# Integration of Fixed- and Flexible-Route Bus Systems

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Temporally integrated bus systems, in which fixed-route services are provided during higher-demand periods and flexible-route services are provided during lower-demand periods, are investigated with analytic optimization models. Threshold analysis is used to determine which option is preferable for a given demand pattern and to identify favorable situations for integrated operation. Optimized vehicle sizes, route spacings, zone areas, and service headways are obtained and compared for fixed-route, flexible-route, and integrated systems.

Conventional bus services are characterized by their fixed routes and schedules and are generally thought to require substantial demand densities to be economically viable; paratransit services have flexible routes or schedules (or both) and are considered most suitable for low-density areas or time periods (*1–11*). The potential for improving public transportation services through coordinated operation of paratransit and conventional transit systems has been recognized (*12,13*). However, most studies on integration of public transportation systems have focused on spatially integrated systems of conventional modes, such as park-and-ride operation coordinated with mass transit systems (*14,15*) and integrated feeder bus–rail transit systems (*16–18*), which are commonly applied in U.S. urban transit systems.

Various types of integration of conventional bus and paratransit services have been attempted in several suburban areas with varying levels of success (*19–23*). Control strategies and issues related to the implementation of integrated systems have also been discussed and evaluated (*10,13,24–27*). However, studies concerning the temporal integration of conventional bus and paratransit services, in which conventional fixed-route services are provided during higher-demand periods and flexible-route door-to-door services are provided during lower-demand periods, are mostly limited to conceptual and qualitative analyses (*5,10,12,25*). A simulation model has been developed and used to evaluate temporal integration options for cities with populations of less than 10,000 (*28*). It was concluded that the net operating costs of alternative dial-a-ride/fixed-route services comprising a mixed bus fleet of 45-seat buses for peaks and 12-seat buses for off-peaks are better than those of either fixed-route or dial-a-ride services. However, the alternatives compared were all prespecified rather than optimized.

In this paper an analytic approach is applied to design and evaluate temporally integrated systems. Two feeder bus systems, a conventional fixed-route and a flexible-route subscription bus system, are considered. Threshold evaluation based on analytic optimization models (*11*) is used to determine favorable situations for operation of temporally integrated systems, and mathematical models of total system costs for an integrated system are formulated and analyzed. Optimized results are presented for vehicle size, route spacing, headway, and service zone areas.

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## BUS SYSTEM CHARACTERISTICS

Figure 1 shows the service areas and their specific route structures for the two feeder systems. The variables and the typical values used in the numerical analysis are given in Table 1. The bus systems with either fixed or flexible routes are assumed to connect a rectangular area of length $L$ and width $W$ to a major generator (e.g., a transportation terminal or an activity center) that is $J$ mi from that area. Analytic optimization models for these two feeder systems developed in earlier work (*11,29*) are applied. The models provide optimized solutions in closed form with time-dependent demand and supply characteristics (vehicle operating cost and speed) and over multiple periods. Route structures and operation attributes for the two services are briefly described.

### Fixed-Route Services

For fixed-route services, the service area is divided into $N$ zones with route spacing $r = W/N$, which is fixed over time, as shown in Figure 1a. A vehicle round-trip in period $t$ consists of (*a*) a line-haul distance $J$ traveled at express speed $yV_t$ from the major terminal to the service area; (*b*) a delivery route $L$ mi long traveled at local speed $V_t$ along the centerline of the zone, stopping for passengers every $s$ mi, with an average delay of $d_t$ hr for each stop; and (*c*) reversal of the previous two phases to collect passengers and carry them to the terminal.

### Flexible-Route Services

The route structure for the flexible-route subscription service is shown in Figure 1b. The service area is divided into $N_t$ equal zones, each of which has area $A_t = LW/N_t$. This service zone structure is more flexible than that for fixed-route service and is allowed to vary over time. In each time period, feeder buses travel from the terminal a line-haul distance $J$ and an average distance $L/2$ mi at express speed $yV_t$ to the center of each

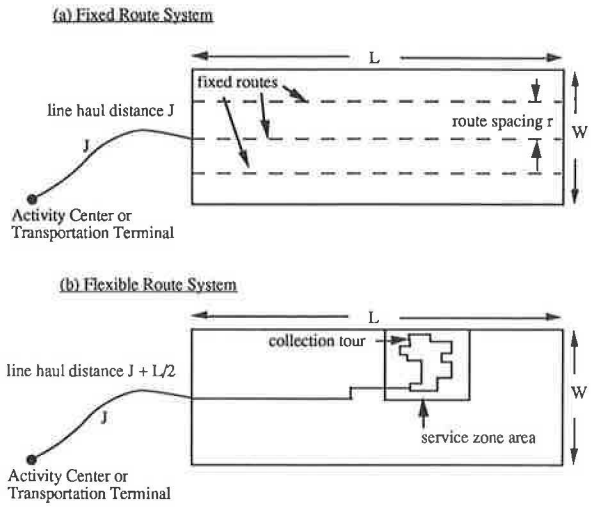



**FIGURE 1 Fixed- and flexible-route feeder bus systems.**

zone. They collect passengers at their doorsteps through a tour of $n_t$ stops with length $E_t$ at local speed $V_t$. The values of $n_t$ and $E_t$ are endogenously determined using Stein's formula (*30,31*). To return to their starting point, the buses retrace an average of $L/2 + J$ mi at $yV_t$ mph. It is assumed that buses operate on preset schedules with variable routing designed to minimize the tour distance $E_t$ and that tours are routed on a rectangular grid street network. Tour departure headways are assumed to be equal for all zones in the service area and uniform within each period. For both service types the average wait time equals a constant factor $z_1$ times the headway $h_t$. As in fixed-route service, vehicle layover time and external costs of bus services are assumed to be negligible.

On the basis of the assumptions that $n_t$ points are randomly and independently dispersed over an area $A_t$ and that an optimal traveling salesman tour has been designed to cover these $n_t$ points, the collection distance $E_t$ in an optimized zone may be approximated by the following result of Stein (*30,31*) for dial-a-ride routing:

$$E_t = \phi(n_t A_t)^{1/2} \tag{1}$$

In Equation 1, $\phi$ is constant and has been estimated to be 0.765 for a Euclidean metric (*31*). Applications of Equation 1 are discussed by Larson and Odoni (*32*) and Daganzo (*33*).

The demand density $q_t$ during each time period $t$ is assumed to be obtained from empirical distributions of demand over time, as shown in Figure 2. The demand distribution over time typically represents a daily demand cycle, as in the four-period demand distribution shown in Figure 2, although it may also be used to analyze noncyclical demand conditions, such as long-term growth patterns. The demand is also assumed to be deterministic, uniformly distributed over time during each specified period, and uniformly distributed over space within each specified service area. The number and duration of time periods are unlimited.

The analytic results for the optimal route structures (route spacings and zone sizes), vehicle sizes, and service headways for the two services derived by Chang and Schonfeld (*11*) are used in this analysis. These optimality relations are presented

TABLE 1 VARIABLE DEFINITIONS

| Variable | Definition | Baseline Value |
|---|---|---|
| $a_t$ | fixed cost coefficient for period t ($/hr) | - |
| | $a_1$, $a_2$, $a_3$, and $a_4$ = 30, 15, 15, and 15, respectively. | |
| $A_t$ | service zone area in period t (sq. miles) | - |
| $b_t$ | variable cost coefficient for period t ($/seat hr) | - |
| | $b_1$, $b_2$, $b_3$, and $b_4$ = 0.4, 0.2, 0.2, and 0.2, respectively. | |
| C | total system cost ($/day); = $C_o + C_u$ | - |
| $C_o$ | operator cost ($/day) | - |
| $C_u$ | total user cost ($/day) | - |
| $C_v$ | user in-vehicle cost ($/day) | - |
| $C_w$ | user wait cost ($/day) | - |
| $C_x$ | user access cost ($/day) | - |
| $d_t$ | average delay per stop during period t (hr/stop) | 0.01 |
| $D_t$ | avg. bus round trip time during period t (hrs) | - |
| $E_t$ | distance of one collection tour in period t (miles) | - |
| $F_t$ | fleet size in period t (vehs) | - |
| g | access speed (miles/hr) | 2.5 |
| $h_t$ | headway in period t (hr) | - |
| J | line haul distance (miles) | 8.0 |
| L | length of corridor (miles) | 4.0 |
| m | number of periods in the analysis time frame | 4 |
| $n_t$ | number of pickup stops in one collection tour during period t | |
| N | number of zones | - |
| $N_t$ | number of zones in period t | - |
| $p_t$ | bus load factor at peak load point during period t | - |
| $q_t$ | potential demand density in period t (trips/sq. mile/hr) | |
| | $q_1$, $q_2$, $q_3$, and $q_4$ = 120, 60, 10, and 5, respectively | |
| r | route spacing (miles) | - |
| s | stop spacing (miles) | 0.25 |
| $T_t$ | duration of period t (hrs) | |
| | $T_1$, $T_2$, $T_3$, and $T_4$ = 3, 6, 6, and 9, respectively. | |
| $u_t$ | avg. no. of passengers per pickup point during period t | 1.2 |
| U | equivalent line haul distance (miles) = 2J/y + L/y | - |
| v | value of in-vehicle time ($/hr) | 5.0 |
| $V_t$ | bus speed during period t (miles/hr); | |
| | $V_1$, $V_2$, $V_3$, and $V_4$ = 10, 12, 15 and 15, respectively | |
| w | value of wait time for conventional bus ($/hr) | 10.0 |
| w' | value of wait time for paratransit ($/hr) | 8.0 |
| W | width of corridor (miles) | 3.0 |
| x | value of access time ($/hr) | 10.0 |
| y | express ratio = express speed/local speed | 2.0 |
| $z_1$ | ratio of wait time/headway | 0.5 |
| $z_2$ | ratio of access distance/route spacing | 0.25 |
| Z | composite variable defined in Table 2 | - |
| $\phi$ | constant in the collection distance equation | 1.15 |
| $\pi$ | composite variable defined in Table 2 | - |
| $\delta$ | composite variable defined in Table 2 | - |
| $\theta$ | composite variable defined in Table 2 | - |

in Table 2. Different effects of demand density and other system parameters can be identified on the basis of the analytic results for single-period cases. From the results for a single period, it is shown that the optimized vehicle sizes are proportional to the ⅓ power and the ⅕ power of demand density for fixed- and flexible-route services, respectively. Fig-

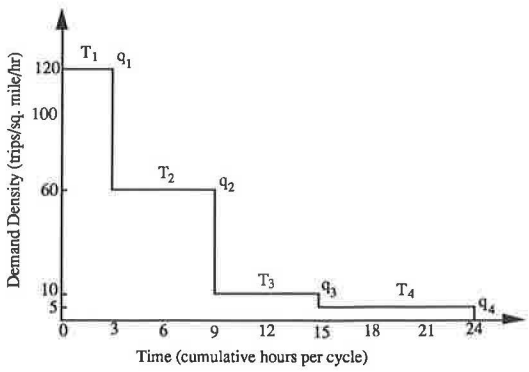



**FIGURE 2 Demand pattern assumed in numerical example.**

TABLE 2 ANALYTIC RESULTS FOR OPTIMIZED BUS SYSTEMS

| (1) Fixed Route Service | Multiple Periods | Single Period |
|---|---|---|
| Vehicle Size (seats/veh) | $\left(\frac{gL\pi\delta^2}{z_1 z_2 wx}\right)^{\frac{1}{3}}$ | $\left(\frac{gLqa^2D^2}{z_1 z_2 wxp^3}\right)^{\frac{1}{3}}$ |
| Route Spacing (miles) | $\left(\frac{z_1 wg^2\delta}{z_2^2 x^2 L\pi}\right)^{\frac{1}{3}}$ | $\left(\frac{z_1 wg^2 aD}{z_2^2 x^2 Lq}\right)^{\frac{1}{3}}$ |
| Headway (hrs) | $\left(\frac{p_i}{q_i}\right)\left(\frac{z_2 x\pi^2\delta}{z_1^2 w^2 gL}\right)^{\frac{1}{3}}$ | $\left(\frac{z_2 xaD}{z_1^2 w^2 gLq}\right)^{\frac{1}{3}}$ |
| **(2) Flexible Route Service** | **Multiple Periods** | **Single Period** |
| Vehicle Size (seats/veh) | $\left(\frac{U^3 Z^3}{2z_1 w\phi^2}\right)^{\frac{1}{5}}$ | $\left(\frac{U^3 a^3 uq}{2z_1 w\phi^2 p^3 (b+vp/2)^2 V}\right)^{\frac{1}{5}}$ |
| Zone Area (sq. miles) | $\beta_i S^*\left(\frac{2z_1 wV_i u_i^{1/2}}{\phi q_i(a_i+b_i S^*+vp_i S^*/2)}\right)^{\frac{2}{3}}$ | $\left(\frac{8auz_1^3 w^3 V^3 U^3}{\phi^4 (b+vp/2)^2 q^3}\right)^{\frac{1}{5}} \theta^{-\frac{2}{3}}$ |
| Headway (hrs) | $\left(\frac{\phi(a_i+b_i S^*+vp_i S^*/2)}{2z_1 wV_i u_i^{1/2} q_i^{1/2}}\right)^{\frac{2}{3}}$ | $\left(\frac{a\phi}{4z_1^2 w^2 V^2}\right)^{\frac{2}{5}}\left(\frac{1}{q}\right)^{\frac{1}{5}} \theta^{\frac{2}{3}}$ |

Note:

$$\theta = \left(\frac{2z_1 wa^2 p^3 V}{u^{3/2} q}\right)^{1/5} + \left(\frac{U^3(b+vp/2)}{u^{3/2}}\right)^{1/5} \quad Z = \frac{\sum_i a_i q_i T_i / p_i V_i}{\sum_i (q_i^2(b_i+vp_i/2)^2/\beta_i V_i^2)^{1/3} T_i} \quad \pi = \frac{\sum_i q_i T_i}{\sum_i p_i T_i} \quad \delta = \frac{\sum_i a_i D_i q_i T_i / p_i}{\sum_i q_i T_i}$$

ure 3 compares optimal vehicle sizes for the two services. It is shown that for demand densities between 1 and 120 trips per square mile per hour, the optimal vehicle size ranges from 9 to 24 seats per vehicle for flexible-route services and from 12 to 58 seats per vehicle for fixed-route services. As mentioned, the vehicle size is less sensitive to demand densities for flexible-route services than for fixed-route services.

The numerical results for the two services are presented in Table 3 on the basis of the demand pattern shown in Figure 2. The average costs are \$6.10 and \$6.23 per trip for fixed- and flexible-route services, respectively. On the basis of these results, fixed-route services are preferable to flexible-route services for the given demand pattern and other assumptions.

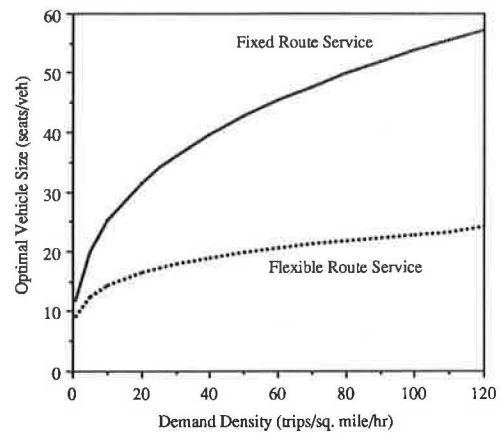
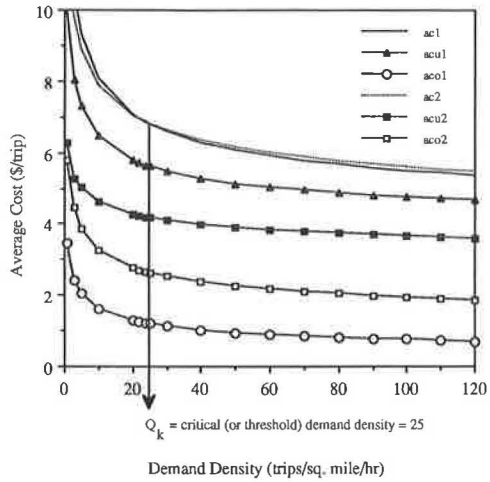



**FIGURE 3 Vehicle size comparison for fixed- and flexible-route services.**

TABLE 3 COMPARISON OF FIXED-ROUTE, FLEXIBLE-ROUTE, AND INTEGRATED SYSTEMS

| Systems | Fixed Route | Flexible Route | Integrated System |
|---|---|---|---|
| Vehicle Size (seats/veh) | 48 | 17 | 37 |
| Route Spacing (miles) | 0.867 | - | 0.683 |
| Zone Area (sq. miles) | - | 0.681<br>1.350<br>4.201<br>6.110 | -<br>-<br>4.054<br>5.021 |
| Headway (hrs) | 0.115<br>0.164<br>0.401<br>0.554 | 0.208<br>0.183<br>0.247<br>0.295 | 0.113<br>0.160<br>0.265<br>0.295 |
| Fleet Size (no. of vehs) | 48<br>29<br>10<br>7 | 115<br>60<br>13<br>8 | 63<br>37<br>13<br>9 |
| Total Cost (\$/day) | 60,390 | 61,633 | 59,390 |
| Avg. Cost (\$/trip) | 6.100 | 6.226 | 5.998 |
| Avg. Operator Cost (\$/trip) | 1.422 | 2.224 | 1.693 |
| Avg. User Cost (\$/trip) | 4.678 | 4.002 | 4.305 |
| Avg. Wait Cost (\$/trip) | 0.906 | 0.819 | 0.737 |
| Avg. In-Veh Cost (\$/trip) | 2.655 | 3.183 | 2.754 |
| Avg. Access Cost (\$/trip) | 1.117 | 0 | 0.814 |

Table 3 and Figure 4 indicate that the average user cost for fixed-route services is considerably higher than that for flexible-route services, whereas the operator cost for fixed-route services is considerably lower than that for flexible-route services. Operators, therefore, on the basis of their own costs, would strongly favor fixed-route services. The optimized ve-




Variable Representation
ac1: average cost for fixed route service = acu1+aco1
acu1: average user cost for fixed route service
aco1: average operator cost for fixed route service
ac2: average cost for flexible route service = acu2+aco2
acu2: average user cost for flexible route service
aco2: average operator cost for flexible route service

**FIGURE 4 Average cost comparison for fixed- and flexible-route services.**

hicle sizes are much smaller for flexible-route services (17 seats versus 48 seats for fixed routes), thus requiring a much larger fleet size (115 rather than 48 vehicles in the peak period).

## THRESHOLD ANALYSIS

A threshold analysis is used to determine which service type is preferable in which situations. Average cost (dollars per trip) is used to identify the critical demand density $Q_k$, below which the flexible-route service is preferable and above which the fixed-route service is preferable. In Figure 4 the optimized average costs of the two services are compared for a wide range of demand densities. The two average cost functions intersect at a demand density of 25 trips per square mile per hour, at which the average cost is $6.8 per trip. Hence, for the given parameter values and related assumptions, flexible-route services are preferable for demand densities below 25 trips per square mile per hour, which is considered to be the critical demand density. However, because the average cost functions for the two services intersect at very slight angles, the threshold value (e.g., 25 trips per square mile per hour) is quite sensitive to various system parameters. System parameters other than demand density, such as service area, operating cost, speed, and value of time, may also be analyzed to determine the values for which one service is better than the other. Sensitivity analyses (*11*) indicate that the relative advantages of flexible-route services generally increase with smaller service areas, higher operating speeds, lower fixed bus costs, lower incremental costs of vehicle size, higher values of access and wait time, and lower values of in-vehicle time.

With the critical demand density, the demand distribution can help determine under what circumstances fixed- or flexible-route bus services should be used exclusively. Figure 5 shows a transit daily demand distribution in which the maximal demand density is $q_{max}$ and the minimal demand density is $q_{min}$. This demand distribution has been processed from the original distribution to produce a distribution of flow versus duration. There are three possible interrelationships among the threshold demand density, maximal demand density, and minimal demand density:

1. If the flexible-route paratransit service is preferable to the conventional bus service at the highest demand density $q_{max}$ (i.e., the threshold demand density is $q_a$), it is preferable to operate the paratransit service exclusively. (See Figure 5, Case *a*.)
2. If the fixed-route bus service is better than the flexible-route bus service at the lowest demand density $q_{min}$ (i.e., the threshold demand density is $q_b$), fixed-route service should be operated exclusively. (See Figure 5, Case *b*.)
3. If the fixed-route service is better at $q = q_{max}$ but the flexible-route service is appropriate at $q = q_{min}$ (i.e., the threshold demand density $q_c$ is between $q_{max}$ and $q_{min}$), an integrated system will be preferable. (See Figure 5, Case *c*.)

Conditions for determining which service is preferable were discussed by Adebisi and Hurdle (*9*), but no strategy for the integration was developed, because only steady demand conditions were modeled. Multiperiod analytic optimization models for designing integrated systems are presented below.

## TEMPORALLY INTEGRATED SYSTEMS

From the threshold analysis, the range of demand densities for which flexible- or fixed-route services are preferable and the situations in which an integrated system is preferable can be identified. In the numerical examples (Figure 4), the flexible-route services were preferable to the fixed-route services at demand densities below 25 trips per square mile per hour. Because the demand distribution includes periods with demand above and below 25 trips per square mile per hour, a temporally integrated system should be preferable. The integrated system provides fixed-route services in the higher-demand periods (e.g., Periods 1 and 2 in the numerical example shown in Figure 2) and flexible-route services during the lower-demand periods (Periods 3 and 4).

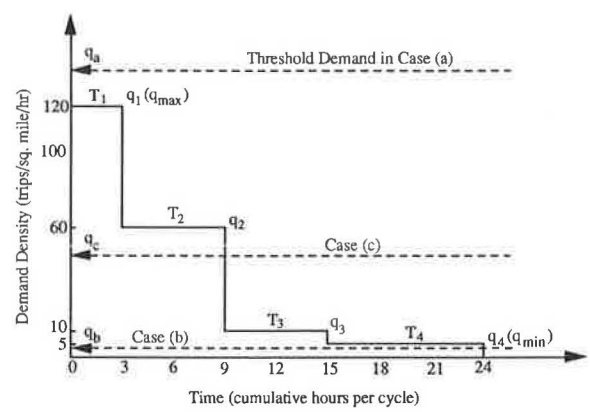



**FIGURE 5 Various cases of threshold demand density.**

This optimization approach seeks to determine the combination of vehicle size, route spacing, zone sizes, and service headways that minimizes total system cost ($C$). $C$, including the operator cost ($C_o$), user wait cost ($C_w$), user access cost ($C_x$), and user in-vehicle cost ($C_v$), can be expressed as a function of the decision variables [i.e., vehicle size ($S$), route spacing ($r$), zone area ($A_t$), and headway ($h_t$)] and system parameters:

$$C = \sum_{t=1}^{j-1} [C_o(S, r, h_t, K_t) + C_w(h_t, K_t) + C_x(r, K_t) + C_v(K_t)] + \sum_{t=j}^{m} [C_o(S, A_t, h_t, K_t) + C_w(h_t, K_t) + C_v(S, A_t, h_t, K_t)] \quad (2)$$

where $K_t = (B_t, V_t, T_t, L, W, w, x, v)$ is a set of system parameters consisting of operating cost ($B_t$); operating speed ($V_t$); duration of time periods ($T_t$); service area dimensions ($L$ and $W$); access speed ($g$); and values of wait, access, and in-vehicle time ($w$, $x$, and $v$, respectively).

The first part of Equation 2 is the cost of operating fixed-route services during Periods 1 to $j - 1$. The second part is the cost of operating flexible-route services during Periods $j$ to $m$. The access cost is assumed to be negligible because users are picked up and dropped off at their doorsteps. Such a formulation relies on the previous threshold analysis to determine that fixed-route services are preferable in Periods 1 to $j - 1$, whereas flexible-route services are preferable in Periods $j$ to $m$. This total cost function can be considered a combined cost function for the two types of service.

The following type of linear function for bus operating cost used by Jannson (*34*) and by Oldfield and Bly (*35*) is adopted for the total cost function:

$$B_t = a_t + b_t S \quad (3)$$

where $S$ is the vehicle size in seats per vehicle and $a_t$ and $b_t$ are parameters that may be estimated statistically. Certain relationships among vehicle size, zone size, and headway are also specified in the total cost function. For fixed-route service, they are expressed as

$$h_t = p_t S / r L q_t \quad (4)$$

and for flexible-route service as

$$h_t = p_t S / A_t q_t \quad (5)$$

In Equations 4 and 5 $p_t$ is the bus load factor at the peak load point. With these relationships, the total system cost of Equation 2 can be formulated for the integrated system as follows:

$$C = \sum_{t=1}^{j-1} \frac{LWD_t q_t T_t (a_t + b_t S)}{p_t S} + \sum_{t=1}^{j-1} \frac{w z_1 p_t S W T_t}{r} + \sum_{t=1}^{j-1} x z_2 L W q_t T_t \left(\frac{r + s}{g}\right) + \sum_{t=1}^{j-1} v L W q_t T_t M_t + \sum_{t=j}^{m} \frac{LWU q_t T_t (a_t + b_t S)}{V_t p_t S} + \sum_{t=j}^{m} \frac{L W q_t T_t \phi (a_t + b_t S) A_t^{1/2}}{V_t (u_t p_t S)^{1/2}} + \sum_{t=j}^{m} \frac{v L W U T_t}{2 V_t} + \sum_{t=j}^{m} \frac{v L W q_t T_t \phi (A_t p_t S)^{1/2}}{2 V_t u_t^{1/2}} + \sum_{t=j}^{m} \frac{w' z_1 L W T_t p_t S}{A_t} \quad (6)$$

Detailed derivations of these relationships are presented by Chang (*36*).

The variables and parameters are defined in Table 1. Different values of wait time, denoted as w and w′ for fixed- and flexible-route services, respectively, are defined for the two services. They allow a lower value of time to be used for indoor waiting at the origin, which may occur for flexible-route pickup. For this integrated system a single vehicle size is jointly optimized for both fixed- and flexible-route services, whereas the route spacing ($r$) and service zone area ($A_t$) are optimized separately for fixed- and flexible-route services.

The solution procedure for this problem is the combination of the solution procedures for the separate fixed- and flexible-route systems (*11*). Detailed derivations for integrated systems are provided by Chang and Schonfeld (*11*) and Chang (*36*). Equation 7 is obtained by solving the first derivatives of the total cost function:

$$\frac{\beta_1}{S^2} - \frac{\beta_2}{S^{1/2}} - \frac{\beta_3}{S^{1/3}} = 0 \quad (7)$$

where

$$\beta_1 = \sum_{t=1}^{j-1} \frac{a_t D_t q_t T_t}{p_t} + \sum_{t=j}^{m} \frac{a_t U q_t T_t}{p_t V_t} \quad (8)$$

$$\beta_2 = \sum_{t=1}^{j-1} q_t T_t \left( \frac{z_1 z_2 w x \sum_{t=1}^{j-1} p_t T_t}{g L \sum_{t=1}^{j-1} q_t T_t} \right)^{1/2} \quad (9)$$

$$\beta_3 = (2 w' z_1 \phi^2)^{1/3} \sum_{t=j}^{m} T_t \left\{ \frac{q_t^2 (b_t + v p_t/2)^2}{u_t V_t [1 + a_t / S (b_t + v p_t/2)]} \right\}^{1/3} \quad (10)$$

If $j = 1$, Equation 7 includes only flexible-route services. In that case the optimized vehicle size shown in Table 2 for flexible-route services can be used. If $j - 1 = m$, the problem is reduced to finding the optimal solution for only fixed-route services, and the analytic results shown in Table 2 for fixed-route services can be applied.

Equation 7 is not difficult to solve numerically, but it has not been solved in closed form. After the optimal vehicle size is obtained, the optimal route spacing ($r^*$) for fixed-route services and zone area ($A_t^*$) for flexible-route services can be obtained with the following equations:

$$r^* = \left( \frac{z_1 w g S^* \sum_{t=1}^{j-1} p_t T_t}{z_2 w L \sum_{t=1}^{j-1} q_t T_t} \right)^{1/2} \quad (11)$$

$$A_t^* = p_t S^* \left( \frac{2z_1 w' V_t u_t^{1/2}}{\phi q_t (a_t + b_t S^* + v p_t S^*/2)} \right)^{2/3}$$

$$t = j, j + 1, \ldots, m \tag{12}$$

The service headway for different periods providing fixed- or flexible-route services can also be obtained by substituting the optimized vehicle size ($S^*$) and route spacing ($r^*$) or zone area ($A_t^*$) into Equations 4 and 5:

$$h_t^* = \frac{S^* p_t}{r^* L q_t} \qquad t = 1, 2, \ldots, j - 1 \tag{13}$$

$$h_t^* = \frac{S^* p_t}{A_t^* q_t} \qquad t = j, j + 1, \ldots, m \tag{14}$$

A compromise vehicle size for providing fixed-route services in the higher-demand periods and flexible-route services in the lower-demand periods can be determined with Equation 6.

## NUMERICAL CASES

### Baseline Value Results

For the four-period example shown in Figure 2, the fixed-route services are provided in the first and second periods. and the flexible-route services are provided in the third and fourth periods. Therefore, Equation 7, in which $\beta_2$ and the first term of $\beta_1$ are components from the first and second periods, whereas $\beta_3$ and the second term of $\beta_1$ are components from the third and fourth periods, becomes

$$\frac{33{,}027.5}{S^2} - \frac{78.1}{S^{1/2}} - \frac{39.5}{S^{1/3}} = 0 \tag{15}$$

By solving Equation 15, the optimal vehicle size for the integrated system is found to be 37 seats per vehicle. By substituting the optimal vehicle size into Equations 11, 12, 13, and 14, respectively, the optimal route spacing, zone area, and headways for the integrated system can be obtained, as given in Table 3.

Comparisons of the temporally integrated systems with pure fixed- and flexible-route systems yield the following observations:

1. The optimized vehicle size of 37 seats for the integrated system lies between those for the two pure systems (48 and 17 for fixed- and flexible-route systems, respectively). Thus, the optimized fleet size of 63 vehicles for the integrated system also lies between those for the two pure systems (48 and 115). It can be verified from Equation 7 that when the demand density and duration of the third period increase, the optimal vehicle size for the integrated system decreases.
2. The average cost for the integrated system is indeed lower than for either pure system. However, its average user cost and the average operator cost both lie between the corresponding pure system values. The cost reduction offered by the integrated system cannot be very high for the systems analyzed in the example, because the average cost functions (Figure 4) for the two pure systems are quite close.
3. The optimal average operator and user costs ($1.693 and $4.305 per trip, respectively) for the integrated system also lie between those for the two pure systems, whereas the optimal average wait cost ($0.737 per trip) is lower than for either pure system.

### Effects of Various Demand Patterns

Three demand patterns, which have the same total demand but different demand fractions in Periods 2 and 3, are shown in Figure 6. Case 1 is the previously computed baseline example. The difference in demand between Periods 2 and 3 decreases in Case 2 and increases in Case 3. Table 4 presents the optimized average costs, vehicle sizes, and fleet sizes for the three cases.

Table 4 indicates that the average costs for integrated systems are lower than for pure systems in all three cases, although the decreases in average costs are small in the cases presented here. Vehicle sizes for both pure systems are nearly the same for different demand patterns. However, they vary considerably for integrated systems. Similar results are found for fleet size.

## CONCLUSIONS

Temporally integrated systems in which fixed-route services are provided during higher-demand periods and flexible-route services are provided during lower-demand periods were evaluated analytically and numerically. Threshold analysis was

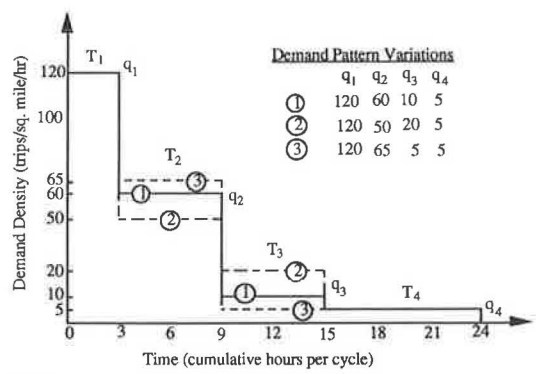



**FIGURE 6 Alternative demand patterns analyzed.**

TABLE 4 OPTIMIZED AVERAGE COST, VEHICLE SIZE, AND FLEET SIZE FOR THREE DEMAND DISTRIBUTIONS

| | Type of System | | |
|---|---|---|---|
| Case | Fixed-Route | Flexible-Route | Integrated |
| Optimized Average Cost ($/trip) | | | |
| 1 | 6.100 | 6.226 | 5.998 |
| 2 | 6.148 | 6.302 | 6.062 |
| 3 | 6.048 | 6.207 | 5.946 |
| Optimized Vehicle Size | | | |
| 1 | 48 | 17 | 37 |
| 2 | 48 | 16 | 33 |
| 3 | 48 | 17 | 41 |
| Optimized Fleet Size | | | |
| 1 | 49 | 115 | 63 |
| 2 | 49 | 121 | 70 |
| 3 | 49 | 115 | 57 |

used to identify the range of demand densities for which purely fixed- or flexible-route services are preferable and the situations in which integrated systems are preferable. It was shown that the threshold is sensitive to system parameters.

Numerical results indicate that the optimal vehicle size in integrated systems (37 seats per vehicle) is a compromise between the optimal vehicle sizes for pure fixed-route and pure flexible-route services (48 and 17 seats per vehicle, respectively). More important, the average system cost per trip for integrated systems can be lower than for either pure system. However, if the total costs per trip for fixed- and flexible-route alternatives are close, the integrated system cannot offer costs that are much lower than for either pure system. In realistic applications, the benefits of temporal integration are expected to increase as the relative duration of low-demand periods (in which flexible-route services are preferable) increases.

Further studies should consider operation and control strategies for transitions between the two service types in an integrated system. Mixed rather than homogeneous bus fleets for integrated operation are also worth analyzing. Further research may consider demand elasticity and many-to-many demand patterns.

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