Dynamic Analysis of User-Optimized Network Flows with Elastic Travel Demand

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An equivalent continuous time optimal control problem is formulated for dynamic user-optimized traffic assignment with elastic travel demand. Using the Pontryagin minimum principle, optimality conditions are derived and economic interpretations that correspond to a dynamic generalization of Wardrop's first principle are provided. The existence and optimality of singular controls are examined. Under steady-state assumptions, the model is shown to be a proper dynamic extension of Beckmann's equivalent optimization problem for static user-optimized traffic assignment with elastic demand. Finally, limitations and extensions of the present model are discussed.

There has been a great deal of interest in dynamic network equilibrium models. The interest derives from a growing recognition that steady-state network equilibrium is not typically reached during peak hours of commuting and that demand and supply characteristics of urban transportation networks are inherently time-varying in certain situations. With this recognition, a number of researchers have developed dynamic network equilibrium models from different perspectives. Friesz (*1*), Alfa (*2*), and Wie et al. (*3*) provide literature reviews of the dynamic network equilibrium models proposed to date.

In this paper, a simplistic dynamic extension of the static user equilibrium traffic assignment model with elastic demand, which was first formulated as an equivalent optimization problem by Beckmann et al. (*4*), is analyzed. The key simplification is related to the assumption that adjustments from one system state to another may occur instantaneously as traffic conditions on the network change. The problem of dynamic user-optimized traffic assignment with elastic demand is not only to predict time-varying traffic flows on congested networks, but also to predict the temporal distribution of travel demand from each origin node in response to dynamic changes in traffic conditions. This paper should be regarded as an extension of the dynamic user-optimized traffic assignment model presented by Friesz et al. (*5*). The particular extension is to include elastic time-varying travel demand, which leads to the implicit consideration of departure time changes. The previous model considered only the route choice decision-making process because travel demand at each instant was assumed to be known and inelastic with respect to changes in travel costs. This paper is a direct extension of Wie (*6*), which was restricted to a very simple network with one origin-destination pair connected by parallel arcs to a network with many origins and a single destination.

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To describe individual behaviors of departure time and route choices, we assume that each driver receives complete information on the current state of the network at each instant through an in-car computer connected to a traffic information center. The current traffic information may include instantaneous measures of arc densities and arc capacity changes due to traffic accidents, weather conditions, or road construction. On the basis of continuously updated traffic information, each driver can estimate the instantaneous expected unit path costs from an origin node or any en route intersection node to the destination node. We further assume that all network users attempt to minimize individual travel costs by changing routes and departure times. The problem considered in this paper corresponds to a type of noncooperatively dynamic game in which network users act independently without collaboration and compete with one another for limited network capacity through route and departure time choices.

Our dynamic model has different behavioral assumptions compared with the dynamic models that have been developed by Hendrickson and Kocur (*7*), de Palma et al. (*8*), Mahmassani and Herman (*9*), Ben-Akiva et al. (*10*), Newell (*11*), and Arnott et al. (*12*). The choice of route or departure time, or both, in these dynamic models is generally based on the trade-off between travel time and schedule delay (i.e., a penalty for late or early arrival). In contrast, our dynamic model cannot explicitly treat schedule delay; in other words, the choice of departure time is not based on the trade-off between travel time and schedule delay. Our dynamic model handles departure time and route choices in a sequential manner. At each instant, travel demand—that is, the rate of departure from each origin node—is endogenously determined as a function of the instantaneous expected travel cost between the associated origin-destination pair. Each driver who decides to depart then chooses the shortest path with minimum instantaneous expected unit travel cost to the destination. It is assumed that no driver has information as to how travel costs for further downstream arcs may change by the time of arrival at those arcs. However, as a driver moves downstream, he is free to revise his route choice at any en route intersection node if his current route is no longer optimal on the basis of updated traffic information.

Another important difference in behavioral assumptions is associated with the dynamic user equilibrium conditions. The instantaneous expected travel cost defined in this paper is not the cost actually experienced on that particular day, but an estimate based on current traffic information. Therefore, our dynamic model cannot predict time-varying traffic flows and elastic travel demands satisfying the dynamic user equilibrium conditions such that all network users with the same desired arrival time and same origin-destination pair experience equal travel cost to the destination regardless of route and departure time chosen. Because the optimality conditions of our dynamic model require only equalization of instantaneous expected unit path costs, it is possible that some drivers can reduce individual travel costs by unilaterally changing routes or departure times.

An equivalent continuous time optimal control problem that corresponds to the problem of dynamic user-optimized traffic assignment with elastic travel demand is formulated. We derive the optimality conditions using the Pontryagin minimum principle, and we examine the existence and optimality of singular controls. The economic interpretation of the optimality conditions is given as a dynamic generalization of Wardrop's first principle. Our dynamic model is also analyzed under steady-state assumptions to show that it is a proper dynamic extension of Beckmann's equivalent optimization problem for static user-optimized traffic assignment with elastic demand. Finally, limitations and extensions of our dynamic model are discussed.

MODEL FORMULATION

Assume a network represented by a directed graph $G(N, A)$, where N is the set of nodes and A is the set of arcs. The cardinality of the set N is denoted by $|N| = n$. Nodes $1, 2, \ldots, n - 1$ are origins, whereas n is the only destination. The set of all origins is denoted by M. In general, we use the index a to denote an arc, k a node, and p a path. The set of all paths connecting Node k and Node n is denoted by P_{kn}. We consider a fixed time horizon of length T; that is, all activities occur at some time $t \in [0, T]$.

Let $x_a(t)$ denote the number of vehicles traveling on Arc a at Time t, which will be referred to as the traffic volume on Arc a at Time t. We assume that the instantaneous expected travel cost for a driver (or drivers) entering Arc a at Time t is dependent on $x_a(t)$ and that the instantaneous expected unit cost functions $c_a[x_a(t)]$ are positive, nondecreasing, differentiable, and convex for all $x_a(t) \geq 0$ and $t \in [0, T]$. Link interactions are not considered in this model. To depict the physical phenomenon of traffic congestion on each arc, the exit functions $g_a[x_a(t)]$ are assumed to be nonnegative, nondecreasing, differentiable, and concave for all $x_a(t) \geq 0$ and $t \in [0, T]$ with the additional restriction that $g_a(0) = 0$ for all $a \in A$. A functional form of the exit functions can be represented as

$$g_a[x_a(t)] = g_a^{\max} \cdot \{1 - \exp[-x_a(t)/\beta_a]\} \qquad \forall\, a \in A \quad \forall\, t \in [0, T] \qquad (1)$$

where $g_a^{\max}$ is the maximum number of vehicles that can exit from Arc a at each instant and β_a is a parameter that varies with road type and traffic signal system.

The dynamic evolution of the state of each arc is described by first-order nonlinear differential equations:

$$dx_a(t)/dt = \dot{x}_a(t) = u_a(t) - g_a[x_a(t)] \qquad \forall\, a \in A \quad \forall\, t \in [0, T] \qquad (2)$$

where

$x_a(t)$ = the state variable denoting the traffic volume on Arc a at Time t;
$u_a(t)$ = the control variable denoting the traffic flow entering Arc a at Time t; and
$g_a[x_a(t)]$ = the traffic flow exiting Arc a at Time t.

Throughout the paper, Equation 2 will be called the state equation. In addition, we assume that the traffic volume on Arc $a \in A$ is a known nonnegative constant at the initial time $t = 0$:

$$x_a(0) = x_a^0(t) \geq 0 \qquad \forall\, a \in A \qquad (3)$$

Different initial values of traffic volumes may lead to different predictions of time-varying traffic flow patterns.

Let $S_k(t)$ denote the instantaneous travel demand generated (i.e., the rate of departure) from an origin node k at Time t. We assume that $S_k(t)$ is endogenously determined as a function of the instantaneous expected unit travel cost between an origin node k and the destination node n at Time t. It follows that

$$S_k(t) = D_k[t, \mu_k(t)] \qquad \forall\, k \in M \quad \forall\, t \in [0, T] \qquad (4)$$

where $D_k(\cdot)$ is the instantaneous demand function for travel between Nodes k and n at Time t and $\mu_k(t)$ is the minimum instantaneous expected travel cost between Nodes k and n at Time t. We assume that the instantaneous demand functions are nonnegative and monotonically decreasing and that they can continuously change in functional form over the time interval $[0, T]$ to represent time-varying price elasticity of demand for travel between each origin-destination pair. We are unable to discuss a functional form of the instantaneous demand functions and their calibrations. A more realistic dynamic model should also consider cross elasticity of demand, implying that $S_k(t)$ is determined as a function of the trajectory of $\mu_k(t)$ over the time interval $[0, T]$. At this point, we are unable to model this case. Furthermore, we assume that the inverse of the instantaneous demand function is well defined and exists as follows:

$$\mu_k(t) = \Phi_k[t, S_k(t)] \qquad \forall\, k \in M \quad \forall\, t \in [0, T] \qquad (5)$$

The flow conservation constraints are stated as follows:

$$S_k(t) + \sum_{a \in B(k)} g_a[x_a(t)] - \sum_{a \in A(k)} u_a(t) = 0 \qquad \forall\, k \in M \quad \forall\, t \in [0, T] \qquad (6)$$

where

$S_k(t)$ = the control variable, denoting the instantaneous travel demand generated from Node k at Time t;
$A(k)$ = the set of arcs whose tail node is k; and
$B(k)$ = the set of arcs whose head node is k.

We ensure that both the state and control variables are nonnegative:

$$x_a(t) \geq 0 \qquad \forall\, a \in A \qquad \forall\, t \in [0, T] \tag{7}$$

$$u_a(t) \geq 0 \qquad \forall\, k \in M \qquad \forall\, t \in [0, T] \tag{8}$$

$$S_k(t) \geq 0 \qquad \forall\, k \in M \qquad \forall\, t \in [0, T] \tag{9}$$

However, we will not consider the nonnegatively of the state variables in an explicit manner because the assumption $g_a(0) = 0$ ensures that the state variables are always nonnegative.

We define $x(t) \equiv (\ldots, x_a(t), \ldots)$, $u(t) \equiv (\ldots, u_a(t), \ldots)$, and $S(t) \equiv (\ldots, S_k(t), \ldots)$. In the sequel, we employ the set of feasible solutions

$$\Omega \equiv \{[x(t), u(t), S(t)]\text{: Expressions 2, 3, 6, 8, and 9 are satisfied}\} \tag{10}$$

for economy of notation. We are now ready to formulate the dynamic user-optimized traffic assignment problem with elastic demand as an equivalent continuous time optimal control problem:

$$\text{Minimize } J = \sum_{a \in A} \int_0^T \int_0^{x_a(t)} c_a(\omega)[dg_a(\omega)/d\omega]\, d\omega\, dt - \sum_{k \in M} \int_0^T \int_0^{S_k(t)} \Phi_k(t, \eta)\, d\eta\, dt \tag{11}$$

subject to

$$[x(t), u(t), S(t)] \in \Omega$$

where ω and η are dummy variables of integration. The performance index J is a scalar function that has no intuitive economic interpretation. It should be viewed strictly as a mathematical construction. The derivation of J is analogous to that of the objective function of Beckmann's equivalent optimization problem for a static user equilibrium traffic assignment with elastic demand (*4*).

OPTIMALITY CONDITIONS

The Pontryagin minimum principle (*13*) is used to derive the necessary conditions for an optimal solution of the control problem (Equation 11). We first construct the Hamiltonian function:

$$\begin{aligned} H[x(t), u(t), S(t), \lambda(t), \mu(t)] &= \sum_{a \in A} \int_0^{x_a(t)} c_a(\omega)[dg_a(\omega)/d\omega]\, d\omega \\ &- \sum_{k \in M} \int_0^{S_k(t)} \Phi_k(t, \eta)\, d\eta + \sum_{a \in A} \lambda_a(t) \Big\{ u_a(t) - g_a[x_a(t)] \Big\} \\ &+ \sum_{k \in M} \mu_k(t) \Big\{ S_k(t) + \sum_{a \in B(k)} g_a[x_a(t)] - \sum_{a \in A(k)} u_a(t) \Big\} \end{aligned} \tag{12}$$

where $\lambda_a(t)$ is the costate variable associated with the state equation (Equation 2) and $\mu_k(t)$ is the Lagrange multiplier associated with the flow conservation constraints (Equation 6). Note that $\lambda(t) \equiv (\ldots, \lambda_a(t), \ldots)$ and $\mu(t) \equiv (\ldots, \mu_k(t), \ldots)$.

The Pontryagin minimum principle states that the dynamic evolution of the costate variables are governed by the following first-order differential equations:

$$\begin{aligned} -\dot{\lambda}_a(t) &= \partial H[x(t), u(t), S(t), \lambda(t), \mu(t)]/\partial x_a(t) \\ &= \Big\{ c_a[x_a(t)] - \lambda_a(t) + \mu_k(t) \Big\} g_a'[x_a(t)] \\ &\forall\, a \in B(k) \qquad \forall\, k \in M \qquad \forall\, t \in [0, T] \end{aligned} \tag{13}$$

where $g_a'[x_a(t)] = dg_a[x_a(t)]/dx_a(t)$. Equation 13 will be called the costate equation. Let $\phi_a[x_a(T)]$ denote the salvage value function when the terminal state is $x_a(T)$. Because we impose no constraint on the values of the state variables at the terminal time T, the value of $\phi_a[x_a(T)]$ must be equal to zero for all $a \in A$. Hence, the terminal boundary conditions on the costate variables are given as follows:

$$\lambda_a(T) = \partial\phi_a[x_a(T)]/\partial x_a(T) = 0 \qquad \forall\, a \in A \tag{14}$$

Equation 14 is often called the transversality conditions. In addition, the state equation (Equation 2) can be expressed in terms of the Hamiltonian as follows:

$$\begin{aligned} \dot{x}_a(t) &= \partial H[x(t), u(t), S(t), \lambda(t), \mu(t)]/\partial \lambda_a(t) \\ &= u_a(t) - g_a[x_a(t)] \\ &\forall\, a \in A \qquad \forall\, t \in [0, T] \end{aligned} \tag{15}$$

In optimal control theory, the differential equations for the state variables and the differential equations for the costate variables plus all boundary conditions are called the canonical equations, which give rise to the two-point boundary value problem.

The Pontryagin minimum principle also requires that the Hamiltonian (Equation 12) be minimized by choice of the optimal control variables at each point along the optimal state trajectories. The control problem (Equation 11) can thus be converted into an infinity of constrained static optimization problems for each instant $t \in [0, T]$ as follows:

$$\text{Minimize } H[x(t), u(t), S(t), \lambda(t), \mu(t)] \tag{16}$$

subject to

$$u_a(t) \geq 0 \qquad \forall\, a \in A$$

$$S_k(t) \geq 0 \qquad \forall\, k \in M$$

While holding $x(t)$ and $\lambda(t)$ constant, the Kuhn-Tucker necessary conditions for $u(t)$ and $S(t)$ to be optimal are readily obtained:

$$\partial H/\partial S_k(t) = \mu_k(t) - \Phi_k[t, S_k(t)] \geq 0 \qquad \forall\, k \in M \qquad \forall\, t \in [0, T] \tag{17}$$

$$S_k(t)\,[\partial H/\partial S_k(t)] = S_k(t)\left\{\mu_k(t) - \Phi_k[t, S_k(t)]\right\} = 0$$

$$\forall\, k \in M \qquad \forall\, t \in [0, T] \qquad (18)$$

$$\partial H/\partial u_a(t) = \lambda_a(t) - \mu_k(t) \geq 0$$

$$\forall\, a \in A(k) \qquad \forall\, k \in M \qquad \forall\, t \in [0, T] \qquad (19)$$

$$u_a(t)[\partial H/\partial u_a(t)] = u_a(t)[\lambda_a(t) - \mu_k(t)] = 0$$

$$\forall\, a \in A(k) \qquad \forall\, k \in M \qquad \forall\, t \in [0, T] \qquad (20)$$

$$\partial H/\partial \mu_k(t) = S_k(t) + \sum_{a \in B(k)} g_a[x_a(t)] - \sum_{a \in A(k)} u_a(t) = 0$$

$$\forall\, k \in M \qquad \forall\, t \in [0, T] \qquad (21)$$

The complementary slackness conditions (Equations 17 and 18) indicate that if the optimal value of the control variable $S_k(t)$ is positive, Equation 17 must hold as an equality. In other words

$$\mu_k(t) = \Phi_k[t, S_k(t)] \qquad \forall\, k \in M \qquad \forall\, t \in [0, T] \qquad (22)$$

Both sides of Equation 22 can be inverted to obtain

$$S_k(t) = D_k[t, \mu_k(t)] \qquad \forall\, k \in M \qquad \forall\, t \in [0, T] \qquad (23)$$

Hence, if the traffic flow generated at Node k at Time t is positive, it must be determined by the instantaneous demand function $D_k[t, \mu_k(t)]$. If, however, $\mu_k(t) > \Phi_k[t, S_k(t)]$, then $S_k(t) = 0$, meaning that the minimum instantaneous expected travel cost between Origin k and Destination n at Time t may be too high to induce any departure. Because the instantaneous demand function $D_k[t, \mu_k(t)]$ is assumed to be monotonically decreasing, it follows that its inverse, $\Phi_k[t, S_k(t)]$, should be a decreasing function. The integral of a decreasing function is strictly concave, and the negative of the sum of concave functions is a strictly convex function. Thus, the second term in the Hamiltonian (Equation 12) is strictly convex, implying that the optimization problem (Equation 16) has a unique solution in terms of $S(t)$.

From the complementary conditions (Equations 19 and 20), we know for all $a \in A(k)$ and $k \in M$ that if $\lambda_a(t) > \mu_k(t)$, then $u_a(t) = 0$; if $\lambda_a(t) = \mu_k(t)$, then $u_a(t) \geq 0$. Obviously, the optimal value of the control variable $u_a(t)$ is influenced by the sign of $[\lambda_a(t) - \mu_k(t)]$, which is called the switching function. If, however, $\lambda_a(t) = \mu_k(t)$ for some $a \in A(k)$ and $k \in M$ during a finite time interval, the minimization of the Hamiltonian (Equation 12) leads to nonunique determination of the optimal value of $u_a(t)$, that is, singular control. As a result, the Pontryagin minimum principle yields no useful information to determine the optimal value of the control variable $u_a(t)$. In this circumstance, an additional necessary condition is required to replace the optimality conditions (Equations 19 and 20) so that the singular controls could be tested for optimality. To this end we proceed to derive an expression for the singular control and to ensure that the generalized Legendre-Clebsch condition is satisfied. If $\lambda_a(t) = \mu_k(t)$ for some $a \in A(k)$ and $k \in M$ during a finite time interval $[t_1, t_2] \subseteq [0, T]$, it follows at once that

$$\frac{d}{dt}\left[\frac{\partial L}{\partial u_a(t)}\right] = \dot{\lambda}_a(t) - \dot{\mu}_k(t) = 0 \qquad (24)$$

$$\frac{d^2}{dt^2}\left[\frac{\partial L}{\partial u_a(t)}\right] = \ddot{\lambda}_a(t) - \ddot{\mu}_k(t) = 0 \qquad (25)$$

Consider an arc whose tail node is k and head node is s, that is, $a = (k, s) \in A$. Using the costate equation (Equation 13), we may rewrite Equations 24 and 25 as follows:

$$-\left\{c_a[x_a(t)] - \lambda_a(t) + \mu_s(t)\right\} g'_a[x_a(t)] - \dot{\mu}_k(t) = 0 \qquad (26)$$

$$-\left\{c'_a[x_a(t)]\,\dot{x}_a(t) - \dot{\lambda}_a(t) + \dot{\mu}_s(t)\right\} g'_a[x_a(t)]$$

$$-\left\{c_a[x_a(t)] - \lambda_a(t) + \mu_s(t)\right\} g''_a[x_a(t)]\dot{x}_a(t) - \ddot{\mu}_k(t) = 0 \qquad (27)$$

where $dc_a[x_a(t)]/dx_a(t) = c'_a\,[x_a(t)]$. For the moment, we suppress the time notation (t) and the traffic volume notation $x_a(t)$. Substitution of Equations 2 and 24 into Equation 27 yields an expression for the singular control as a feedback control:

$$u_a[x(t), \lambda(t), \mu(t)] =$$

$$\frac{(c'_a g_a + \dot{\mu}_k - \dot{\mu}_s)g'_a + (c_a - \mu_k + \mu_s)g''_a g_a - \ddot{\mu}_k}{c'_a g'_a + (c_a - \mu_k + \mu_s)g''_a}$$

$$\forall\, a = (k, s) \in A \qquad \forall\, k \in M \qquad \forall\, s \in M$$

$$\forall\, t \in [t_1, t_2] \subseteq [0, T] \qquad (28)$$

When a singular control exists, an additional test is needed to determine whether this singular control is optimizing or not. Usually, the optimality of singular controls given by Equation 28 can be tested by using the generalized Legendre-Clebsch condition (*14*) as follows:

$$\frac{\partial}{\partial u_a(t)}\left\{\frac{d^2}{dt^2}\left[\frac{\partial L}{\partial u_a(t)}\right]\right\} = -(c_a - \mu_k + \mu_s)g''_a - c'_a g'_a \leq 0$$

$$\forall\, a = (k, s) \in A \qquad \forall\, k \in M \qquad \forall\, s \in M$$

$$\forall\, t \in [t_1, t_2] \subseteq [0, T] \qquad (29)$$

Because $c_a[x_a(t)]$ and $g_a[x_a(t)]$ are assumed to be nondecreasing for all $x_a(t) \geq 0$, it follows that $c'_a[x_a(t)]\, g'_a[x_a(t)]$ is nonnegative. We also know that $g''_a[x_a(t)] \leq 0$ because $g_a[x_a(t)]$ is concave for all $x_a(t) \geq 0$. It remains to be proven that $c_a[x_a(t)] - \mu_k(t) + \mu_s(t) \leq 0$. However, this condition is not strictly satisfied because it requires from the costate equation (Equation 13) that $d\lambda_a(t)/dt$ always be nonnegative. Certainly, we know that $d\lambda_a(t)/dt$ can be negative during a finite time period. Hence, we are not able to conclude that the singular controls expressed in Equation 28 are optimal. We reserve this issue for future research.

ANALYSIS OF OPTIMALITY CONDITIONS

Let us consider a path p connecting an origin node k and the destination node n, expressed in generic form as

$$p \equiv [k = k_0, a_1, k_1, \ldots, k_{m-1}, a_m, n = k_m] \tag{30}$$

Using the costate equation (Equation 13), we define the instantaneous unit path travel cost function:

$$\Psi_p(t) = \sum_{a \in A(p)} \left\{ c_a[x_a(t)] + \frac{\dot{\lambda}_a(t)}{g_a'[x_a(t)]} \right\}$$

$$\forall\, p \in P_{kn} \qquad \forall\, t \in [0, T] \tag{31}$$

where $A(p)$ denotes the set of arcs composing a path p. We are now ready to state and prove the following theorem:

Theorem 1. If $u_a(t) > 0$ for all $a \in A(p)$ at some time $t \in [0, T]$, then $\Psi_p(t) = \inf \{\Psi_b(t)\colon \forall\, b \in P_{kn}\}$ for an optimal solution of the dynamic user-optimized traffic assignment problem with elastic demand (Expression 11).

Proof: Using the costate equation (Equation 13), we may rewrite Equation 31 as

$$\Psi_p(t) = \sum_{i=1}^{m} [\lambda_{a_i}(t) - \mu_{k_i}(t)] \tag{32}$$

We know from Equation 20 that if $u_a(t) > 0$ for all $a \in A(p)$,

$$\lambda_{a_i}(t) = \mu_{k_{i-1}}(t) \qquad \text{for } i = 1, \ldots, m \tag{33}$$

Substituting Equation 33 into Equation 32, it follows at once that for a path $p \in P_{kn}$

$$\begin{aligned} \Psi_p(t) &= [\mu_{k0}(t) - \mu_{k1}(t)] + [\mu_{k1}(t) - \mu_{k2}(t)] + \ldots \\ &\quad + [\mu_{k_{m-1}}(t) - \mu_{k_m}(t)] \\ &= \mu_k(t) - \mu_n(t) \end{aligned} \tag{34}$$

The theorem follows immediately. Q.E.D.

Provided that $\mu_n(t) = 0$, the economic interpretation of the Lagrange multiplier $\mu_k(t)$ can be given as a minimum instantaneous expected unit travel cost between Origin k and Destination n at Time t:

$$\mu_k(t) = \inf \left(\sum_{a \in A(p)} \left\{ c_a[x_a(t)] + \frac{\dot{\lambda}_a(t)}{g_a'[x_a(t)]} \right\} \colon \forall\, p \in P_{kn} \right) \tag{35}$$

Substituting Equation 35 into Equation 13 yields

$$\begin{aligned} \lambda_a(t) &= c_a[x_a(t)] + \left\{ \dot{\lambda}_a(t)/g_a'[x_a(t)] \right\} + \mu_s(t) \\ &= c_a[x_a(t)] + \left\{ \dot{\lambda}_a(t)/g_a'[x_a(t)] \right\} \\ &\quad + \inf \{\Psi_p(t)\colon \forall\, p \in P_{sn}\} \end{aligned}$$

$$\forall\, a = (k, s) \in A \qquad \forall\, s \in M \qquad \forall\, t \in [0, T] \tag{36}$$

The costate variable $\lambda_a(t)$ can be interpreted as a minimum instantaneous expected unit travel cost between Origin k and Destination n at Time t with only the restriction that $a = (k, s)$ must be the first arc to traverse. Therefore, $\Psi_p(t)$ may be viewed as consisting of static and dynamic components. The term $\Sigma_{a \in A(p)}\, c_a[x_a(t)]$ is considered to be static. On the other hand, the term $\Sigma_{a \in A(p)}\, [d\lambda_a(t)/dt]g_a'[x_a(t)]$ is considered to be dynamic in that it includes the rate of change of $\lambda_a(t)$ with respect to time, which represents the dynamics of the costate variables, whereas $g_a'\,[x_a(t)]$ is interpreted as a scaling of $d\lambda_a(t)/dt$.

Theorem 1 requires equilibration of instantaneous unit travel costs for all the paths that are being used at each instant in time for a given origin-destination pair as follows:

$$\Psi_1(t) = \ldots = \Psi_j(t) \leq \Psi_{j+1}(t) \leq \ldots \leq \Psi_J(t) \tag{37}$$

$$u_a(t) > 0 \text{ for all } a \in \{A(p) \mid p = 1, \ldots, j\} \tag{38}$$

$$u_a(t) = 0 \text{ for some } a \in \{A(p) \mid p = j + 1, \ldots, J\} \tag{39}$$

where J is the cardinality of the set P_{kn}. Hence, Theorem 1 can be interpreted as a dynamic generalization of Wardrop's first principle (*15*) such that if, at each instant in time, for each origin-destination pair, the instantaneous unit travel costs for all the paths that are being used are identical and equal to the minimum instantaneous unit travel cost, the corresponding time-varying traffic flow pattern is said to be user optimized.

EQUIVALENCY UNDER STEADY-STATE ASSUMPTIONS

We shall establish that the control problem (Equation 11) is a proper dynamic extension of Beckmann's mathematical programming problem for a static user equilibrium traffic assignment with elastic demand (*4*). To this end we analyze our dynamic model under the following steady-state assumptions: first, that $S_k(t)$ and $c_a[x_a(t)]$ are time-invariant for all $a \in A$ and $k \in M$, and second, that $dx_a(t)/dt = 0$ and thus $u_a(t) = g_a[x_a(t)]$ for all $a \in A$ and $t \in [0, T]$. By changing the variables of integration, we may rewrite the first term in the Hamiltonian (Equation 12) and have the following relation:

$$\sum_{a \in A} \int_0^{x_a} c_a(\omega)[dg_a(\omega)/d\omega]\, d\omega = \sum_{a \in A} \int_0^{g_a(x_a)} c_a(\xi)\, d\xi \tag{40}$$

Let f_a denote $g_a(x_a)$, meaning the traffic flow rate on Arc a. Under the steady-state assumptions, the continuous time optimal control problem (Equation 11) becomes a series of constrained static optimization problems that are identical at each instant during the fixed time interval $[0, T]$ as follows:

Minimize $Z(x)$ =

$$\sum_{a \in A} \int_0^{f_a} c_a(\omega) d\omega - \sum_{k \in M} \int_0^{S_k} \Phi_k(\eta)\, d\eta \tag{41}$$

subject to

$$S_k + \sum_{a \in B(k)} f_a - \sum_{a \in A(k)} f_a = 0 \qquad \forall\, k \in M$$

$$f_a \geq 0 \qquad \forall\, a \in A$$

$$S_k \geq 0 \qquad \forall\, k \in M$$

The Kuhn-Tucker necessary conditions for the problem (Equation 41) can be readily obtained as

$$f_a[c_a(f_a) - \mu_k] = 0 \qquad \forall\, a \in A \tag{42}$$

$$c_a(f_a) - \mu_k \geq 0 \qquad \forall\, a \in A \tag{43}$$

$$S_k[\mu_k - \Phi_k(S_k)] = 0 \qquad \forall\, k \in M \tag{44}$$

$$\mu_k - \Phi_k(S_k) \geq 0 \qquad \forall\, k \in M \tag{45}$$

$$S_k + \sum_{a \in B(k)} f_a - \sum_{a \in A(k)} f_a = 0 \qquad \forall\, k \in M \tag{46}$$

$$f_a \geq 0 \qquad \forall\, a \in A \tag{47}$$

$$S_k \geq 0 \qquad \forall\, k \in M \tag{48}$$

where μ_k is the Lagrange multiplier associated with the flow conservation constraints, denoting the minimum unit travel cost between Origin k and Destination n. Because the optimality conditions (Equations 42 through 48) are identical to user equilibrium conditions (*16*), the control problem (Equation 11) is proven to be a proper dynamic extension of Beckmann's equivalent optimization problem for a static user equilibrium traffic assignment with elastic demand.

CONCLUSION

We have shown that an equivalent continuous time optimal control problem can be formulated to model the problem of dynamic user-optimized traffic assignment with elastic travel demand. The optimality conditions were derived and given economic interpretations. We have also shown that the model presented in this paper is, under steady-state assumptions, a proper dynamic extension of Beckmann's equivalent optimization problem for static user-optimized traffic assignment with elastic demand.

The dynamic model encounters several limitations. First, the instantaneous costs are used as a criterion for departure time and route choices. A more realistic model should use the anticipated costs as the approximations of the actually experienced costs. Second, the instantaneous travel demand is determined as a function of the instantaneous perceived unit cost between the associated origin-destination pair. This assumption implies that no cross elasticity of demand is considered in our dynamic model. Third, the choice of departure time and route is not based on the trade-off between travel time and schedule delay. It means that our model cannot explicitly handle schedule delay as a penalty for early or late arrival at the destination. Fourth, the state equation (Equation 2) is not empirically tested to answer whether it provides an adequate representation of reality. No consideration of cross-link interactions further simplifies the dynamics of traffic flow on each arc. Finally, a functional form of the instantaneous demand function is not specified in this paper. The question arises as to whether time-varying elasticity of demand can be well represented in its functional form.

Our future research includes the following important issues. First, we should be able to test the optimality of singular controls using the generalized convexity condition, and we need to investigate transition process from a singular control to a nonsingular control. Second, our dynamic model should be extended to analyze traffic flows in a congested network with multiple origins and multiple destinations. However, as discussed by Wie et al. (*3*) and Wie (*17*), there are some difficulties in generalizing the present model to a multiple destination case. Its generalization requires the linear exit function assumptions to keep the property of separability, which is crucial to show equilibration of instantaneous unit path costs. Last, an efficient algorithm must be developed for the computation of our dynamic model. Recently, a gradient algorithm based on the discrete maximum principle was developed by Wie (*18,19*) to solve the problem of dynamic user-optimized traffic assignment with fixed demand. It is hoped that the algorithm can be modified to solve the problem with elastic demand.

REFERENCES

1. T. L. Friesz. Transportation Network Equilibrium, Design and Aggregation: Key Developments and Research Opportunities. *Transportation Research*, Vol. 19A, 1985, pp. 413–427.
2. A. S. Alfa. A Review of Models for the Temporal Distribution of Peak Traffic Demand. *Transportation Research*, Vol. 20B, 1986, pp. 491–499.
3. B. W. Wie, T. L. Friesz, and R. L. Tobin. Dynamic User Optimal Traffic Assignment on Congested Multidestination Networks. *Transportation Research*, Vol. 24B, No. 6, 1990, pp. 431–442.
4. M. Beckmann, C. McGuire, and C. Winston. *Studies in the Economics of Transportation*. Yale University Press, New Haven, Conn., 1956.
5. T. L. Friesz, F. J. Luque, R. L. Tobin, and B. W. Wie. Dynamic Network Traffic Assignment Considered as a Continuous Time Optimal Control Problem. *Operations Research*, Vol. 37, No. 6, 1989, pp. 893–901.
6. B. W. Wie. An Application of Optimal Control Theory to Dynamic User Equilibrium Traffic Assignment. In *Transportation Research Record 1251*, TRB, National Research Council, Washington, D.C., 1989, pp. 66–73.
7. C. Hendrickson and G. Kocur. Schedule Delay and Departure Time Decisions in a Deterministic Model. *Transportation Science*, Vol. 15, 1981, pp. 62–77.
8. A. de Palma, M. Ben-Akiva, C. Lefevre, and N. Litinas. Stochastic Equilibrium Model of Peak Period Traffic Congestion. *Transportation Science*, Vol. 17, 1983, pp. 430–453.
9. H. S. Mahmassani and R. Herman. Dynamic User Equilibrium Departure Time and Route Choice on Idealized Traffic Arterials. *Transportation Science*, Vol. 18, 1984, pp. 362–384.
10. M. Ben-Akiva, A. de Palma, and P. Kanaroglou. Dynamic Model of Peak Period Traffic Congestion with Elastic Arrival Rates. *Transportation Science*, Vol. 20, 1986, pp. 164–181.
11. G. F. Newell. The Morning Commute for Nonidentical Travelers. *Transportation Science*, Vol. 21, 1987, pp. 74–82.
12. R. Arnott, A. de Palma, and R. Lindsey. Schedule Delay and Departure Time Decisions with Hetereogeneous Commuters. In *Transportation Research Record 1197*, TRB, National Research Council, Washington, D.C., 1989, pp. 56–67.
13. L. S. Pontryagin, V. A. Boltyanski, R. V. Gankrelidge, and E. F. Mishenko. *The Mathematical Theory of Optimal Process*. John Wiley and Sons, New York, 1962.
14. A. E. Bryson and Y.-C. Ho. *Applied Optimal Control* (revised version). John Wiley and Sons, New York, 1975.

15. J. G. Wardrop. Some Theoretical Aspects of Road Traffic Research. *Proc., Institution of Civil Engineers*, Vol. 2, No. 1, 1952, pp. 235–378.
16. Y. Sheffi. *Urban Transportation Networks: Equilibrium Analysis with Mathematical Programming Methods*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1985.
17. B. W. Wie. *Dynamic Models of Network Traffic Assignment: A Control Theoretic Approach*, Ph.D. dissertation. University of Pennsylvania, 1988.
18. B. W. Wie. *A Gradient Algorithm for Dynamic User Optimized Traffic Assignment*. Working paper. University of Hawaii, Honolulu, 1990.
19. B. W. Wie. *A Comparison of Dynamic User-Optimum vs System-Optimum Traffic Assignment*. Working paper. University of Hawaii, Honolulu, 1990.

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