Equilibrium Allocation Model of Urban Activity and Travel with Some Numerical Experiments

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An equilibrium model of urban shopping activity allocation/travel distribution is developed, with endogenous travel costs and zonal prices of goods sold. At equilibrium, revenues in each zone balance the cost of operating zonal facilities supporting the activity. This cost is assumed to be a function of the level of activity (shopping trip ends), whereas zonal demands are a gravity-type function of the prices of goods and costs of travel. A simple "quasi-balancing" algorithm is used to illustrate the sensitivity of the equilibrium solution to values of the system's parameters. The resulting shopping activity/trip ends distributions are in conformance with standard location theory results. Also, when diseconomies of scale are present in activity supply, the equilibrium solution is always unique. Otherwise, discontinuities in trip ends and interzonal travel distributions may take place, depending on the magnitude of the zonal trip ends in the zones. Thus, the model is able to reproduce rich and complex spatial patterns of activity on the basis of the interaction of economic-type variables. In conclusion, further refinements are discussed.

The standard model of travel distribution Y_{ij} may be formulated as

$$Y_{ij} = R_i e_i \frac{X_j^\sigma \exp(\beta C_{ij})}{\Sigma_j X_j^\sigma \exp(\beta C_{ij})} \quad (1)$$

where R_i is the (given) level of residential population in zone i and e_i is the (given) residents' "propensity to travel" (i.e., the trip rate). C_{ij} is the cost of travel from zone i to zone j. Parameter β represents the marginal disutility of generalized cost of travel and parameter σ represents the sensitivity of the traveler to the zone's attractive attribute, X_j (e.g., shopping facility size in the case of shopping travel).

The model also may be interpreted as an activity location (land use) model. Consequently, a possible definition of equilibrium may be that zonal demands (i.e., number of trip ends) balance the cost of supplying the particular activity in a given zone. This latter quantity may be assumed to be a function of the size of zonal facilities, leading to the following equilibrium conditions:

$$Y_j = \Sigma_i \left[R_i e_i \frac{X_j^\sigma \exp(\beta C_{ij})}{\Sigma_j X_j^\sigma \exp(\beta C_{ij})} \right] = bX_j^m \qquad \text{for } j = 1, \ldots, n \quad (2)$$

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where the inverse of b represents the marginal cost of supporting one unit of activity (e.g., a trip end) and m is a parameter representing economies of scale in activity supply. [It is worthwhile noting that this equilibrium also corresponds to the maximum of a certain consumer surplus (*1*).]

In spite of the simple form of the model, the nonlinear nature of Equation 2 implies that the behavior of the solution will depend in a complex manner on the parameter values. Several authors have, in particular, investigated the possibility of discontinuities in the equilibrium distribution of activity/travel. For instance, Harris and Wilson (*2*) have shown that for values of σ less than 1, only one stable, "nonzero" equilibrium solution (i.e., one in which there is activity in each zone) will exist, and for values greater than 1, two will exist. When this parameter goes through this critical value, a discontinous "jump" in the value(s) of (at least) one zonal activity level(s) will result.

Furthermore, other potential discontinuous changes in the values of the X_i's may be induced by changes in the value of parameter b. In this case such discontinuous changes are qualitatively similar to the "fold catastrophe" (*3*). This similarity also was demonstrated formally in the simple case of a two-zone spatial system by Kaashoek and Vorst (*4*), who showed that for values of σ less than 1, the equilibrium depends continuously on σ. In addition, if the number of destination zones is larger than the number of origin zones, there cannot exist a nonzero stable equilibrium for $\sigma > 1$. Eilon et al. (*5*) have shown that there are several nonzero solutions when $\sigma = 1$. Rijk and Vorst (*6*) have shown that for $\sigma \neq 1$, there is always a nonzero equilibrium solution, and there are at least three when the number of zones is equal to two.

The above model thus has the ability to reproduce rather rich and complex spatial patterns of activity—most importantly, the type of discontinous change that has been observed in the recent past (*7*). Nevertheless, the formulation above presents a number of drawbacks. First, zonal attractiveness is represented only by the size of the zone's facilities. It is likely that, on the contrary, travelers will be at least as sensitive to the cost of conducting the activity in the destination zone (e.g., price of goods in the case of shopping). Perhaps an even more serious drawback is that because there is no relationship between the travel times C_{ij} and the interzonal trips Y_{ij}, congestion effects are not represented.

The purpose of the research reported in this paper was to refine the above model of activity allocation/travel distribution. In this case the model is applied to the case of commercial activity/shopping travel. The fundamental improvements consist of incorporating two endogenous cost variables: the zonal prices of goods sold, P_j, and the interzonal travel times, C_{ij}. The model's properties under this more realistic formulation then could be analyzed.

THE MODEL

It is now assumed that the endogenous zonal activity cost, P_j (here the price of goods sold in zone j), is a significant zonal attribute in the choice of destination (shopping) zone. Other relevant zonal attribute(s), X_j, are now assumed to be given exogenously. The demand for interzonal travel from zone i to zone j is then

$$Y_{ij} = R_i e_i \frac{X_j^{\sigma} P_j^{\varepsilon} \exp(\beta C_{ij})}{\sum_j X_j^{\sigma} P_j^{\varepsilon} \exp(\beta C_{ij})} \quad \text{for } j = 1, 2, \ldots, j, \ldots, n \qquad (3)$$

The value of parameter ε, which translates the travelers' sensitivity to the zonal price of goods, normally would be negative. Parameter σ, which represents the travelers' sensitivity to the zone's attractiveness, normally would be positive. Parameter β, which represents the travelers' marginal disutility of travel ("distance deterrence") is negative.

In addition, it is now assumed that the cost of supplying the activity in zone j is primarily a function of the level of trip ends, of the form kY_j^{ω}. The parameter ω translates economies of scale in this connection; k has the dimension of a cost per trip end.

Finally, the interzonal cost of travel C_{ij} is made a function of Y_{ij}, the interzonal level of (shopping) travel as specified above, through the standard "B.P.R." link performance function (*8*):

$$C_{ij} = C_{ij}^{0}\left[1 + a\left(\frac{Y_{ij} + Y_{ij}^{0}}{Cap_{ij}}\right)^{b}\right] \qquad \text{for } i,j = 1,2,\ldots,n \qquad (4)$$

where a and b have the values of 0.15 and 4, respectively, for typical urban conditions. Cap_{ij} is the "practical capacity" [i.e., the volume at which link cost of travel—travel time—is 15 percent higher than the travel time at zero volume—that is, C_{ij}^{0} ("free-flow travel time")]. Y_{ij}^{0} is the fixed amount of interzonal travel corresponding to purposes other than shopping.

The condition for equilibrium is again that in each zone the revenue (i.e., trip ends times activity price) equals the cost of operating the facility:

$$P_j Y_j = kY_j^{\omega} \qquad \text{for } j = 1, 2, \ldots, n \qquad (5)$$

MODEL PROPERTIES

This section summarizes the existence, uniqueness, and stability properties of the equilibrium activity and travel configurations produced by Models 3, 4, and 5. Details of the derivation of these results may be found in Oppenheim (*9*).

Existence

A nonlinear mathematical program may be devised, in which the necessary conditions are equivalent to the model. An optimal solution will always exist to this program.

Uniqueness and Stability

It is also possible to show that for values of ω greater than 1 (which imply economies of scale in zonal facility operating cost), the uniqueness of the solution values for Y_{ij} and the corresponding zonal prices and travel costs is guaranteed. In this case also, continuous changes in the values of the system's parameters should result in continuous changes in the equilibrium solution.

When $\omega < 1$ but $\varepsilon\ (\omega - 1) < 1$, this will remain true. However, when $\omega < 1$ but $\varepsilon\ (\omega - 1) > 1$, there may now be several equilibrium activity and travel states, depending on the value of the zonal trip ends. The equality $\varepsilon\ (\omega - 1) = 1$ thus defines a critical boundary whose passage may trigger discontinuities in trip ends distribution. (This effect is illustrated in the section below.)

Properties of the Solution

If the utility of a shopping zone is measured by

$$1 + \log Y_{ij} - \sigma \log X_j - \beta C_{ij} - \varepsilon \log P_j$$

then the allocation of shopping trip ends is such that no traveler can decrease the disutility of his/her choice of destination. In that sense the macroequilibrium (i.e., with respect to zonal facilities), which originally defined the allocation of trip ends, also implies a microequilibrium, (i.e., with respect to the users of the zonal facilities and the transportation network).

SIMULATION OF THE ACTIVITY AND TRAVEL SYSTEM BEHAVIOR

In order to get some insight into the nature of the model's output, the effect of changes in the values of key parameters on the equilibrium zonal distributions of activity prices and trip ends was investigated. A hypothetical, simple system of 16 square zones of equal size was used to that effect. The zonal distributions of population (R_i values) and zonal attractiveness (X_j) initially were assumed to be uniform in all 16 zones.

The simplifying assumption that there is only one link connecting any couple of zones was also made, so that the issue of route choice and network equilibrium did not intervene. The free-flow travel time between two given zones was assumed to be proportional to the distance between zone centroids. Intrazonal travel time for all zones was arbitrarily set at 10 percent of the highest interzonal time. The fixed travel flows Y_{ij}^{0} for purposes other than shopping were the same on all links. Finally, the capacity of all links initially was assumed to be constant as well and approximately equal to the highest travel volume on the network's links.

A simple algorithm was used to provide illustrative solutions to the model. The algorithm essentially "recycles" successive values of the P_j's according to Equation 5a:

$$P_j = k(Y_j)^{\omega-1} \qquad \text{for } j = 1, 2, \ldots, n \tag{5a}$$

Given initial values Y_0 and travel costs C_0, the first estimates P_1 are evaluated. Given those, the next values Y_1 are then estimated from Equation 3. Given those, the updated travel costs C_1 are estimated from Equation 4, and the process is iterated until convergence. It may be shown that if

$$-1 < \varepsilon(1 - \omega) < 1$$

the above algorithm will converge. A more detailed discussion of convergence issues and their connection with solution uniqueness may be found in Oppenheim (*9*).

Effect of Travelers' Sensitivity to Travel Cost

β, the distance deterrence parameter, was first varied in the range from 0 to −1 for a large combination of (fixed) values of ε and ω. The results are represented in Figures 1 and 2.

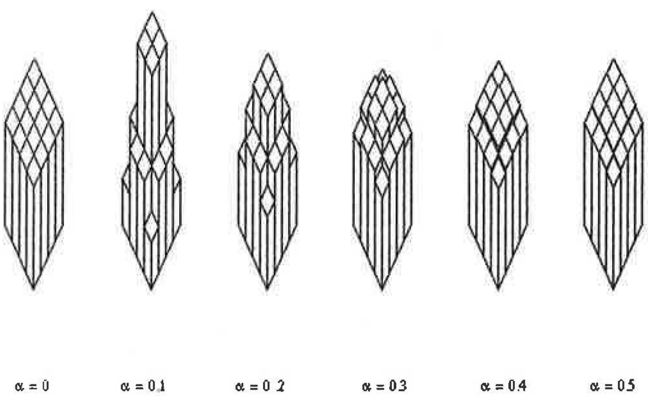

FIGURE 1 Sensitivity of trip ends distribution to changes in travel cost deterrence level.

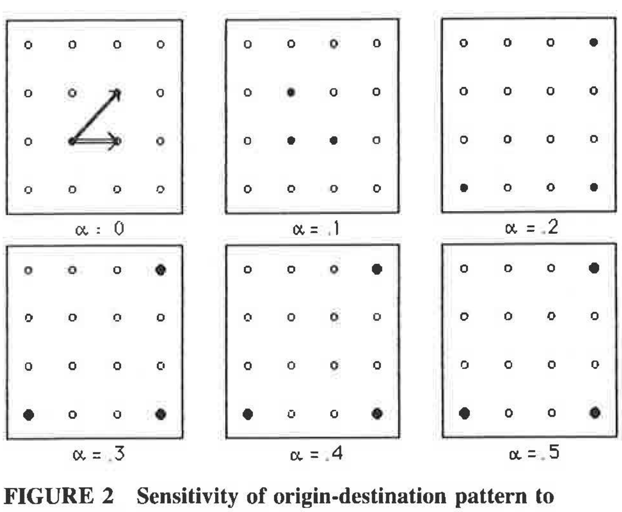

FIGURE 2 Sensitivity of origin-destination pattern to changes in travel cost deterrence level.

They may be explained in the following manner. (In all figures representing interzonal travel flows, the thickness of the arrow is log-proportional to the magnitude of the flow. A circle represents intrazonal travel, with the same meaning concerning the thickness of the diameter.) At either end of the range of values above, the effect of travel cost is negligible. When β = 0, travel cost is very low. When β = 1, it is very high so that only intrazonal travel (the cost of which is the same in all zones) is taking place. In both cases each zone should then share equally in the area's total activity, because zones are not differentiated with respect to any of the other intervening factors. Consequently, the distribution of shopping trip ends and prices should be uniform.

As the sensitivity of travelers to distance increases, all other things remaining the same, the spatial distribution of prices and trip ends becomes more concentrated at the center. The distribution of zonal prices, however, is affected differently, depending on the value of ω. For ω < 1, it also becomes more peaked in the center, because the travelers incur a higher activity price as a result of less efficient economies of scale in activity supply (owing to a more uniform zonal distribution of trip ends). Conversely, for ω > 1, the distribution of zonal prices dips in the center. These variations clearly illustrate the compensatory relationship between the travelers' activity and travel costs.

Also, the mean trip length (MTL) decreases monotonically as α decreases from its maximum at β = 0 (no-cost travel).

Effect of Sensitivity to Activity Price

Next, parameter ε, which represents a traveler's sensitivity to activity price, was varied in the range from 0 to −4, with a value of ω set at 0.6. The effects on the zonal distribution of trip ends are shown on Figure 3. It can be seen that as the activity price becomes increasingly important in the choice of destination zone, trip ends become more and more concentrated in the area's center. Conversely, when price becomes less important, the distribution of trip ends tends to become uniform.

This is due to the fact that at one end of the range, when activity price is all important, the concentration of all activity in a single zone minimizes prices through (positive) supply economies of scale. This is, in turn, a necessary response of zonal facility operators to travelers' increasing sensitivity to price. However, as travelers become less and less willing to

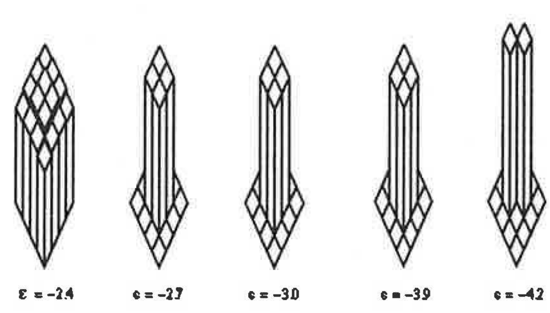

FIGURE 3 Changes in spatial structure of trip ends for changes in travelers' sensitivity to activity price.

pay high prices in nearby zones, they must travel increasingly longer distances.

At the other end, when price is not a factor ($\varepsilon = 0$) and when interzonal travel time does not intervene (e.g., for a sufficiently large value of α so that intrazonal travel, which has a constant cost, is the dominant form of travel), all zones should have the same share of the total trip ends, since a choice of destination is based on factors (i.e., zonal attractiveness and intrazonal travel) that are equal for all zones.

The passage through the critical surface $\varepsilon(\omega - 1) = 1$ identified above occurs when ε goes through the value $1/(\omega - 1) = 1/0.75 = 1.33$. The corresponding sudden change in the nature of the spatial distribution may be considered a discontinuity (*7*).

As shown in Figure 4, the spatial pattern of interzonal travel also goes through two major changes—from intrazonal travel concentrated on the periphery to intrazonal travel concentrated in the inner area and then to travel mainly from the periphery to the center. In this case, a sudden change from a predominantly intrazonal to an interzonal travel pattern takes place. Also, it is worthwhile noting that the MTL increases monotonically with a decreasing ε but does not reflect this discontinuity.

Other experiments, not represented here, confirm the expectation that in the case of negative economies of scale in activity supply (i.e., $\omega > 0$), increasing sensitivity to price implies uniformization of trip ends distribution as a mechanism for minimizing operational costs.

Variations in Economy of Scale for Zonal Activity Supply

Parameter ω next was varied in the range from 0.1 to 1.9 (the value of ε was set at -1.8). The results are represented in Figure 5. As expected from the discussion above, as economies of scale diminish (i.e., ω increases continuously), the spatial distribution of trip ends becomes uniform, although significantly more slowly after ω goes over the value 1.

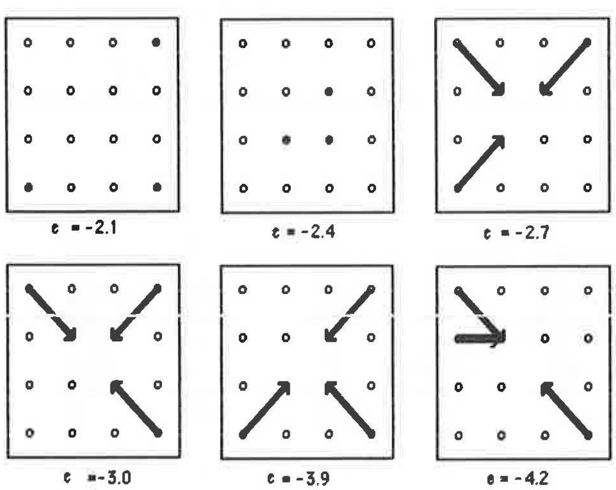

FIGURE 4 Changes in spatial structure of interzonal travel for changes in travelers' sensitivity to activity price.

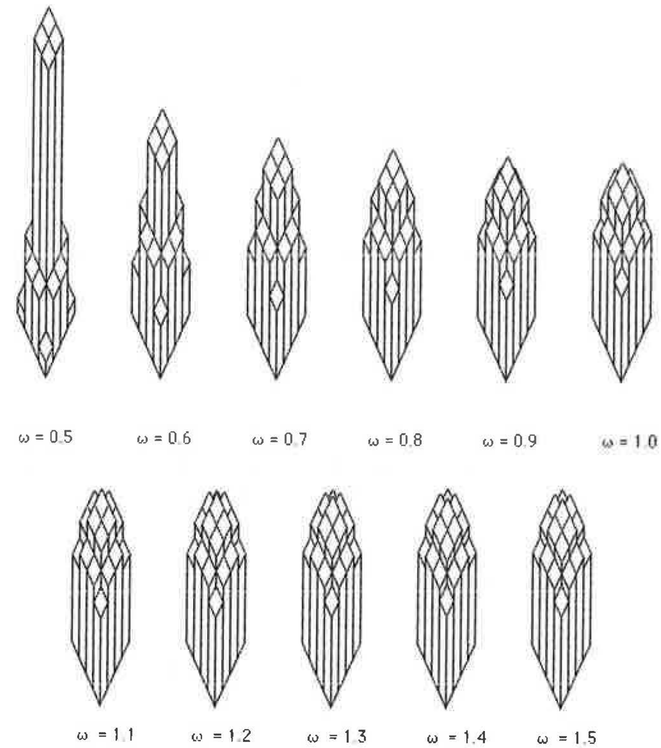

FIGURE 5 Changes in spatial structure of trip ends as a function of activity suppliers' economies of scale.

SUMMARY AND CONCLUSIONS

A one-way production-constrained model of equilibrium trip ends distribution with endogenous zonal prices and costs of travel was developed. A simple algorithm of the "quasi-balancing factor" type was used to investigate the sensitivity of equilibrium activity configurations to changes in the values of the system's parameters. The results are in general agreement with classical location theory concepts. Discontinuities were identified corresponding to critical parameter values.

The model thus is able to reproduce rich and complex spatial patterns of activity/travel on the basis of the interaction of economic-type variables. The potentiality for the occurrence of discontinuities in the model's output as a result of continuous variations in its parameter values suggests that precise parameter identification is critical for this type of model.

Because of its basic simplicity, the model lends itself to several possible extensions. First, the simplifying assumption of a single route from a given origin to a given destination might be relaxed to allow for a more accurate representation of the effects of network congestion on the joint activity/travel equilibrium. This may easily be achieved by introducing another level of spatial choice—that of the travel route—in a "nested" logit formulation. (The demand function is not strictly of the logit type in the present formulation but can easily be made of that type by changing P_i to $\ln P_{i\cdot}$.) Also, the zonal trip production rates (e_i in the model), instead of being fixed, might be determined endogenously—for instance, as a function of the activity prices and/or travel costs.

Finally, the extension of the above framework to multiple activity systems would provide new insights, into comprehensive land use/travel interactions.

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