Computational Experience with a Simultaneous Transportation Equilibrium Model Under Varying Parameters

K. NABIL A. SAFWAT AND MOHAMAD K. HASAN

Safwat and Magnanti have developed a combined trip generation, trip distribution, modal split, and traffic assignment model that can predict demand and performance levels on large-scale transportation networks simultaneously—that is, a simultaneous transportation equilibrium model (STEM). Safwat and Brademeyer have developed a globally convergent algorithm for predicting equilibrium on the STEM. The objective of this paper is to investigate the relative computational efficiency of the algorithm as a function of demand, performance, and network parameters for two small, sample networks and one large-scale, real-world network. The algorithm was found indeed to be sensitive to the values of several variables and constants of the model. Many of the results were as expected and could be generalized. As the values of demand parameters increase, the algorithm tends to take more iterations, on the average, to arrive at a given accuracy level. Beyond maximum "practically feasible" values, however, the algorithm may require a considerable computational effort to satisfy a given tight level of accuracy. Network configuration may have a considerably greater influence on convergence rate than network size. These results should further encourage application of the STEM approach to large-scale urban transportation studies.

Safwat and Magnanti (*1*) have developed a combined trip generation, trip distribution, modal split, and traffic assignment model that can predict demand and performance levels on large-scale transportation networks simultaneously—that is, a simultaneous transportation equilibrium model (STEM). The model achieves a practical compromise between behavioral and computational aspects of modeling the equilibrium problem. It is formulated as an equivalent convex optimization problem, yet it is behaviorally richer than other models that can be cast as equivalent convex programs. Although the model is not as behaviorally rich as the most general equilibrium models, it has computational advantages. It can be solved with a globally convergent algorithms [see Safwat and Brademeyer (*2*) for proof of convergence of the logit distribution of trips (LDT), algorithm under milder assumptions compared with the strict "norm" conditions required for convergence of existing algorithms for general asymmetric models], that is also computationally efficient for large-scale networks [see Safwat and Walton (*3*) for computational experience with an application of the STEM model to the urban transportation network of Austin, Texas]. It is not clear, however, how the computational efficiency of the LDT algorithm would be influenced by variations in demand, performance, and network characteristics of the STEM model for different applications.

The objective of this paper is to investigate the relative computational efficiency of the LDT algorithm as a function of demand, performance, and network parameters for selected example networks as well as the large-scale Austin network. This sensitivity analysis should provide useful guidelines for future applications of the approach.

In the following section a brief summary of the STEM model and the LDT algorithm is presented. The next section includes the sensitivity analysis procedures, results, and interpretations. The final section contains the summary and major conclusions.

A STEM METHODOLOGY

Following is a brief description of a STEM model and the LDT algorithm that predicts equilibrium on the STEM model by solving an equivalent convex program (ECP). For a detailed description of the methodology, the reader may refer to work of Safwat and Magnanti (*1*). Proof of convergence of the LDT algorithm may be found in work by Safwat and Brademeyer (*2*).

A STEM Model

In this subsection, a STEM model that describes users' travel behavior in response to system's performance on a transportation network is presented as follows:

$$G_i = \alpha S_i + E_i \qquad \text{for all } i \in I \tag{1}$$

$$S_i = \max \{0, \ln \sum_{j \in D_i} \exp(-\theta U_{ij} + A_j)\} \qquad \text{for all } i \in I \tag{2}$$

$$T_{ij} = G_i \exp(-\theta U_{ij} + A_j) / \sum_{k \in D_i} \exp(-\theta U_{ik} + A_k) \qquad \text{for all } ij \in R \tag{3}$$

$$C_p \begin{cases} = U_{ij} \text{ if } H_p > 0 \\ \geq U_{ij} \text{ if } H_p = 0 \end{cases} \qquad \text{for all } p \in P_{ij}, \text{ all } ij \in R \tag{4}$$

$$C_p = \sum_{a \in A} \delta_{ap} C_a(F_a) \qquad \text{for all } p \in P_{ij}, \text{ all } ij \in R \tag{5}$$

Department of Urban & Regional Planning, Texas A&M University, College Station 77843.

In this model, the demand variables are

G_i = the number of trips generated from origin i,
T_{ij} = the number of trips distributed from origin i to destination j,
H_p = the number of trips traveling via path p from any given origin i to any given destination j, and
F_a = the number of trips using link a.

The performance variables are

S_i = an accessibility variable that measures the expected maximum utility of travel on the transport system as perceived from origin i;
U_{ij} = the average minimum "perceived" cost of travel from i to j;
C_p = the average cost of travel via path p from any given i to any given j; and
C_a = the average cost of travel on link a expressed as a function of the number of trips (F_a) on that link.

The rest of the quantities are

E_i = a composite measure of the effect that the socioeconomic variables, which are exogenous to the transport system, have on trip generation from origin i;
A_j = a composite measure of the effect that the socioeconomic variables, which are exogenous to the transportation system, have on trip attraction at destination j;
α = a parameter that measures the additional number of trips that would be generated from any given origin i if the expected maximum utility of travel, as perceived by travelers at i, increased by unity;
θ = a parameter that measures the sensitivity of the utility of travel between any given origin-destination pair ij as a result of changes in the system's performance between that given O-D pair;

$$\delta_{ap} = \begin{cases} 1 & \text{if link } a \text{ belongs to path } p \\ 0 & \text{otherwise;} \end{cases}$$

and the defined sets are

I = set of origins,
R = set of destinations,
P_{ij} = set of simple paths from i to j, and
D_i = set of destinations accessible from origin i.

The basic assumptions of this STEM model may be summarized as follows:

1. Trip generation (G_i) is given by any general function as long as it is linearly dependent on the system's performance through an accessibility measure (S_i) based on the random utility theory of travel behavior (i.e., the expected maximum utility of travel).
2. Trip distribution (T_{ij}) is given by a logit model where each measured utility function includes the average minimum perceived travel cost (U_{ij}) as a linear variable.
3. Modal split and trip assignment are simultaneously user optimized. Notice that the STEM framework allows for the modal split to be given by a logit model or (together with trip assignment) to be system optimized [see Safwat (*4*)].

LDT Algorithm

The LDT algorithm belongs essentially to the class of feasible direction methods. At any given iteration r, the method involves two main steps. The first step determines a direction for improvement (d^r). The second step determines an optimum step size (λ^*) along that direction. The current solution x^r is then updated, that is, $x^{r+1} = x^r + \lambda^* d^r$, and the process is repeated until a convergence criterion is met. Feasible direction algorithms differ mainly in the way feasible directions are determined and may not always converge to the optimum solution.

The feasible direction d^r, in the LDT algorithm, is determined as follows:

Step 1. Update link cost by calculating $C_a^r = C_a(F_a^r)$ for all $a \in A$. Set $i = 1$ in an ordered set of origin I.

Step 2. Find the shortest path tree from i to all $j \in D_i$. Let U_{ij}^r be the cost of the shortest path from i to j.

Step 3. Find $d^r = Y^r - X^r$ where the vector $X^r = (S^r, T^r, F^r)$ and the vector $Y^r = (L^r, Q^r, V^r)$ are given by

$$L_i^r = \max \{0, \ln \sum_{j \in D_i} \exp(-\theta_i U_{ij}^r + A_j)\} \qquad \text{for all } i \in I$$

$$Q_{ij}^r = (\alpha_i L_i^r + E_i) \exp(-\theta_i U_{ij} + A_j) / \sum_{k \in D_i} (-\theta_i U_{ik}^r + A_k) \quad ij \in R$$

$$B_p^r = \begin{cases} Q_{ij}^r & \text{if } p = p^* \in P_{ij} \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for all } p \in P_{ij},\ ij \in R$$

$$V_a^r = \sum_{ij \in R} \sum_{p \in P_{ij}} \delta_{ap} B_p^r \qquad \text{for all } a \in A$$

Then the feasible direction at iteration r is the vector d^r with the following components:

$$d_i^r = L_i^r - S_i^r \qquad \text{for all } i \in I$$

$$d_{ij}^r = Q_{ij}^r - T_{ij}^r \qquad \text{for all } ij \in R$$

$$d_a^r = V_a^r - F_a^r \qquad \text{for all } a \in A$$

Safwat and Brademeyer (*2*) proved that the LDT algorithm is globally convergent under the same mild assumptions as with the STEM model.

SENSITIVITY ANALYSIS PROCEDURES AND RESULTS

Several major factors may influence the convergence rate of the LDT algorithm:

1. Trip generation parameter (α),
2. Minimum trip generation (E_i),
3. Trip distribution parameter (θ),
4. Attractiveness measure (A_{ij}),
5. Link performance function (C_a),
6. Network configuration,
7. Network size,
8. Convergence criterion, and
9. Accuracy level.

It is very clear that the combinations of values for these factors are enormous; hence, we have to be selective and more focused, particularly when initial experimental results revealed that the LDT algorithm is indeed sensitive to the selected values. This required a systematic approach and additional care in the "selection process."

Two small example networks and one large, real-world network were used in the analysis. The first small example network (Network 1) was obtained from work by Nguyen and Dupuis (*5*) and the second (Network 2), from the work of Nagurney (*6*); both were proposed for testing algorithms for the asymmetric traffic assignment problem. Network 1 consists of 19 links, 13 nodes, 4 origin-destination pairs, and 2 origins (see Figure 1); and Network 2 consists of 36 links, 22 nodes, 12 origin-destination pairs, and 4 origins (see Figure 2). Tables 1 and 2 include the "observed" interzonal demand volumes on Networks 1 and 2, respectively. Note that these networks are, however, different from the "original" ones in terms of their demand and link performance functions.

The trip generation parameter E_i was selected as the "observed" trip generation. Two values of the attractiveness measure A_{ij} were tested.

1. A_{ij} equals the natural logarithm of the observed trip distribution from i to j (this is a reasonable estimate that is based on theoretical grounds), and
2. A_{ij} equals five times the value in item 1.

Two link performance functions were considered: linear and the usual BPR (i.e., Bureau of Public Roads) 4th power function. These are

Cost** 1: $C_a = t_{oa}\,[1 + b\,(F_a/\text{CAP}_a)]$ and

Cost** 4: $C_a = t_{oa}\,[1 + b\,(F_a/\text{CAP}_a)^4]$

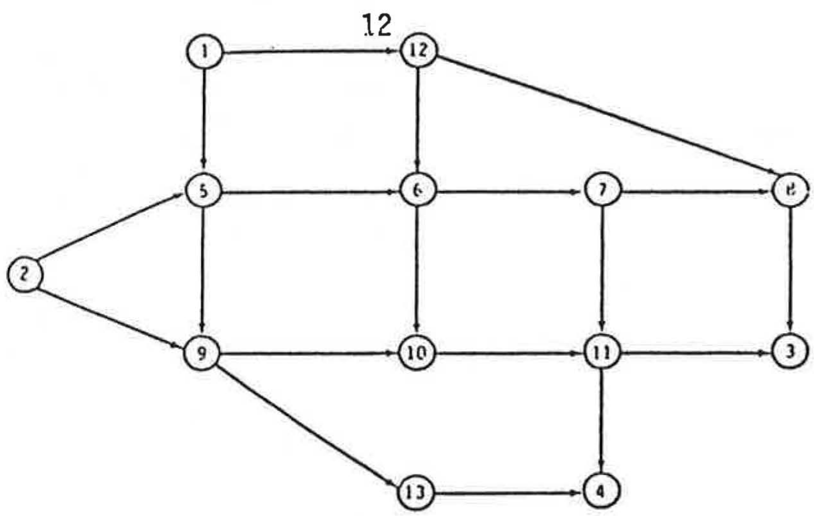

FIGURE 1 Network 1.

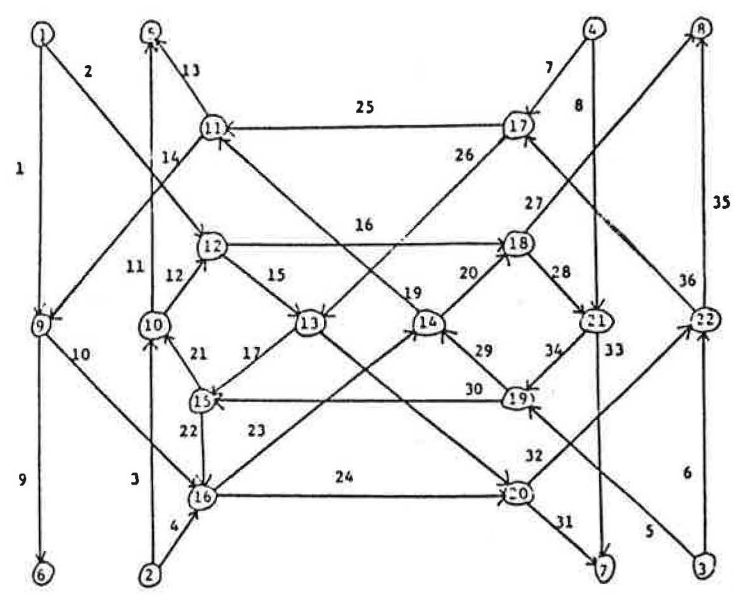

FIGURE 2 Network 2.

TABLE 1 TRIP DISTRIBUTION MATRIX FOR NETWORK 1

	3	4
1	400	800
2	600	200

TABLE 2 TRIP DISTRIBUTION MATRIX FOR NETWORK 2

	5	6	7	8
1	—	235	230	220
2	240	—	235	225
3	230	220	—	235
4	235	225	240	—

where t_{oa} is the free-flow travel time on link a, b is the link congestion parameter, and CAP_a is the practical capacity of link a. These "parameters" were selected at "reasonable" values for all links of a given network such that the average volume-to-capacity ratio on the network at equilibrium is approximately 0.6 (i.e., t_{oa} = 1 and b = 1.15 for both networks, and CAP_a = 700 for Network 1 and 400 for Network 2).

The third network (i.e., the large-scale urban transportation network of Austin, Texas) consists of 7,096 links, 2,137 nodes, 19,213 origin-destination pairs, and 520 origins. The network was used earlier by Safwat and Walton (*3*), and no changes were made in its demand or performance functions. The average volume-to-capacity ratio on the Austin network was approximately 0.2; this is quite conceivable because the network includes existing, committed, and proposed improvements for the year 2000.

The analysis focused on the two major travel demand parameters α and θ. For the two example networks, possible values of these two parameters were considered at two different values of the other one. That is, the values of α varied between 0.001 and 50 while values for θ were set at 0.05 and 0.12, and the values of θ varied between 0.01 and 0.9 while values of α were set at 0.001 and 10. For the Austin network, the values of α varied between 1 and 50 while the value of θ was set at 0.05, and the values of θ ranged between .05 and 0.14 while the value of α was set at 1. The ranges of values were selected to capture "significant" variability in the computational efficiency of the algorithm, as reflected by the number of iterations required to arrive at a prescribed accuracy level based on a given convergence criterion. In some cases, however, there were "practically maximum" values of the parameters beyond which the algorithm could not arrive at the prescribed accuracy level (which was selected to be tight) in thousands of iterations.

Two convergence criteria and two accuracy levels were included in the analysis. Notice that at any iteration in the LDT algorithm the following equation holds true for all $ij \in R$ [see Safwat and Magnanti (*1*)]:

$$T^r_{ij} = \frac{G^r_i \exp(-\theta U^r_{ij} + A_j) \exp(\theta C^r_{ij})}{\sum_{k \in D_i} (-\theta U^r_{ik} + A_k)}$$

where

$$C^r_{ij} = 1/\theta\,[S^r_i - \ln(\alpha S^r_i + E_i) + \ln T^r_{ij} - A_j] + U^r_{ij} \qquad \text{for all } ij \in R$$

It is obvious that at equilibrium $\theta C^r_{ij} = 0$ for all $ij \in R$; hence, two convergence criteria may be specified as follows:

1. Stop whenever $-\varepsilon_1 < \theta C^r_{ij} \leq +\varepsilon_1$ for all $ij \in R$ or
2. Stop whenever TERMS $= \sqrt{\sum} (\theta C^r_{ij})^2 < \varepsilon_2$

where $\varepsilon_1, \varepsilon_2 > 0$ are small accuracy levels (selected at 0.05 and 0.1 in our analysis) and TERMS is the Total Equilibrium Root Mean Squares error.

The convergence rate of the LDT algorithm was measured in terms of the number of iterations required to achieve a given level of accuracy. This is a proxy measure for the CPU time as it was more or less constant for each iteration. In particular, for the example networks, the CPU time for input and initial solution was 0.09 sec and, per iteration, 0.01 sec on a VAX 8650 minicomputer that was used for analysis. For the Austin network the CPU times were about 190 and 170, respectively.

Because the emphasis in analysis is on the demand parameters α and θ, values of other factors were selected on the basis of their respective influence on the effect of changes in these two parameters on the convergence rate of the algorithm. For instance, to select the appropriate value for the attractiveness measure A_{ij}, Figure 3 shows the effect of θ on the number of iterations to arrive at a prespecified accuracy level (which was selected at ε_1 = 0.05 as determined by the sensitivity analysis procedure itself, as is explained later) for the two different values of the attractiveness measure A_{ij} already

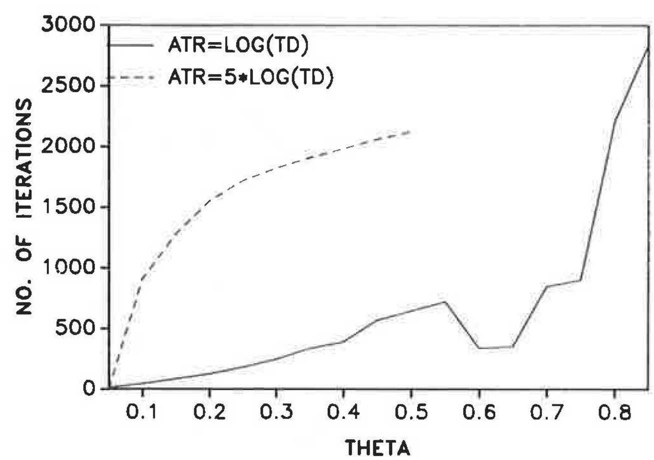

FIGURE 3 Effect of theta on convergence rate (Network 1, Cost 4, alpha = 0.001, epsilon = 0.05).**

indicated. In Figure 3, the parameter α was set at a small value of 0.001 to reduce its influence on results to a minimum; the usual BPR 4th power link cost function (Cost** 4) was selected because it is more realistic than the linear cost function (Cost** 1); and Network 1 was used because its configuration was found to have more influence on the results than Network 2 (see Figure 4).

The graphs in Figure 3 show very clearly that using five times the value of a "reasonable estimate" for A_{ij} caused the number of iterations to increase considerably for all values of θ; the increase becomes more significant as θ increases. On the basis of these results, the attractiveness measure for the remainder of the analysis was set at its more "reasonable" value—that is

$$A_{ij} = \ln (\text{"observed" trips from } i \text{ to } j)$$

To select a convergence criterion, Figure 5 shows the sensitivity of results with respect to the two proposed criteria. As expected, the second criterion (i.e., TERMS) was always met before the first, "stricter" one, and the patterns of convergence are similar. This is so because $\varepsilon_2 = 0.1$ implies achieving an average value of $\varepsilon_1 = 0.05$, whereas the first criterion allows a maximum value of 0.05 on each individual link. The first criterion was used in the remainder of the analysis to achieve more accurate results. As for the accuracy level, Figure 6 shows the results for two different values of ε_1 (i.e., 0.05 and 0.1). Again the results were as expected in terms of the magnitudes and shapes of the two curves in the figure. The value of 0.05 was used throughout the analysis to obtain more accurate results.

The effect of network configuration and size is shown in Figure 4 for the two example networks. Surprisingly, the "larger" Network 2 always converged considerably more quickly regardless of the change in the parameter θ, whereas the "smaller" Network 1 revealed relatively slower convergence rates, particularly at higher values of θ. It seems that Network 2 has a significantly "simpler" configuration than Network 1 in terms of layout, traffic circulation, and travel demand data (see Figures 1 and 2). These results indicate that network configuration may be a significant factor that could override the effect of network size.

The convergence rates of the algorithm with respect to changes in θ are shown in Figures 7, 8, 9, and 10. For the example networks, Figures 7 and 8 show that regardless of the value of α and network configuration, the number of iterations would on the average increase as θ increases, as would be expected, because larger values of θ imply higher

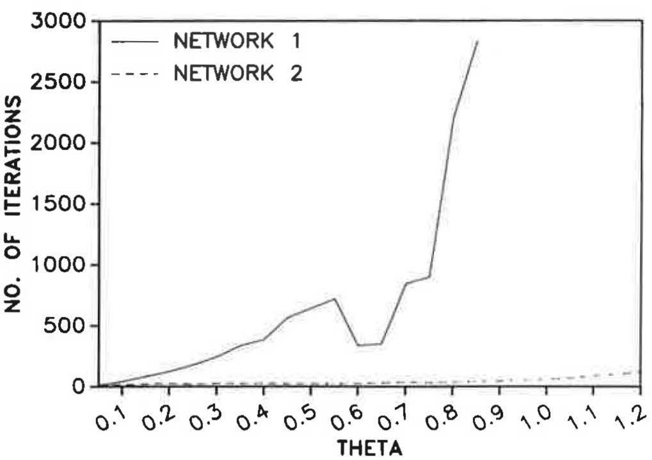

FIGURE 4 Effect of theta on convergence rate (Cost 4, alpha = 0.001, epsilon = 0.05).**

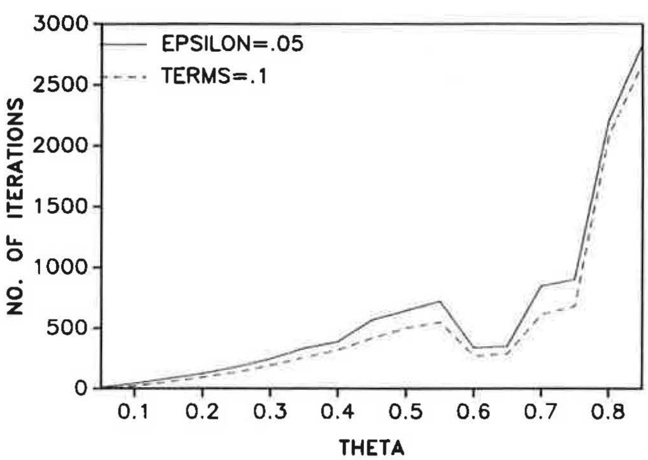

FIGURE 5 Effect of theta on convergence rate for epsilon = 0.05 and TERMS = 0.1 (Network 1, Cost 4, alpha = 0.001).**

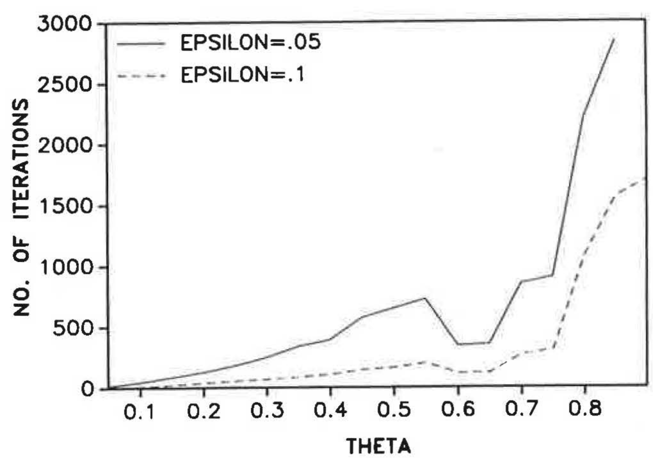

FIGURE 6 Effect of theta on convergence rate for epsilon = 0.05 and epsilon = 0.1 (Network 1, Cost 4, alpha = 0.001).**

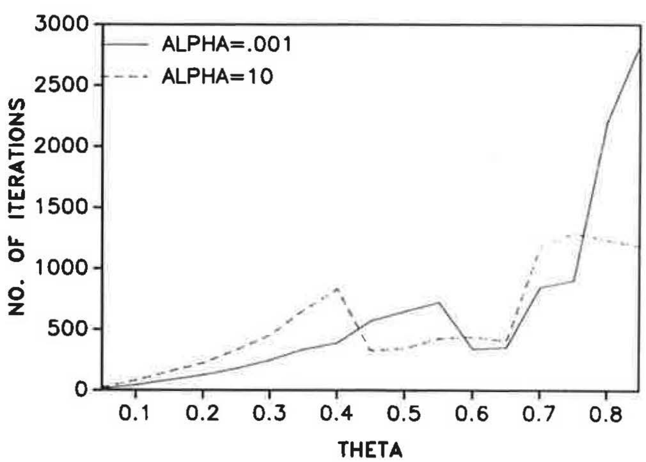

FIGURE 7 Effect of theta on convergence rate (Network 1, Cost 4, epsilon = 0.05).**

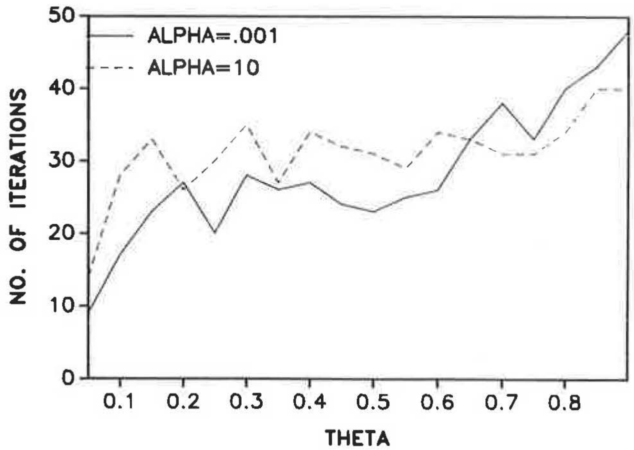

FIGURE 8 Effect of theta on convergence rate (Network 2, Cost** 4, epsilon = 0.05).

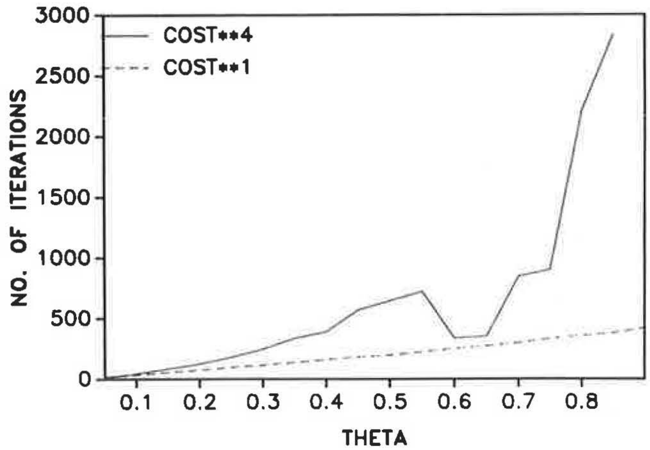

FIGURE 9 Effect of theta on convergence rate (Network 1, alpha = 0.001, epsilon = 0.05).

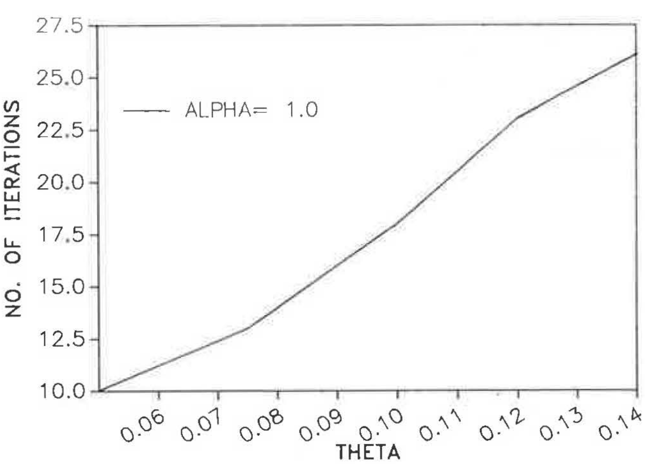

FIGURE 10 Effect of theta on convergence rate (Austin network, epsilon = 0.05).

sensitivity of travel demands to changes in the system's performance. The rate of increase, however, may depend on network configuration and, more important, the shape of the link performance function; as the cost function becomes steeper, Figure 9 shows that, again as expected, the rate of convergence becomes nonlinearly slower.

Figure 10 shows that for the Austin network the results are monotonic and confirm the same trend. The relatively faster convergence on the Austin network may be due to the fact that it is far less congested than the two example networks. Also, network configuration may have been a significant factor that superseded the effect of network size, which does not seem to be a significant factor.

The results for the effect of the demand parameter α on convergence rate are shown in Figures 11 through 14. Figure 11 shows the effect of α for two different values of θ (0.05 and 0.12). It is very clear that the decrease in the value for θ has dampened the effect of α on convergence rate. This behavior is in conformity with our intuition. A similar trend was observed for different cost functions (see Figure 12) and network configuration (see Figures 13 and 14). In Figure 12, then BPR 4th power function adversely influenced the rate of convergence nonlinearly, whereas the linear cost function

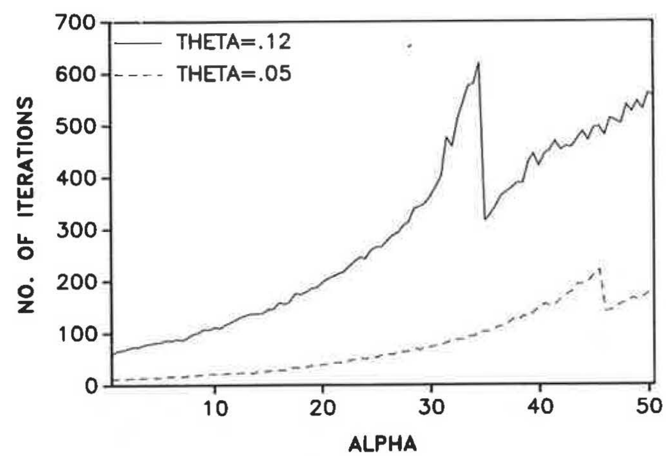

FIGURE 11 Effect of alpha on convergence rate (Network 1, Cost** 4, epsilon = 0.05).

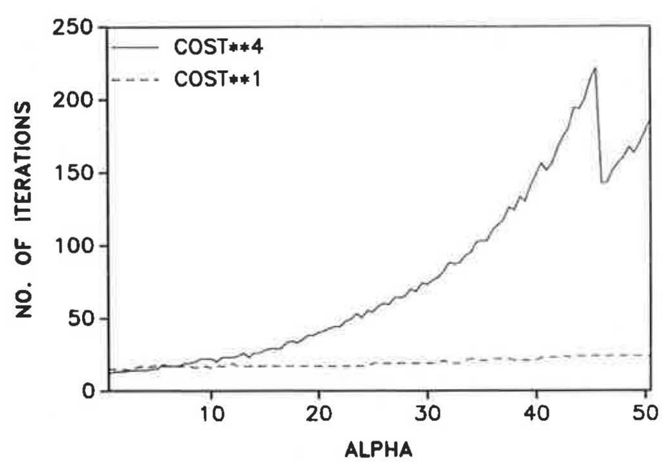

FIGURE 12 Effect of alpha on convergence rate (Network 1, theta = 0.05, epsilon = 0.05).

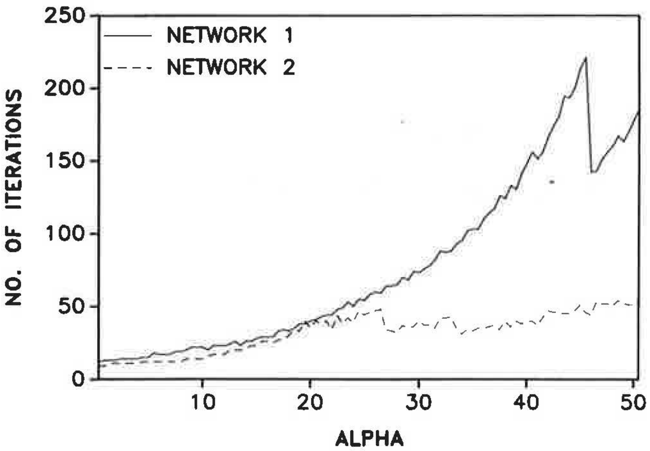

FIGURE 13 Effect of alpha on convergence rate (Cost 4, theta = 0.05, epsilon = 0.05).**

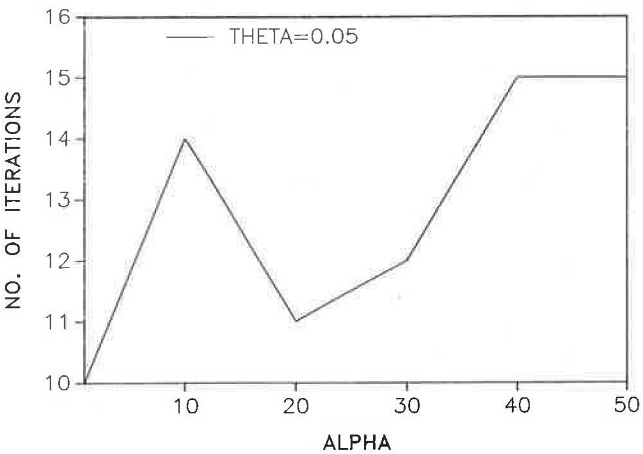

FIGURE 14 Effect of alpha on convergence rate (Austin network, epsilon = 0.05).

had virtually no effect. Figure 13, consistent with Figure 4, shows that configuration of Network 2 appears to be "simpler" than that of Network 1. The results of the Austin network shown in Figure 14 are also consistent with those of the example networks.

SUMMARY AND CONCLUSIONS

The objective of this paper was to investigate the computational efficiency of the LDT algorithm for predicting equilibrium on a simultaneous transportation equilibrium model (STEM) as influenced by several demand and performance parameters of the STEM model as well as network characteristics. The sensitivity analysis considered several major factors, including demand parameters (α, θ, E_i and A_{ij}), performance functions (linear and 4th power), convergence criterion, accuracy level, and network configuration and size. The focus, however, was on the two major demand parameters α, θ.

The main conclusions of this paper may be summarized as follows:

1. The effect of each of the two major parameters α, θ on convergence rate was found to be, as expected, sensitive to the values of the other one in addition to the values of other major variables and constants of the STEM model and the network configuration and size.
2. In general, as the value of the parameter increases, the number of iterations to arrive at a prespecified accuracy level will tend to increase as expected. The effect of θ seems to be more significant than that of α. The combined effect of both parameters is considerably greater than that of the individual parameters separately.
3. There are maximum "practically feasible" values of α, θ beyond which the algorithm may take a considerable computational effort to satisfy a given tight level of accuracy. These maximum values may differ from one application to another. The possible reason for the existence of such practically "upper bounds" on the values of parameters may be related to the flatness of the objective function of the equivalent convex program that is being solved by the LDT algorithm, particularly when the network is less congested.
4. Network configuration may have considerable effects on the convergence rate whereas network size may not.

These results, especially those of the Austin network, further encourage the application of the STEM approach to real-world urban transportation studies. Actual calibration of demand and performance parameters will certainly provide additional insights into the practicality of the proposed method.

ACKNOWLEDGMENTS

The authors would like to express their deep appreciation to Hani Mahmassani, Chairman of TRB Committee on Transportation Supply Analysis, and three anonymous referees for their invaluable comments on an earlier version of this paper.

This work is supported by a research grant from the National Science Foundation.

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