Highway Vertical Alignment Analysis by Dynamic Programming

T. F. FWA

In this paper, an optimization program developed to produce an optimum vertical highway profile for a preselected horizontal alignment is described. The aim of the program was to provide local highway engineers and planners in Singapore with a practical aid for highway geometric design and location analysis. A dynamic programming formulation was adopted to minimize the overall cost, which includes earthwork, land acquisition, materials, and vehicle operating costs. The program's versatility allows it to model most of the cost items important in highway construction and can also be used to provide quick information on the effects of different geometric design parameters on project costs and to determine the relative importance of different cost categories. A real-life numerical example is presented to illustrate program application.

In the process of constructing new highways and relocating existing highways, a highway location study is usually performed to position the proposed highway within a corridor according to engineering, social, economic, and environmental considerations. Such an analysis typically involves a number of conflicting construction and operational requirements. A highway designer is required to achieve an economical design that is adequate operationally.

Vertical alignment design, a subproblem of the broad three-dimensional highway location problem, is of great practical significance because its solution has direct bearing on highway construction and vehicle operating costs. Vertical alignment design has attracted much research interest in the last two decades, particularly in the area of highway vertical alignment optimization. Comparisons with conventional manual designs have shown that an average savings of 15 percent of construction costs could be achieved by using computer-derived optimum alignments (*1*,*2*).

The optimization techniques that have been used for vertical road alignment studies include linear programming (*2*), quadratic programming (*3*), dynamic programming (*4*), and relaxation methods (*5*). Methods of search—direct search, random search, and gradient search (*6*,*7*)—have also been used.

In this paper, an optimization program that was developed to produce a preliminary vertical highway profile for a preselected horizontal alignment is described. The aim was to provide local highway engineers and planners in Singapore with a practical aid for highway geometric design and location analysis. A dynamic programming formulation was adopted to minimize the overall cost, which included earthwork, land acquisition, materials, and vehicle operating costs. Constraints were carefully selected and modeled to suit local conditions and design and construction practices. Parametric and sensitivity studies were conducted, based on a real-life example, to illustrate the salient features of the program.

Department of Civil Engineering, National University of Singapore, Kent Ridge, Singapore 0511.

MODELING OF ALIGNMENT PROBLEM

Three basic elements can be identified in modeling the problem of vertical alignment optimization of a highway: (a) representation of input ground profile and output road alignment, (b) formulation of objective function, and (c) definition of construction constraints and operational requirements. Detailed descriptions of these aspects of the optimization program are included below.

Ground Profile and Road Alignment

The natural ground profile is established along the central axis of the proposed highway, the horizontal alignment of which has been predetermined and fixed. This profile is input as a series of straight-line segments, each spanning an equal horizontal distance measured along the central axis of the highway. These equal horizontal intervals, which are determined by the grid size selected for a particular problem, also determine the line segment intervals over which the output road alignment will be represented. Smoothing of the piecewise-linear alignment by a design engineer is required to give a practical curved profile.

The dynamic programming optimization process uses a set of vertical data grid lines spaced at the horizontal grid intervals mentioned above, as shown in Figure 1. Both the piecewise-linear input ground profile and output road alignment pass through one grid point each on each vertical grid line. The optimization algorithm determines one point on each grid line to represent the output alignment that satisfies specified constraints and yields the lowest overall cost.

Objective Function

The objective function is the overall cost that represents the sum of the following four cost components: (a) earthwork cost, (b) pavement cost, (c) land acquisition cost, and (d) vehicle operating cost.

Earthwork Cost

Earthwork cost may be divided into two major components: (a) cost of cutting when the computed road alignment level

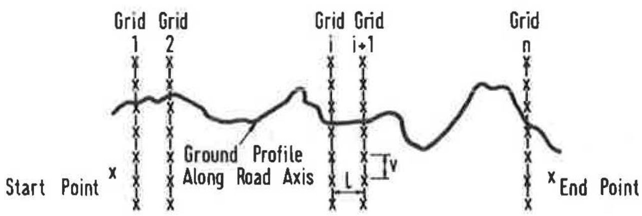

FIGURE 1 Data grid for vertical alignment optimization.

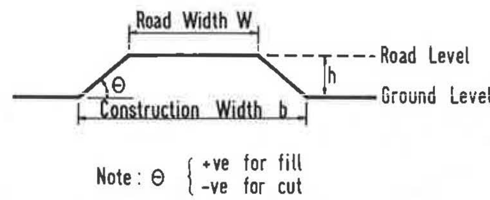

FIGURE 2 Simplified cross section for earth volume computation.

is lower than the existing ground level and (b) cost of filling or embankment construction when the required road alignment lies above ground level. These two cost components can be further subdivided into excavation or backfilling cost, transport cost, and roadbed preparation cost according to the size of project.

Excavation and backfilling costs are basically functions of the volume of earth involved, although the unit rate of excavation may vary with the depth of cut and the type of soil. Transport costs can be computed once the hauling distance is known. Locations of dumping sites and borrow pits are required as input to the optimization program. Roadbed preparation costs vary with the type of soil upon which the proposed road is to be constructed.

For the purpose of cut- or fill-earth volume calculation, the cross sections of finished roadbed were simplified (see Figure 2). The cross-sectional area of cut or fill at any grid point is completely defined by three variables: (a) roadbed width, w; (b) cut or embankment slope represented, θ; and (c) the difference between computed road alignment level and ground level, h. End area method was used to compute the volume of cut or fill required between any two grid lines.

Pavement Cost

Pavement cost, computed from pavement thickness and unit costs of pavement materials used, is basically a function of the quality of roadbed soil. In most highway projects, soil type and strength are found to vary along the length of the route. Pavement thickness requirement is also likely to vary between fill and cut sections, as well as among different depths of cut at a given point.

Because the type and strength of soil is a function of grid line location, as well as the relative vertical position of roadbed with respect to ground surface, it is necessary to compute unit pavement cost for each grid point. The pavement cost for each road segment between two vertical grid lines was computed by multiplying the length of the road segment by the arithmetic mean of the unit pavement costs at the two grid points which defined the road segment.

Land Acquisition Cost

Land acquisition cost was defined as the product of unit area land cost multiplied by the area of land requirement for highway right-of-way. The unit cost of land may vary along the length of the proposed route. Depending upon the type of construction (cut or fill) and the nature of land development on the two sides of road, right-of-way width may also vary along the length of the route.

When the unit land cost and width of right-of-way requirement are known, the land acquisition cost at each grid point can be computed. Figure 2 shows that the right-of-way width at a grid point depends on the construction width, b, at the ground level. The magnitude of width b is, in turn, related to the difference between road alignment and ground level. To obtain the land acquisition cost for any line segment, the arithmetic mean of land acquisition costs at the two corresponding grid points was computed, and the mean value was multiplied by the road segment length to calculate the desired land acquisition cost.

Vehicle Operating Cost

Two elements that influence the cost incurred by road users were considered—length and gradients. These factors can be conveniently combined into a single input variable expressed in terms of total user cost/percent gradient/kilometer (or mile) of road length. This cost was computed based on a period equal to the design service life in the highway. The total highway traffic volume expected over this period must be estimated for computing the unit user cost.

Constraints

The following constraints were included in the optimization program: (a) a maximum gradient, (b) curvature requirements, and (c) control points with fixed levels. The numerical control values for each constraint were based on the design standards and requirements provided by the Singapore Public Works Department (*8*).

Gradient control is a traffic operational requirement to ensure smooth vehicle movement. The maximum allowable gradient is a function of design highway speed, as well as the types of vehicles included in the design traffic stream. The constraint on gradient can be dealt with in the following manner:

$$|Y_i - Y_{i-1}| \leq G\,d \qquad i = 1, 2, \cdots, N \tag{1}$$

where

Y_i = road alignment level at grid line i;
G = maximum allowable gradient;
d = horizontal distance between two successive grid lines; and
N = total number of grid intervals.

Curvature requirements were achieved by controlling the magnitude of algebraic change in gradient between two consecutive line segments. Adopting the recommended practice of Singapore Public Works Department (8), the absolute value of gradient change, g, which was derived from considerations of highway safety, aesthetics, and comfort of ride, were specified separately for crest and sag vertical curves as follows:

For crest vertical curves:

$$g \leq 425/(2S - L) \qquad \text{when } L \leq S \tag{2}$$

$$g \leq 425L/S^2 \qquad \text{when } L > S \tag{3}$$

For sag vertical curves:

$$g \leq (122 + 3.5S)/(2S - L) \qquad \text{when } L \leq S \tag{4}$$

$$g \leq (122 + 3.5S)L/S^2 \qquad \text{when } L > S \tag{5}$$

where

g = absolute algebraic difference in gradient (%);
L = length of vertical curve (m); and
S = sight distance (m).

The sight distance, S, is a function of design speed, vehicle type and roadway gradient. The allowable gradient change, g, would therefore also vary with the geometric parameters selected by the designer.

The third constraint, control points with fixed levels, is commonly encountered in actual highway design problems. The levels of the start and end points of a new highway are typically fixed. Intermediate fixed-level control points are needed where a new highway intersects existing roads.

DYNAMIC PROGRAMMING FORMULATION

The dynamic programming formulation for the vertical alignment problem is summarized in this section. The objective was to minimize the total sum of selected costs incurred in the construction of a new highway. The objective function was

$$\text{Min} \sum_{k=0}^{N-1} [C_1(U_k) + C_2(U_k) + C_3(U_k) + C_4(U_k)] \tag{6}$$

Subject to the following constraints:

Slope

$$|U_k| \leq G\,d \qquad k = 0, 1, \cdots, N - 1 \tag{7}$$

Curvature

$$|U_{k+1} - U_k| \leq 2d(g/100) \qquad k = 0, 1, \cdots, N - 1 \tag{8}$$

System equations

$$U_k = Y_{k+1} - Y_k \qquad k = 0, 1, \cdots, N - 1 \tag{9}$$

For boundary conditions, Y_o and Y_N are prescribed.

All the variables and symbols in the constraint equations 7, 8, and 9 have been explained in the preceding section and are graphically represented in Figure 3. $C_1(U_k)$, $C_2(U_k)$, $C_3(U_k)$, and $C_4(U_k)$ represent earthwork, pavement, land acquisition, and vehicle operating costs, respectively, for the line segment k bounded by grid lines k and $k + 1$. Each of the four cost components is a function of the positions of ground level $f(x)$ and road alignment $y(x)$, where x is the horizontal distance measured from the start point along the roadway central axis. The four cost components can be computed once the relative position of road alignment with respect to ground surface is known, as represented by $h_k = y(x_k) - f(x_k)$ in Figure 3.

APPLICATION

The work reported in this paper was a response to a need to provide local highway engineers and planners in Singapore with a practical aid for highway location analysis and preliminary geometric design. The program can be used to provide quick solutions concerning the impacts of different geometric design parameters on project costs and to determine the relative importance of different cost categories. A real-life numerical example is presented here to illustrate program application.

Numerical Example

The optimization program was used to conduct a preliminary construction cost analysis for a proposed four-lane highway connecting an industrial town to a residential area with a population of 500,000. The horizontal alignment of the proposed highway had been fixed. The highway would shorten the one-way traveling distance between the two locations from 11.5 km to 5.9 km. Figure 4 shows the ground surface profile along the central axis of the proposed route. The total horizontal length was 5875 m (19,275 ft).

The cost data used in the optimization analysis are summarized in Table 1. The proposed road was located within government land reserved for highway construction. Because sufficient right-of-way width had been reserved, land acquisition constraints of cost and width became irrelevant and were not considered in this example. The following discussion presents the results of vertical alignment analysis in the fol-

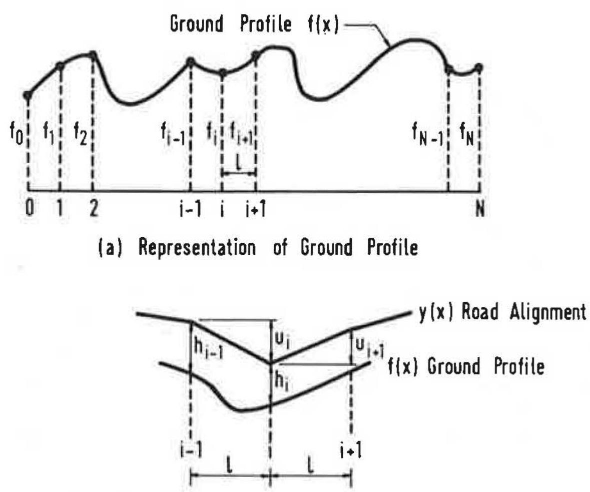

FIGURE 3 Dynamic programming model.

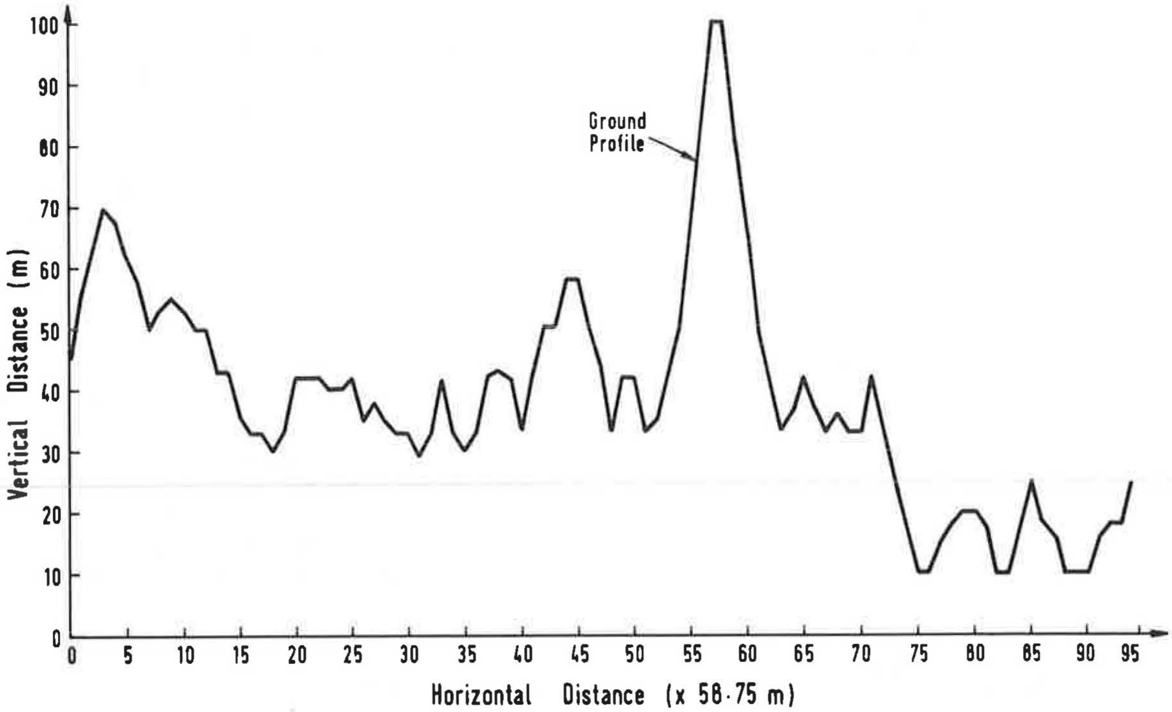

FIGURE 4 Ground profile input for example problem.

TABLE 1 COST DATA AND CONSTRAINT CONDITIONS FOR EXAMPLE PROBLEM

Parameter	Value
Vertical gradient control	4% maximum
Vertical curvature	(See equations 2–5)
Filling cost	S$10.00/m^3
Cutting costs	
Depth < 1.5 m	S$10.00/m^3
1.5–3.0 m	S$14.40/m^3
3.0–4.5 m	S$18.20/m^3
4.5–6.0 m	S$25.00/m^3
6.0–7.5 m	S$30.00/m^3
> 7.5 m	S$50.00/m^3
Pavement cost	S$80/m^2
Vehicle operating costs	Tangent running costs on grades recommended by AASHTO (*9*) for passenger cars on multilane highways.

NOTE: One Singapore dollar (S$) is approximately 0.5 U.S. dollar.

lowing aspects: (a) sensitivity of results in terms of total cost with respect to horizontal and vertical grid sizes; (b) effect of maximum gradient constraint on optimum total cost; (c) effect of minimum curvature control on optimum total cost; (d) effect of earth filling cost on optimum total cost; (e) effect of earth cutting cost on optimum total cost; and (f) effect of vehicle operating cost on optimum total cost.

Following the local practice of project cost computation, the first five analyses did not consider vehicle operating cost. The sixth analysis was performed to give an indication of the effect of including vehicle operating cost. Because no information was available on the magnitude of vehicle operating costs in Singapore, values for passenger cars recommended in an AASHTO Manual (*9*) were used as examples only. Only tangent running costs on grades were included in the analysis.

Selection of Horizontal and Vertical Grid Intervals

The input ground profile and the final computed road alignment are each represented by a series of line segments passing through one grid point on each vertical grid line. As shown by the data grid in Figure 1, the accuracy of the computation depends to a great extent on the horizontal grid interval, d, as well as the length of each division, v, on the vertical grid lines.

A sensitivity analysis of the computed total cost with respect to the grid variables v and d was performed. The results of this analysis are presented in Tables 2 and 3. Data in Table 2 indicate that with a horizontal grid interval, d, fixed at 62.5 m, sufficiently accurate solution could be obtained with vertical division spacing v equal to 0.5 m or less. When fixing v at 0.25 m (Table 3), acceptable results were obtained for values of d equal to 62.5 m or less.

Analyses conducted on highways in other regions of Singapore also showed similar ranges of acceptability for horizontal and vertical grid intervals. For Singapore terrain it was, therefore, recommended that the horizontal and vertical grid intervals be not more than 60 m and 0.5 m, respectively.

It is of importance to impose another control on the values of horizontal and vertical grid intervals selected. The values of d and v used must be such that their ratio (v/d) is less than the maximum gradient constraint specified. The results in Tables 2 and 3 indicated that a (v/d) ratio of less than 1.0 percent would provide an acceptable solution for a maximum vertical gradient control of 4 percent. Because the allowable

TABLE 2 EFFECT OF VERTICAL GRID DIVISION ON COST ANALYSIS

Vertical Division, v(m)	Horizontal Interval, d(m)	Total Cost (S$)	$(v/d) \times 100\%$	Remarks
1.0	62.5	18,106,219	1.60	No good
0.75	62.5	17,030,170	1.20	Marginal
0.5	62.5	16,244,489	0.80	Satisfactory
0.25	62.5	16,239,328	0.40	Satisfactory

NOTE: Input data and constraint conditions are given in Table 1. One Singapore dollar (S$) is approximately 0.5 U.S. dollar.

TABLE 3 EFFECT OF HORIZONTAL GRID INTERVAL ON COST ANALYSIS

Horizontal Interval d(m)	Vertical Division v(m)	Total Cost (S$)	$(v/d) \times 100\%$	Remarks
250.0	0.25	11,907,072	0.10	No good
125.0	0.25	14,421,626	0.20	No good
62.5	0.25	16,239,328	0.40	Satisfactory
31.25	0.25	16,423,647	0.80	Satisfactory

NOTE: Input data and constraint conditions are given in Table 1. One Singapore dollar (S$) is approximately 0.5 U.S. dollar.

range of maximum vertical gradient is between 4 and 8 percent in Singapore (*8*), it has been suggested that a (v/d) ratio of not more than 1 percent be used.

Effect of Maximum Gradient Constraint

A highway designed with a stricter gradient control—by imposing smaller values of maximum vertical gradient constraint on a hilly terrain—allows for a more comfortable ride and a smoother flow of traffic. Savings on vehicle operating costs could also be achieved by using stricter gradient control, a direct consequence of which is the increase in highway construction cost due primarily to the increased earthworks required. Cost computation of highway construction for different maximum gradient controls could therefore provide useful information for a benefit/cost analysis to aid in the selection of gradient control in highway planning and geometric design.

Figure 5 shows the variation of construction cost (excluding vehicle operating costs) with different values of maximum allowable vertical gradient. As expected, highway construction cost decreased as higher maximum gradient was allowed. It should be noted that the computed road profiles were different for different values of gradient control imposed. Figure 6 shows the profiles for maximum gradient control of 4 percent and 6 percent. The 6-percent gradient profile conformed more closely to the ground surface, resulting in a saving on the amount of earthwork cutting and filling.

Although considerable changes in construction costs were observed when vertical gradient control was changed in this example, it is clear that the sensitivity of computed construction costs against gradient control is a function of ground terrain. The construction cost of a highway to be built on a flat terrain would not vary much with changes in vertical gradient control.

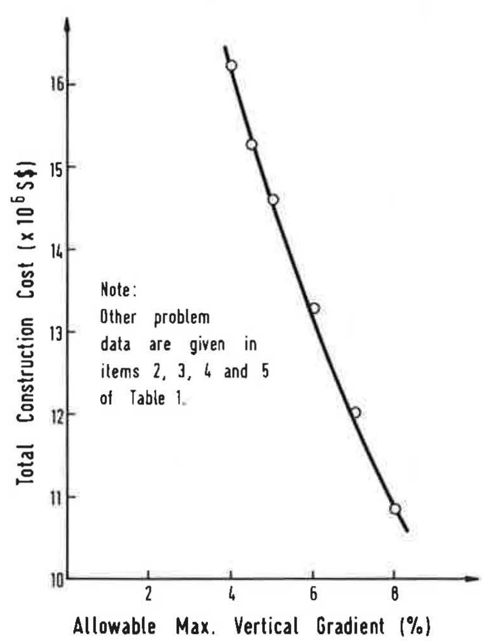

FIGURE 5 Effect of vertical gradient control on highway construction cost.

Effect of Minimum Curvature Control

Curvature control was achieved by setting an upper limit on the value of allowable gradient change, g, which is defined in equations 2 through 5. Small g values allow gradual changes of vertical road alignment, leading to design of highways with smooth riding profile. Construction costs, however, tend to be higher with smaller g values because of the higher quantity of earthwork needed to satisfy the constraint.

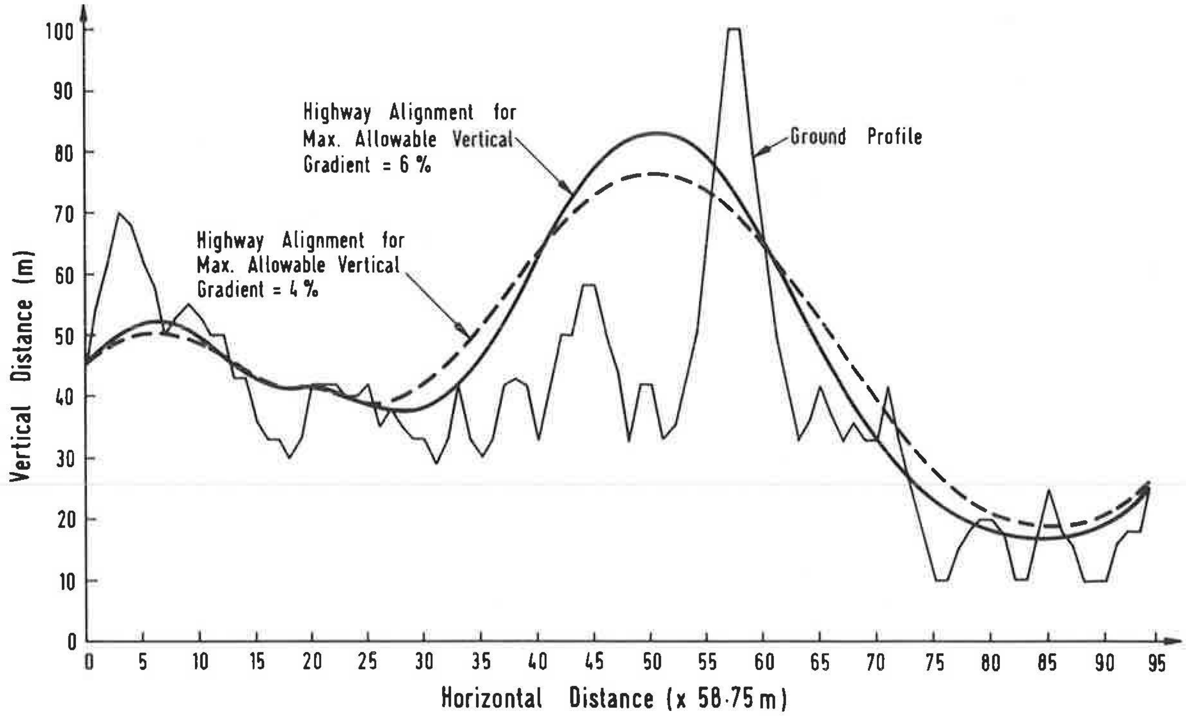

FIGURE 6 Effect of maximum vertical gradient control on highway alignment in example problem.

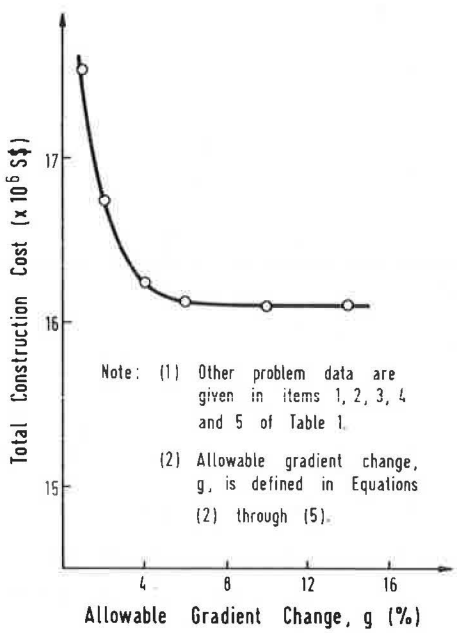

FIGURE 7 Effect of curvature control on highway construction cost.

The relationship between construction cost and curvature control was studied using the example problem, and the results were plotted in Figure 7 for the range of *g* values commonly used in highway design. The changes in construction costs caused by varying the value of *g* were relatively small compared with the changes caused by varying vertical gradient control. This finding has also been found true for highways constructed in other regions of Singapore.

Figure 8 shows the effect of curvature control on the computed optimum road alignment. The alignment computed with a higher value of minimum curvature value, *g*, conformed better to the natural ground profile.

Effect of Earth Filling Costs

Earth filling cost may vary with the type of filling material and hauling distance, when more than one borrow pit is available for a construction project. Construction cost analyses for different borrow materials and hauling distances are useful for highway designers and project planners.

Figure 9 shows the impact of varying earth filling costs for the example problem. The construction cost increased with rising earth filling cost. The amount of increase for each unit rise in filling cost became lower at higher filling costs as the optimum alignment moved toward a profile with less and less volume of fill. Figure 10 illustrates this trend by comparing the optimum profiles for two cases of different earth filling costs.

Effect of Earth Cutting Costs

The effect of varying earth cutting costs on highway construction cost is similar to the effect of earth filling costs in many aspects, as illustrated by Figures 11 and 12. The curve in Figure 11 tended to level off as cutting costs were increased, reflecting the smaller unit impact on construction cost when cutting costs became higher. This trend could be explained by the fact, as depicted in Figure 12, that the volume of cut diminished as cutting costs escalated.

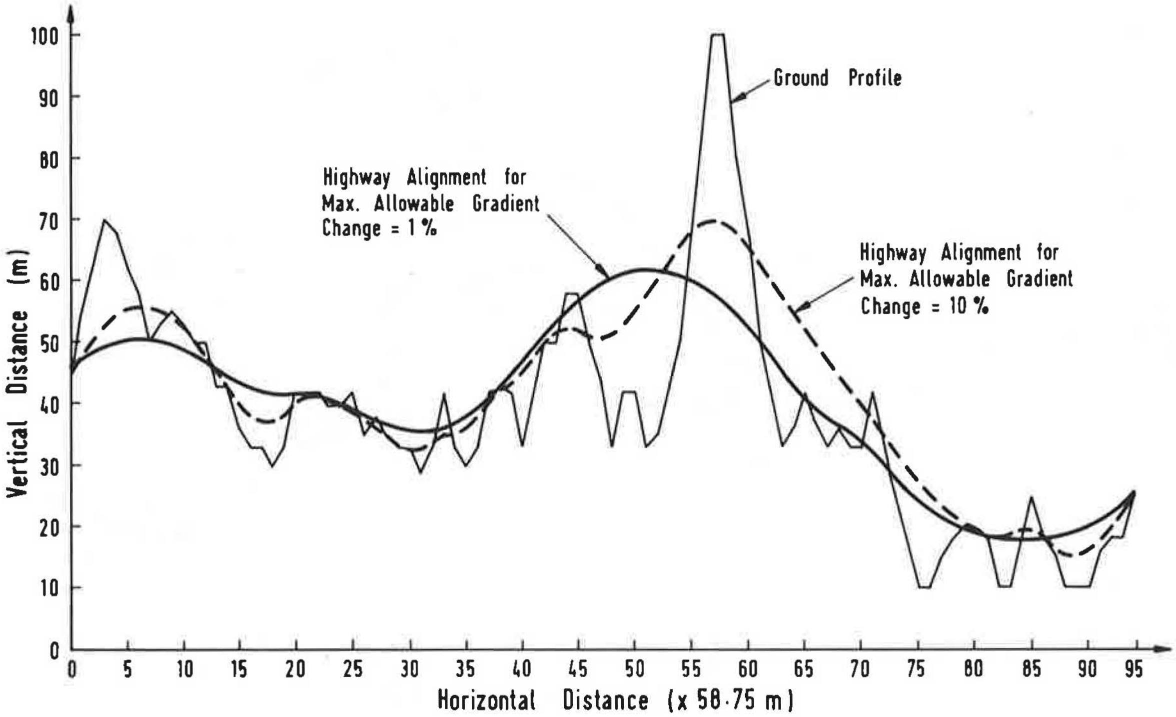

FIGURE 8 Effect of gradient change constraint on highway alignment in example problem.

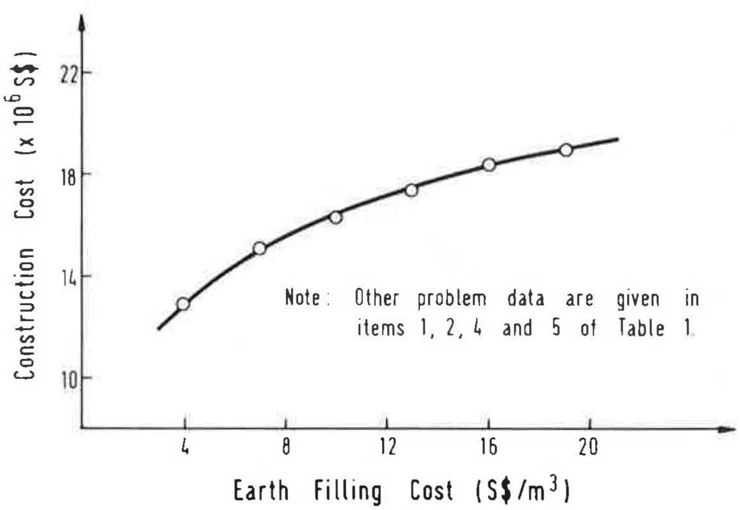

FIGURE 9 Effect of earth filling cost on highway construction cost.

Effect of Vehicle Operating Costs

Although vehicle operating costs have not been commonly considered in cost analyses for highway construction projects in Singapore, examination of how the inclusion of vehicle operating costs would affect engineering designs and project costs is instructive.

For the purpose of illustration, the cases analyzed in Figure 5 were recomputed with the addition of vehicle operating costs defined in item 6 of Table 1. The results of this analysis were plotted in Figure 13. Vehicle operating cost increased as higher values of vertical gradient were permitted in highway design, which had the effect of offsetting the savings in construction cost caused by reduced earthwork when higher vertical gradients were used. As a result, in contrast to the trend in Figure 5, the total costs shown in Figure 13 were relatively insensitive to the changes in allowable maximum vertical gradient used in highway geometric design.

The impact on total highway cost of construction prices and vehicle operating costs depends very much on the relative magnitudes of these two major cost items. In countries where vehicle operating costs are very high, it is possible on certain terrain that the total highway cost might increase as a higher maximum vertical gradient is allowed in design.

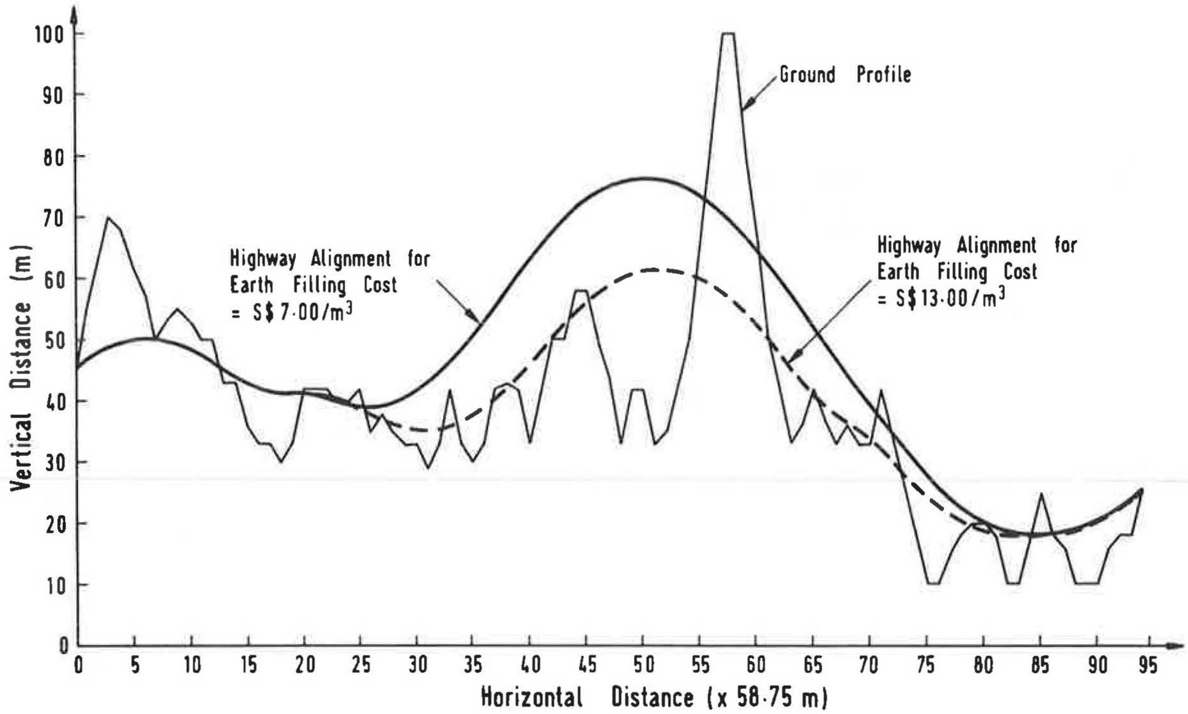

FIGURE 10 Effect of earth filling cost on highway in example problem.

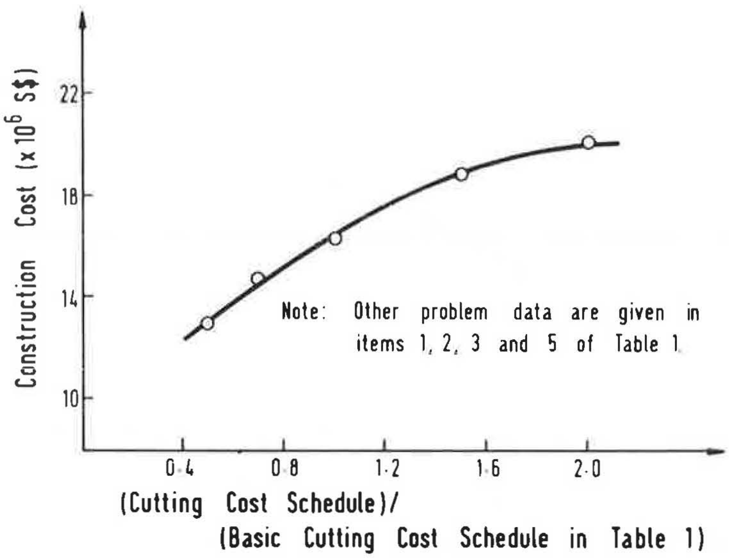

FIGURE 11 Effect of earth cutting cost on highway construction cost.

CONCLUSIONS

Highway vertical alignment analysis is useful for highway designers and planners in the location study, benefit/cost analysis, and geometric design of new highways. The dynamic programming formulation developed in Singapore has been found to be a valuable tool for local highway planners and designers. The formulation of the optimization program is simple in concept, yet has the versatility of modeling practically all of the cost items important in highway construction. The numerical example presented, based on a real-life ground profile and cost data, illustrated these points.

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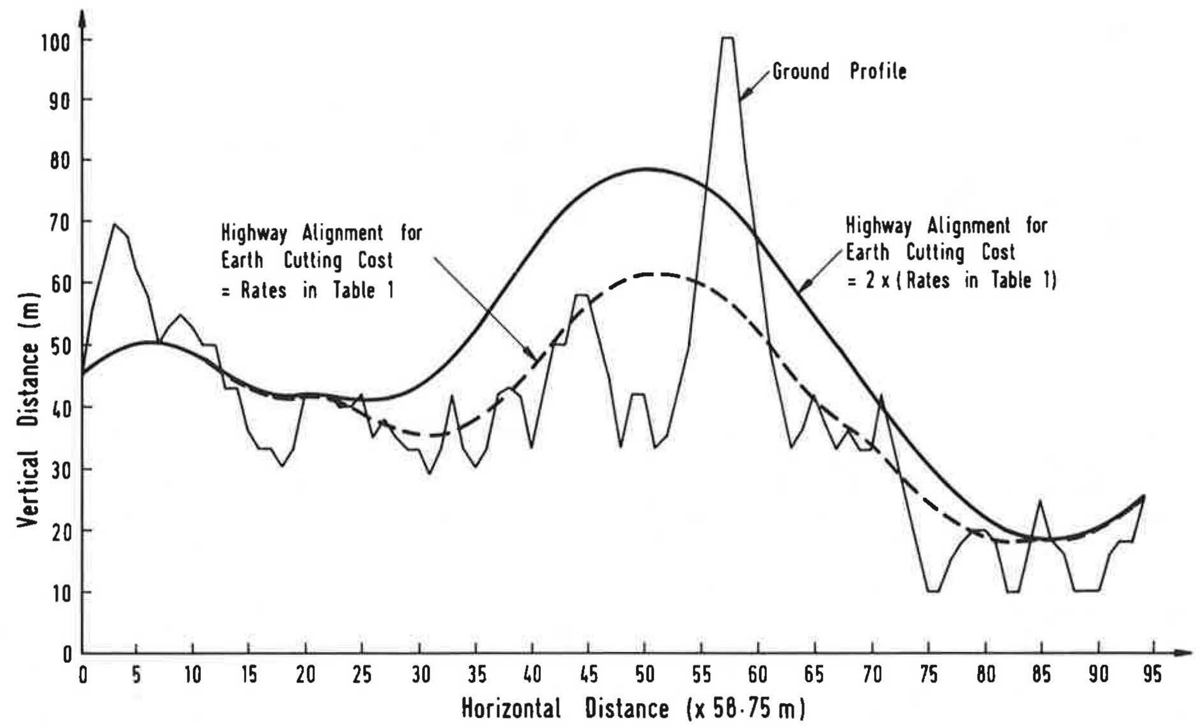

FIGURE 12 Effect of earth cutting cost on highway alignment in example problem.

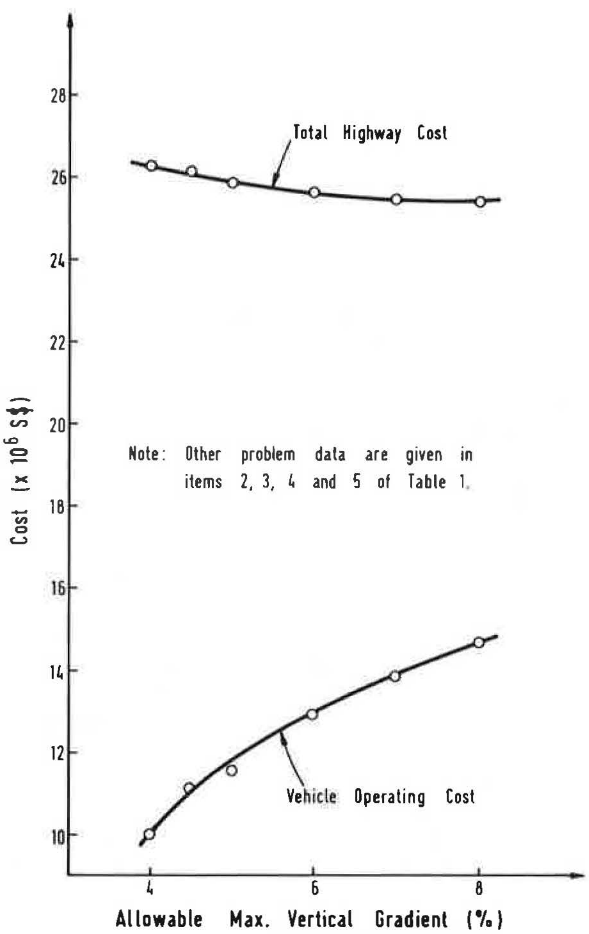

FIGURE 13 Effect of vehicle operating cost on total highway cost.

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