sequence $\{x_i\}$ can be transformed into frequency domain sequence $\{H(\omega_k)\}$, and the characteristics of sequence $\{x_i\}$ can be analyzed in the frequency domain, i.e., knowing $H(\omega_k)$, one is able to analyze the characteristics of $\{x_i\}$.

Because $H(\omega_k)$ is an imaginary sequence, define two real functions:

$$S(\omega_k) = |H(\omega_k)|^2 \tag{7}$$

$$G(\omega_n) = \sum_{k=0}^{n} S(\omega_k) \quad 0 \leq n \leq N-1 \tag{8}$$

where $S(\omega_k)$ is called the spectral density function of sequence $\{x_i\}$, and $G(\omega_n)$ is the spectral cumulative function of sequence $\{x_i\}$.

From Equation 6, we know that the summation is from $-\infty$ to $+\infty$. In practical cases, the sequence length N is finite; that is, we cannot obtain a sequence $\{x_i\}$ with i from $-\infty$ to $+\infty$. In this case, the spectral density function $S(\omega_k)$ should be estimated from $\{x_i\}$.

Now, the problem is how to estimate $S(\omega_k)$. In the area of spectral function estimation, several mathematical methods are available. One of the best methods is the maximum entropy spectral estimation method [see Appendix and Haykin (*2*)]. Mathematically, the maximum entropy spectral density function is expressed by

$$S(\omega_k) = \frac{P_M}{\left|1 + \sum_{m=1}^{M} a_m e^{jm\omega_k}\right|^2} \tag{9}$$

where $P_M, a_1, a_2, \ldots, a_M$ are the parameters estimated by the maximum entropy spectral estimation algorithm. However, this algorithm is very complicated. Haykin (*2*) gives the detailed procedures of the maximum entropy spectral estimation algorithm; they are not included in this paper.

MESA of Crack Spacing Characteristics

From the above discussion, it is clear that spectral density function $S(\omega_k)$ represents the frequency density distribution characteristics of crack spacing sequence $\{x_i\}$, and spectral cumulative function $G(\omega_n)$ represents the frequency cumulative distribution characteristics. For example, if the crack spacing sequence $\{x_i\}$ changes smoothly, then $\{x_i\}$ contains relatively many low-frequency components. This means that the magnitude of $S(\omega_k)$ in the low-frequency region is relatively higher than that in the high-frequency section. On the other hand, if the crack spacing sequence $\{x_i\}$ changes abruptly, then $\{x_i\}$ contains relatively many high-frequency components. Therefore, the magnitude of $S(\omega_k)$ in the high-frequency region is relatively higher than that in the low-frequency region.

Now consider a concept of uniformity of crack spacing. In this paper, the uniformity of crack spacing sequence $\{x_i\}$ is used as a qualitative index. It is said that the uniformity of the crack spacing sequence is relatively good if $\{x_i\}$ changes smoothly. On the other hand, it is said that the uniformity of the crack spacing sequence is relatively poor if $\{x_i\}$ changes abruptly. For example, consider two crack spacing sequences, $\{x_i\}$ and $\{y_i\}$, such that $\{x_i\} = \{1, 2, 3, 4, 5\}$, $\{y_i\} = \{2, 5, 1, 4, 3\}$; $\{x_i\}$ has good uniformity, and $\{y_i\}$ has poor uniformity. From the viewpoint of spectral analysis, good uniformity means a relatively high magnitude of spectral density function, $S(\omega_k)$ in the low-frequency region, and relatively low magnitude of spectral density function, $S(\omega_k)$ in the high-frequency region.

As stated earlier, the uniformity of crack spacing sequence $\{x_i\}$ cannot be assessed in the space domain, but it can be assessed in the frequency domain (see Figure 2: test section 50051N has poor uniformity, but section 130154E has good uniformity).

Let us consider another index of crack spacing characteristics, represented by the variability of the crack spacing sequence. In Equation 8, the spectral cumulative function was defined. When $n = N/2$, then $\omega_n = \pi$. Now define spectral cumulative value Cv as follows:

$$Cv = G(\omega_n)_{n=N/2} = G(\pi)$$

or

$$Cv = \sum_{k=0}^{N/2} S(\omega_k) \tag{10}$$

As will be seen in this paper, the spectral cumulative value, Cv, is statistically related to the variance of the crack spacing sequence. If Cv is defined as the relative variability of the crack spacing sequence, then, statistically, the variability reflects the probability distribution characteristics of the crack spacing sequence. Obviously, a small Cv means less variability.

It is worth mentioning that uniformity and variability are independent concepts. No certain relationship exists between them. For example, as mentioned before, the uniformity of sequence $\{x_i\}$ and $\{y_i\}$ is different, but the variabilities of $\{x_i\}$ and $\{y_i\}$ are the same because they have the same variance.

APPLICATIONS OF MESA

The method described above was used to analyze CRCP crack spacing data collected from pavements in Texas. The data base that contains the information is maintained by the Center for Transportation Research, at the University of Texas at Austin. This data base contains information on CRCP con-

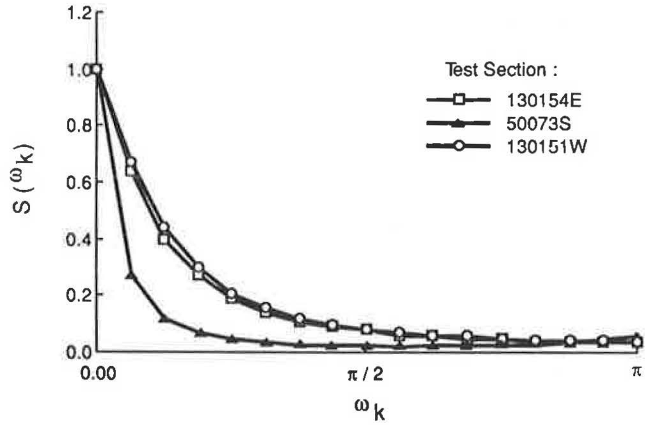

FIGURE 3 Normalized spectral density functions—aggregate type LS.

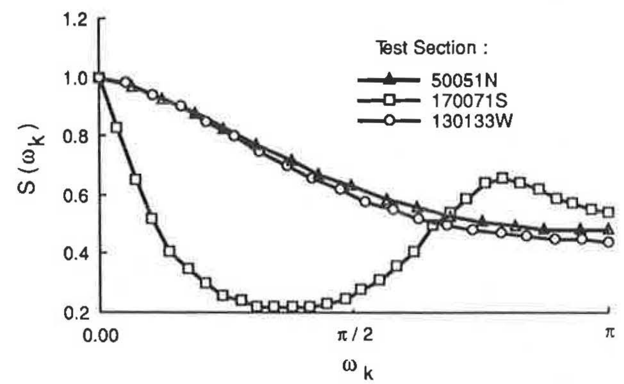

FIGURE 4 Normalized spectral density functions—aggregate type SRG.

ditions since 1974. However, transverse crack spacing data used in this study were collected in 1986.

Uniformity Analysis of Transverse Crack Spacing Characteristics of CRCP in Texas

About 300 pavement test sections were analyzed for this study. Six typical normalized spectral density functions of the crack spacing sequence are shown in Figures 3 and 4. The normalized spectral density functions shown in Figure 3 represent CRCP containing LS aggregates, and the normalized spectral density functions shown in Figure 4 represent CRCP that contain SRG aggregates. As discussed earlier in this paper, good uniformity means a relatively high magnitude of normalized spectral density function in the low-frequency region and a relatively low magnitude in the high-frequency region. From the typical normalized spectral density functions shown in Figures 3 and 4, it is clear that the magnitude of spectral density functions for CRCP with LS aggregates is relatively lower than that with SRG aggregates in the high-frequency section. Therefore, it can be concluded from these spectral density functions that the uniformity of the crack spacing sequence of pavements with LS aggregates is better than with SRG aggregates (see Table 1).

TABLE 1 RESULTS OF CORRELATION ANALYSIS

Factors	F-Values	Correlation
Aggregate Type	70.65	Good
Age of Pavement	3.40	
Subgrade Type	2.83	
Rainfall	2.36	
Cut/Fill Position	1.63	↓
Curve Position	0.34	Poor

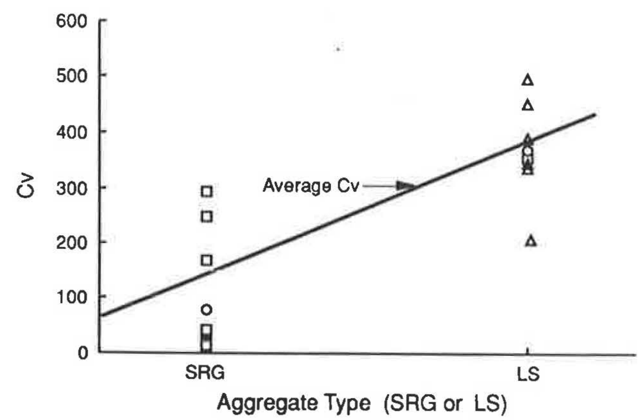

FIGURE 5 Cumulative value of spectral function versus aggregate type.

Variability Analysis of Transverse Crack Spacing Sequence in CRCP

Figure 5 shows the relationship between the cumulative value *Cv* and aggregate type. The test sections selected for this figure are listed in Table 1. As discussed earlier in this paper, less variability of the crack spacing sequence means a small *Cv* value. From Table 1, it can be found that most test sections with LS aggregates have high *Cv* values. Therefore, statistically, it can be stated that the variability of the crack spacing sequence of pavements with SRG aggregates is less than that with LS aggregates.

Effect of Pavement-Related Factors on *Cv* Values

From the standpoint of statistics, it can be hypothesized that several pavement-related factors might affect *Cv* value. To estimate the correlation of the *Cv* value and other factors, the linear regression analysis method was used. In this study, about 300 CRCP sections were analyzed. The results of correlation analysis are shown in Table 1.

As expected, aggregate type is the most important factor affecting the characteristics of the crack spacing sequence.

Characteristics Statistics of Transverse Crack Spacing in CRCP

The data analysis results for 18 typical CRCP test sections are listed in Table 2. As stated early in this paper, uniformity

TABLE 2 ESTIMATES OF VARIOUS PARAMETERS FOR CRCP SECTIONS

Test Sections	Aggregate Types	Uniformity	Variability	$P(3.5 \le X \le 8)$	σ^2	Cv
170074S	SRG	1	11	.38	2.28	43.26
50052N	SRG	2	12	.21	1.85	34.11
50052S	SRG	3	16	.23	1.75	13.82
20982W	LS	4	4	.76	3.89	351.38
170071S	SRG	5	13	.33	1.91	22.15
50051S	SRG	6	15	.18	1.15	14.14
50051N	SRG	7	7	.68	5.26	292.30
170073S	SRG	8	17	.24	1.88	13.15
130133W	SRG	9	8	.65	6.09	247.60
170075S	SRG	10	14	.04	0.51	18.63
20982E	LS	11	5	.66	8.80	347.64
50071S	LS	12	2	.50	8.84	391.59
130153W	LS	13	9	.66	4.66	204.97
170072S	SRG	14	18	.30	1.40	7.56
130134W	SRG	15	10	.40	7.35	168.07
50073S	LS	16	1	.50	6.07	499.88
130154E	LS	17	6	.56	5.60	341.10
130151W	LS	18	3	.56	4.66	353.811

and variability are related to the spectral density function and *Cv* value, and are affected by several factors. However, the aggregate type is the most important factor, and this factor is listed in Table 2 to clarify the analysis.

The uniformity and variability degrees are explained as follows.

Because 18 test sections are listed in Table 1, uniformity and variability are arranged according to 18 degrees. A larger number means a better degree.

In Table 1, $P(3.5 \le x \le 8)$ represents the probability that crack spacing is within the range of 3.5 to 8 ft, and σ^2 is the variance of crack spacing probability distribution. Statistically, from the table, the following conclusions can be obtained:

1. The uniformity of the crack spacing sequence of pavements with LS aggregates is better than with SRG aggregates because most of the pavements with LS aggregates have a large degree of uniformity;
2. The variability of the crack spacing sequence of pavements with SRG aggregates is better than with LS aggregates because most of the pavements with SRG aggregates have a large degree of variability;
3. $P(3.5 \le x \le 8)$ of the crack spacing sequence of pavements with LS aggregates is larger than with SRG aggregates statistically;
4. σ^2 of the crack spacing probability distribution of pavements with SRG aggregates is lower than with LS because most of the pavements with SRG aggregates have low σ^2 values; and
5. σ^2 is statistically related to *Cv* values (see Figure 6).

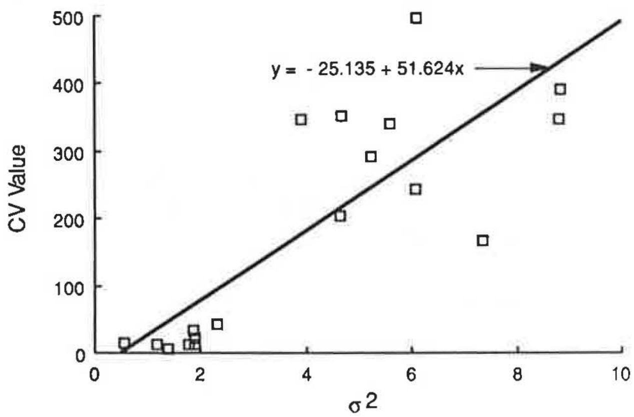

FIGURE 6 Relationship between CV value and variance of crack spacing sequence.

CONCLUSIONS

Maximum entropy spectral analysis is an analysis method of parameter estimation and characteristics evaluation in the frequency domain. It provides a new analysis domain for the analysis and evaluation of transverse crack spacing characteristics in CRCP and can overcome some limitations met by space domain analysis of transverse crack spacing in CRCP. Particularly, the uniformity and variability of crack spacing sequences can be intuitively evaluated by spectral analysis.

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APPENDIX

Simple Description of Maximum Entropy Spectral Estimation

The maximum entropy spectral estimation (MESE) method was introduced by Burg in 1968 (*3*). As is the maximum likelihood spectral estimation, MESE is a sort of estimator of parameter estimation. Consider a discrete sequence $\{x_i\}$ with sequence length N and sample interval T. If the sequence is a stationary, zero mean, approximately normally distributed, and band-limited stochastic process, then the entropy of the sequence is defined as

$$H = \frac{1}{2} \ln (2B) + \frac{1}{4B} \int_{-B}^{B} \ln [S(\omega)]\, dw \qquad \text{(A-1)}$$

where B is the band width of the sequence, and $S(\omega)$ is the spectral density function of the sequence, or

$$S(\omega) = T \sum_{m=-\infty}^{+\infty} R(m)\, e^{-jmT\omega} \qquad \text{(A-2)}$$

In Equation A-2, $R(m)$ is defined as the autocorrelation function of sequence $\{x_i\}$

$$R(m) = E\{X_i \cdot X_i + m\} \qquad \text{(A-3)}$$

Combining Equations A-1 and A-2, one can obtain the entropy

$$H = \frac{1}{2} \ln (2B) + \frac{1}{4B} \int_{-B}^{B} \ln \sum_{m=-\infty}^{+\infty} R(m)\, e^{-jmT\omega} \qquad \text{(A-4)}$$

Suppose we are given the values of autocorrelation $R(m)$ for $m = 0, 1, 2, \ldots, M$, then the corresponding extension of the autocorrelation function is defined by the convolution sum

$$R(m) = -\sum_{k=1}^{M} R(m-k)\, a_k \; (m > M) \qquad \text{(A-5)}$$

or, equivalently,

$$\sum_{k=0}^{n} R(m-k)\, a_k = 0 \; (m > M)$$

The method that Burg introduced is to maximize the entropy H with respect to $R(m)$ ($|m| > M$) with restrained condition Equation A-5, so that the parameters ($a_1, a_2, \ldots, a_M$) can be obtained. Mathematically, this can be expressed as

$$\left.\begin{array}{l} \dfrac{\partial H}{\partial R(m)} = 0 \; (|m| > M) \\ \displaystyle\sum_{k=0}^{M} R(m-k)\, a_k = 0 \end{array}\right\} \qquad \text{(A-6)}$$

It can be proved that, with the conditions in Equation A-6, sequence $\{x_i\}$ can be related by the following autoregression model [$AR(M)$ model]:

$$X_i = -a_1 X_{i-1} - a_2 X_{i-2} - \ldots a_M X_{i-m} + e_i \qquad \text{(A-7)}$$

where M is the order of the $AR(M)$ model, $\{e_i\}$ is approximately normally distributed disturbance with zero mean value. Omitting the mathematical derivation, we can obtain the estimate of the parameters ($a_1, a_2, \ldots, a_M$) by the Yule-Welker equation

$$\mathbf{R} \cdot \mathbf{A} = \mathbf{P} \qquad \text{(A-8)}$$

where $\mathbf{R}$ is the autocorrelation matrix of sequence $\{x_i\}$ and $\mathbf{R}$ is called Toeplitz matrix:

$$\mathbf{R} = \left\{\begin{array}{ccccc} R(0) & R(-1) & \ldots & R(1-M) & R(-M) \\ R(1) & R(0) & \ldots & R(2-M) & R(1-M) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ R(M-1) & R(M-2) & \ldots & R(0) & R(-1) \\ R(M) & R(M-1) & \ldots & R(1) & R(0) \end{array}\right\}$$

and

$$\mathbf{A} = \left\{\begin{array}{c} 1 \\ a_1 \\ a_2 \\ \ldots \\ a_{M-1} \\ a_M \end{array}\right\} \quad \mathbf{P} = \left\{\begin{array}{c} P_M \\ 0 \\ 0 \\ \ldots \\ 0 \\ 0 \end{array}\right\}$$

where

$$P_M = E\{e_i^2\}$$

Finally, with all the parameters estimated by the MESE algorithm, the maximum entropy spectral density function can be expressed by

$$S(\omega) = \frac{P_M \cdot T}{\left|1 + \displaystyle\sum_{m=1}^{M} a_m\, e^{-jmT\omega}\right|^2} \qquad \text{(A-9)}$$

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