Limit Equilibrium Stability Analyses for Reinforced Slopes

Stephen G. Wright and J. M. Duncan

Limit equilibrium slope stability analysis procedures have been successfully adapted and used for analyses of reinforced slopes. A potential source of inaccuracy in these analyses is the limit equilibrium procedure used, specifically the assumptions that are made to satisfy static equilibrium and the equilibrium conditions that are satisfied. A second possible source of inaccuracy is related to the manner in which the reinforcement forces are assumed to be distributed in the soil mass and the direction in which they are assumed to act. Stability computations have been performed using the logarithmic spiral, Bishop simplified, Spencer's, and force equilibrium procedures to evaluate the magnitudes of these inaccuracies. Methods that satisfy complete static equilibrium (logarithmic spiral and Spencer's) were found to result in essentially the same values for the factor of safety. Bishop's simplified procedure also produces very nearly the same values for the factor of safety, although the procedure does not satisfy complete static equilibrium. Force equilibrium procedures produce factors of safety that are sensitive to assumptions made about the inclination of side forces between slices.

Geotextiles, geogrids, and steel reinforcing elements are being used with increasing frequency to reinforce slopes and embankments, making possible construction of steeper slopes and higher embankments. Design of these slopes and embankments is usually based on equilibrium stability analyses. The reinforcing elements are represented in the analyses as stabilizing forces of known magnitude, and the necessary amount and distribution of reinforcement within the slope is determined by trial and error.

Although equilibrium analyses have been used successfully for designing many slopes and for developing design charts for reinforced slopes and embankments, two possible sources of potential inaccuracy are not well understood:

1. How are the factors of safety for reinforced slopes affected by whether or not the method satisfies all conditions of equilibrium and by the assumptions involved in the method? For example, Bishop's simplified procedure does not satisfy horizontal equilibrium, and it assumes that there are no vertical side forces. Are the factors of safety calculated using this method accurate or not?
2. How are the factors of safety for reinforced slopes affected by the way in which the reinforcement forces are represented in the analyses? Reinforcement forces have been represented as concentrated forces on the base of single slices, as concentrated forces at the face of the slope, and as forces distributed within the reinforced soil mass. Do these methods lead to the same value of safety factor or to different values; if different values, which is more correct?

This paper answers these questions through comparative analyses of reinforced slopes and embankments using the logarithmic spiral procedure of analysis, Bishop's simplified procedure, Spencer's procedure, and a force equilibrium procedure of slope stability analysis.

ADAPTATION OF LIMIT EQUILIBRIUM PROCEDURES

All limit equilibrium analysis procedures are based on solving one or more static equilibrium equations for one or more unknowns, including the factor of safety, or F. The factor of safety is defined with respect to soil shear strength as

\[
F = \frac{\text{available shear strength}}{\text{shear strength required for equilibrium}}
\]

The number of unknowns must be equal to the number of equations to achieve a statically determinate solution. Various assumptions regarding the unknowns are made to achieve a balance between equations and unknowns. The number of equilibrium conditions satisfied also varies, depending on which limit equilibrium procedure is used.

When reinforcement is introduced into an analysis, it presents little in the way of additional complexity. Because the reinforcement force is considered to be known and is prescribed for purposes of the analysis, it introduces no additional unknowns. The reinforcement forces are additional known forces that are included in the appropriate equilibrium equations.

Four limit equilibrium procedures are considered in this paper. The first is the logarithmic spiral procedure. This procedure satisfies all conditions of equilibrium for a free-body consisting of the soil mass bounded by the shear surface and the surface of the slope. The three remaining procedures are procedures of slices: the Bishop's simplified (1), Spencer's (2), and force equilibrium procedures. The force equilibrium procedure satisfies equilibrium of forces only; equilibrium of moments is not considered.

Logarithmic Spiral Procedure

The logarithmic spiral procedure assumes that the shear surface is a logarithmic spiral. The spiral is defined by its center point and a radius (r) of the form
\[ r = r_o \exp(\Theta \tan \Phi_m) \]

where

- \( r_o \) = radius to some prescribed reference point (often the toe of the slope),
- \( \Theta \) = angle between the reference radius and the radius \( r \) at some other point on the spiral, and
- \( \Phi_m \) = "mobilized" friction angle (\( \Phi_m = \tan(\Phi/F) \)).

Summation of moments about the center of the spiral results in an equilibrium equation that involves only one unknown, \( \Phi_m \) (that is, the factor of safety). The moment equilibrium equation is solved for the factor of safety.

When reinforcement is included in an analysis, the moments about the center of the spiral include moments due to the reinforcement as well as moments due to the weight of the soil mass and any pore water pressures or cohesive component of shear strength. The moments due to these forces can be computed from the known conditions and an assumed factor of safety. A trial-and-error procedure is used to determine the factor of safety that produces static equilibrium.

Unlike the procedures of slices, the logarithmic spiral procedure requires no assumptions about the internal forces. Accordingly, it provides a useful independent check of the validity of the procedures of slices.

**Bishop's Simplified Procedure**

The Bishop's simplified procedure is based on equilibrium of moments about the center of a circular shear surface and equilibrium of forces in the vertical direction. Equilibrium of moments is considered only for the entire free-body composed of all slices; vertical force equilibrium is considered for individual slices. Horizontal reinforcement forces contribute to the equation of moment equilibrium but do not directly enter or affect the equilibrium equation for forces in the vertical direction. Inclined reinforcement forces affect equilibrium of moments and equilibrium of forces in the vertical direction. In some cases the effect of inclined reinforcement on vertical force equilibrium has been ignored (3,4). For the analyses in this paper, the contribution of inclined reinforcement to both moments and vertical force equilibrium has been accounted for.

**Spencer's Procedure**

Spencer's procedure assumes that the side forces are parallel, that is, that all side forces have the same inclination. Complete static equilibrium of moments and forces is satisfied. The solution involves evaluating the factor of safety, the inclination of the side forces, and the other unknown forces and their locations. Reinforcement forces contribute to the moments on slices and to the forces in the vertical and horizontal directions.

**Force Equilibrium Procedure**

Several force equilibrium procedures differ with regard to the assumption made concerning the inclination of forces between slices. The force equilibrium procedure chosen for the present analyses is based on the assumption that the side forces are horizontal, that there is no shear between slices. This assumption is sometimes referred to as the "simplified Janbu" assumption; it was chosen because of its simplicity and because it has been found to give relatively low, conservative values for the factor of safety.

**EXAMPLE PROBLEMS**

Two example problems are considered to illustrate the differences among the various procedures. The first example consists of a cohesionless fill slope constructed on a firm foundation; any potential for failure is assumed to be restricted to the slope itself. The second example consists of a cohesionless fill slope constructed on a very weak foundation where the foundation governs the stability and necessitates reinforcement. These examples represent two of the most common slope conditions in which reinforcement is likely to be employed.

**Example 1**

The first example slope is a 1:1 (45-degree) slope, 38 ft high, as shown in Figure 1. The soil is cohesionless and has an angle of internal friction of 32 degrees and a total unit weight of 120 psf. The slope contains 17 layers of reinforcement, varying from 23.9 to 29.2 ft long and spaced vertically as shown in Figure 1.

This example was taken from Tensar (5). The original example presented by Tensar had a surcharge of 240 psf, equivalent to about 2 ft of additional slope height, which was neglected for the current analyses.

Each layer of reinforcement has an axial force of 1,000 lb. Although the force would actually decrease to zero near the embedded ends of the reinforcement, the force was assumed to be constant along the entire length of the reinforcement for the current stability computations.

The computed factors of safety for the first example slope are summarized in Table 1. Factors of safety were computed for the reinforcement force acting in two directions. In the first case the reinforcement force acted horizontally; in the second case it was assumed that the reinforcement had rotated such that the force acted tangentially to the shear surface.

The critical shear surface producing the minimum factor of safety was located for each case and for each of the limit

![FIGURE 1 Example slope 1.](image-url)
equilibrium analysis procedures. Although Spencer’s procedure and the force equilibrium procedure can accommodate noncircular shear surfaces, only circular shear surfaces were considered to permit comparisons to be made with the Bishop’s simplified procedure and the logarithmic spiral procedure. Experience has shown that the most critical logarithmic spiral shear surface is very similar to the most critical circle and, thus, circles and logarithmic spirals can be considered comparable shapes.

The factors of safety computed by the four limit equilibrium procedures are summarized in Table 1. The three procedures that satisfy moment equilibrium (Bishop simplified, Spencer’s and logarithmic spiral) produced factors of safety that agree within approximately 2 percent.

The force equilibrium procedure produced factors of safety that were approximately 10 percent lower than the values that the others produced. These lower values are consistent with what the authors have often found using the force equilibrium procedure for unreinforced slopes; they indicate the conservative nature of the procedure in this instance.

The differences between the factors of safety computed for horizontal reinforcement forces as compared to forces tangent to the shear surface were in all cases negligible. This observation is contrary to results of Low and Duncan (6), who found that the factor of safety with the reinforcement force tangent to the shear surface was always larger than the factor of safety with the reinforcement force horizontal. However, the analyses performed by Low and Duncan did not include the vertical component of the reinforcement force in the equations of vertical equilibrium, and this is believed to be the cause of the differences in the results shown in Table 1 and the findings of Low and Duncan.

An inclined force tends to produce a larger moment than a horizontal force, but it has a smaller contribution to the normal forces (and shear strength) along the shear surface. The two effects tend to compensate for each other and the net effect is thus small.

The remarkably close similarity between the factors of safety computed by the Bishop’s simplified, Spencer’s, and logarithmic spiral procedures was unexpected. In particular, the close agreement between the factors of safety computed by procedures satisfying complete equilibrium and the Bishop’s simplified procedure was surprising, because the Bishop’s simplified procedure ignores forces in the horizontal direction, which is the direction of all or much of the reinforcement force, depending on its inclination.

The close agreement between the Bishop’s simplified and complete equilibrium procedures apparently stems from the fact that moment equilibrium is considered and force equilibrium is satisfied in at least one direction by all three methods. This can be seen by first considering the illustration in Figure 2. Equilibrium of moments involves the known moments due to the weight of the soil and the reinforcement force. The only other moment is produced by the shear stresses along the shear surface. Because the weight and reinforcement produce a given, known moment, the average shear stress along the shear surface is fixed, regardless of any assumptions made in the procedures of slices that satisfy moment equilibrium. Now considering the equilibrium of forces in the vertical direction and the illustration in Figure 3, equilibrium of forces in the vertical direction involves the weight (W), the shear stresses (τ), and the normal stresses (σ) along the shear surface. The weight is fixed and, as discussed earlier, the average shear stress is fixed by the requirement of moment equilibrium. The normal stress is the only remaining quantity contributing to vertical equilibrium. Accordingly, the normal stress is also, for practical purposes, fixed—at least in an average sense. The shear and normal stresses may vary in different ways along the shear surface, depending on the specific limit equilibrium procedures used; however, if the procedures satisfy moment equilibrium and, at least in the vertical direction, equilibrium of forces, the average shear and the average normal stresses must be about the same. For this reason the factor of safety, which depends on the stresses along the shear surface, is apparently almost independent of the assumptions made in limit equilibrium procedures, such as Spencer’s and the Bishop’s simplified, that satisfy moment equilibrium and force equilibrium in at least one direction. Only the force equilibrium procedure, which does not satisfy moment equilibrium, produces results that are not in agreement with the other procedures.

**Example 2**

The second example consists of a 10-ft-high cohesionless fill resting on a 10-ft layer of saturated (Φ = 0) clay, as shown...
in Figure 4. Much stronger soils are assumed to exist below the clay. The fill has an angle of internal friction ($\phi$) of 35 degrees and a total unit weight of 105 pcf. The clay has a uniform undrained shear strength of 200 psf.

One layer of reinforcement is placed at the base of the fill on the surface of the clay. The reinforcement carries a constant force of 3,000 lb/linear-ft of slope. The factor of safety of the unreinforced slope is approximately 1.1; reinforcement was selected to increase the value to an acceptable level.

Two slightly different locations of the reinforcement were examined. First, the reinforcement shown in Figure 4 is considered to be located just above the clay so that it crosses the shear surface in the sand. The reinforcing force is thus applied to a slice with an entirely frictional strength, where normal stresses and the contribution of the reinforcement to the normal stresses would be expected to have the greatest influence. Second, the reinforcement shown in Figure 4 is considered to be located just below the top surface of the clay. In this case the reinforcement crosses the shear surface in the clay. The reinforcement force is thus applied to a slice with a purely cohesive strength, where normal stresses on the shear surface have no affect on the shear strength and computed factor of safety. The two positions for the reinforcement were selected because they were expected to illustrate the maximum effect of treating the reinforcement force as a concentrated force.

In one case, in which the reinforcement is in the clay, the concentrated force is applied to a slice where the strength is independent of the normal stress on the shear surface; in the other case the force is applied to a slice where the shear strength is directly proportional to the normal forces on the shear surface.

The factors of safety for the second example slope were computed by the Bishop simplified, Spencer’s, and force equilibrium procedures for horizontal reinforcement forces and forces tangent to the shear surface. The logarithmic spiral is, for practical purposes, restricted to homogeneous slopes ($\phi = \text{constant}$), so it could not be used for this second example.

The factors of safety for horizontal reinforcement forces are summarized in Table 2. It can again be seen that the computed factors of safety differ very little. The extreme range in the values computed was less than 2 percent. The force equilibrium procedure produced factors of safety almost identical to the ones computed by the other procedures for this case, because the side force inclination satisfying moment equilibrium, 2 to 3 degrees, is very close to horizontal. Thus, the force equilibrium procedure, which assumed horizontal side forces, gave results in close agreement with Spencer’s procedure, which determined as part of the solution that the side forces are close to horizontal for this case.

The factors of safety for reinforcement forces that are tangent to the shear surface are summarized in Table 3. The values for the reinforcement located in the sand agree closely with those computed for horizontal reinforcement, shown in Table 2. This reflects the compensating effects of the inclined reinforcement forces on the moments and the normal stresses (shear strength) in the sand, similar to what was shown for Example 1.

The factors of safety for the reinforcement in the clay are significantly higher when the reinforcement forces are tangent to the shear surface. This indicates that the reinforcement has approximately twice the stabilizing effect; the factor of safety increased from approximately 1.1 to 1.57 with the reinforcement forces tangent to the shear surface as compared with an increase from 1.1 to 1.35 for horizontal reinforcement. The higher factors of safety are due to the added effect of the reinforcement on the moment, when the forces are tangent to the shear surface rather than horizontal. The reinforcement forces had no effect on the shear strength when the reinforcement was assumed to be located in the clay because $\phi$ was equal to zero.

The effects of the inclination of the reinforcement forces can be significant when $\phi = 0$ because compensating effects on strength and moment do not exist. However, it seems unlikely that such a case could exist in practice. There is no practical way to install the reinforcement below the surface of the saturated clay. Reinforcement is usually placed on the surface of a sand or gravel pad on top of the clay and is thus embedded in cohesionless material.

### Table 2 Comparison of Factors of Safety Computed for Example 2 with Reinforcement Forces Horizontal

<table>
<thead>
<tr>
<th>Analysis Procedure</th>
<th>Reinforcement in Sand</th>
<th>Reinforcement in Clay</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spencer</td>
<td>1.37</td>
<td>1.36</td>
</tr>
<tr>
<td>Simplified Bishop</td>
<td>1.36</td>
<td>1.37</td>
</tr>
<tr>
<td>Force Equilibrium</td>
<td>1.35</td>
<td>1.35</td>
</tr>
</tbody>
</table>

### Table 3 Comparison of Factors of Safety Computed for Example 2 with Reinforcement Forces Tangent to Shear Surface

<table>
<thead>
<tr>
<th>Analysis Procedure</th>
<th>Reinforcement in Sand</th>
<th>Reinforcement in Clay</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spencer</td>
<td>1.36</td>
<td>1.57</td>
</tr>
<tr>
<td>Simplified Bishop</td>
<td>1.36</td>
<td>1.58</td>
</tr>
<tr>
<td>Force Equilibrium</td>
<td>1.33</td>
<td>1.52</td>
</tr>
</tbody>
</table>

**INTERNAL DISTRIBUTION OF FORCES**

All the stability computations and procedures discussed up to this point have been based on the assumption that each layer of reinforcement can be represented by a concentrated force applied to the base of the slice. Although the reinforcement also creates internal forces between slices, the internal forces were not considered directly. In procedures such as Spencer’s and force equilibrium, the side forces, which are evaluated as unknowns, include the forces transmitted through the soil as well as the force in the reinforcement where it crosses the slice boundary. The assumptions that are made and counted on in the limit equilibrium procedures to properly handle the
forces between slices must also be counted on to properly handle the reinforcement forces between slices.

In reality, the forces in reinforcement are developed and transferred to the soil along the length of the reinforcement. The forces that are actually transferred to individual slices represent the difference between the force at the left and right sides where the reinforcement crosses the slice. Depending on how the reinforcement forces are developed and transferred to the soil, the forces on each slide will vary in magnitude and direction. For example, if the forces gradually increase and decrease along the length of the reinforcement as shown in Figure 5 (top), they will be distributed to individual slices as represented by the net forces on each slice in Figure 5 (bottom).

A realistic way of applying the forces to slices in a limit equilibrium analysis is, thus, to apply the net reinforcement force to each slice. This is done by applying the reinforcement force not only where the reinforcement crosses the shear surface, but also where it crosses the boundaries between slices. Such a set of applied forces is illustrated schematically in Figure 6. The internal reinforcement forces between slices have no net effect on the overall equilibrium of the soil mass; equal and opposite forces on each internal slice boundary of adjacent slices simply cancel. However, the internal reinforcement forces (the differences in reinforcement force on the left and right sides of each slice) do have an effect on the distribution of forces among individual slices, including the distribution of shear and normal stresses along the shear surface.

To examine the effect of the way in which reinforcement forces are applied to the soil mass, another series of stability computations was performed. In these analyses the reinforcement force was considered to be constant along the length of the reinforcement, and internal as well as external forces were applied. The consequence of assuming a constant force was to make the net reinforcement force zero wherever the reinforcement crosses two boundaries of a slice (including the shear surface). The net force is equal to the reinforcement force at the point where the reinforcement terminates. Each layer of reinforcement that is intersected by the shear surface causes a single force to be applied to the face of the slope where the reinforcement is terminated (Layers 1, 2, and 3 in Figure 7).

Reinforcement that does not cross the shear surface and is contained entirely within the soil mass causes two forces of equal magnitude but opposite direction to be applied at the two points at which the reinforcement is terminated (Layers 4, 5, and 6 in Figure 7).

The assumption of a constant force in the reinforcement for the current computations is not considered to be realistic.
TABLE 4 COMPARISON OF FACTORS OF SAFETY COMPUTED FOR EXAMPLE 1 WITH REINFORCEMENT FORCES APPLIED TO SHEAR SURFACE ONLY AND BOTH TO SHEAR SURFACE AND INTERNALLY

<table>
<thead>
<tr>
<th>Analysis Procedure</th>
<th>Forces Applied to Shear Surface Only</th>
<th>Forces Applied Internally and at Shear Surface</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spencer</td>
<td>1.34</td>
<td>1.34</td>
</tr>
<tr>
<td>Simplified Bishop</td>
<td>1.45</td>
<td>1.45</td>
</tr>
<tr>
<td>Force Equilibrium</td>
<td>1.30</td>
<td>1.28</td>
</tr>
</tbody>
</table>

However, it provides an extreme case for comparison with the case in which the reinforcement forces were applied as concentrated forces at the shear surface, as described earlier. Factors of safety were computed for both of the example slopes. The values are shown in Tables 4 and 5. Values are shown for the reinforcement forces distributed internally as well as for the reinforcement force applied as a concentrated force at the shear surface. It can be seen from the results in these two tables that the manner in which the reinforcement force is applied has virtually no effect on the computed factor of safety for either slope. Had the reinforcement forces been assumed to vary along the length of the reinforcement, the differences between the factors of safety computed with the concentrated force applied to the shear surface and those computed as described earlier in this section would have been even smaller.

CONCLUSIONS

The analyses described in the preceding sections provide a basis for some important conclusions concerning the use of equilibrium methods for analysis of the stability of reinforced slopes:

1. Methods that satisfy all conditions of equilibrium result in essentially the same value of factor of safety regardless of the assumptions they may involve. This conclusion is illustrated by the close similarity between the values of factor of safety calculated using the logarithmic spiral procedure and Spencer's procedure of analysis.

2. Bishop's simplified method, although it does not satisfy all conditions of equilibrium, results in values of safety factor that are essentially the same as values calculated using methods that do satisfy all conditions of equilibrium. This is true for steep reinforced slopes on firm foundations and for embankments on weak foundations. The key to its accuracy appears to be the fact that it satisfies moment equilibrium.

3. Factors of safety calculated using force equilibrium procedures (which do not satisfy moment equilibrium and which require assumption of the inclination of the side force inclination) result in factors of safety that are sensitive to the assumed orientation of the side forces. If the assumed inclination is close to the one that satisfies moment equilibrium, the computed factor of safety will be different also. Calculating accurate values of factor of safety using force equilibrium methods is thus largely a matter of choosing the correct side force inclination. This is unnecessary if a method of analysis is used that satisfies all conditions of equilibrium. Bishop's procedure is such a method; it applies to either circular or noncircular (wedge-type) failure surfaces. Alternatively, if only circular shear surfaces are to be analyzed, Bishop's simplified method can be used with equally accurate results.

4. The manner in which the reinforcement force is distributed along the length of the reinforcement has no significant effect on the calculated values of factor of safety. Whether the force is applied as a concentrated force acting at the base of the slice through which it crosses the slip surface or as a concentrated force at the surface of the slope where it terminates, the calculated factor of safety is for all practical purposes the same. These two points of application represent the extremes possible, and it can thus be inferred that any other reasonable method of representing the reinforcement force will also lead to the same value of safety factor.

5. Whether the reinforcement force is applied horizontally or parallel to the shear surface, the calculated factor of safety is essentially the same, provided that the method of analysis includes the horizontal and vertical components of the reinforcement force in the equations of force equilibrium as well as the equation (or equations) of moment equilibrium and provided that $\Phi$ is not equal to zero. Reinforcement would almost always be located in a material in which $\Phi$ is not zero, so the effects of inclination can usually be ignored.

6. In Bishop's simplified procedure, which satisfies vertical force equilibrium, the vertical components of the reinforcement forces should be included in the equations of vertical equilibrium. In Spencer's procedure, which satisfies both horizontal and vertical equilibrium, both horizontal and vertical reinforcement forces should be included in the force equilibrium equations.

7. The studies described in this paper are concerned with factors of safety with respect to sliding along curved surfaces (log spirals or circular arcs). When the frictional resistance to sliding between the reinforcement and the soil is smaller than the shearing resistance for sliding through the soil mass, slip surfaces with a planar portion that coincides with the soil and reinforcement interface may be more critical than continuously curved surfaces, and they should be studied.

REFERENCES


Publication of this paper sponsored by Committee on Mechanics of Earth Masses and Layered Systems.