Welfare Comparison of Fixed- and Flexible-Route Bus Systems

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Analytic models are used to conduct a comparison under equilibrium demand conditions of welfares for fixed-route conventional bus and flexible-route subscription bus systems for providing feeder services. Optimization models are formulated to maximize welfare for the two feeder bus systems, subject to a break-even constraint. Service zone size, headway, and fare are the decision variables in these analyses. For break-even operation it is shown that the equilibrium demands for the two systems are different due to their specific service attributes and that the optimized fare for flexible-route systems is generally higher than for fixed-route systems. The differences in welfare and fares between the two systems tend to decrease as line-haul distance increases and as service area decreases. The flexible-route bus system is generally favored in cases with lower demand densities, larger service areas, and higher local travel speeds.

Various public transportation modes have their own operating characteristics and thus provide different service qualities. Decision makers face the problem of selecting the best service option for a given environment. Therefore it is desirable to compare the options to determine under what conditions each of these systems is preferable. Full cost comparisons of various public transit systems have been conducted by many researchers and in different ways (*1–16*). The general critique for these studies is that they all assume a fixed demand (i.e., demand is perfectly inelastic or insensitive to service quality). This paper attempts to compare fixed- and flexible-route paratransit systems with elastic demand assuming that the bus systems are optimized for the maximum welfare objective, subject to a break-even constraint.

Analytic models have been developed and used in comparing fixed-route conventional bus and flexible-route subscription bus systems (*16*). It was recognized that different demand levels may be generated for service attributes of different systems. A method for comparing the two systems when their service levels generate different passenger volumes was presented in that study. However, that proposed method was still based on the results obtained for the perfectly inelastic demand conditions, and fare was not considered in that analysis.

In this paper, an analytic approach is used to compare the fixed- and flexible-route bus systems under their break-even conditions. The route structures and system characteristics used here are substantially similar to those used by Chang and Schonfeld (*16*), except that elastic demand is considered in this paper. Thus, optimization models with demand elasticity are needed for the comparison. Analytic results for the decision variables (e.g., headway, route spacing, fare) at break-even conditions have been obtained for the fixed-route system (*17*). Therefore, these results are directly used in the comparison. For the flexible-route system, however, since analytic results are difficult to obtain, an algorithm is developed to incorporate the analytic results found for inelastic demand conditions and obtain the equilibrium results for the break-even operation.

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BUS SYSTEM CHARACTERISTICS

Figure 1 shows the service areas and their specific route structures for the two feeder systems. The variables and the typical values used in the numerical analyses are defined in Table 1. Basically, the bus systems with either fixed routes or flexible routes are assumed to connect a rectangular area of length L and width W to a major generator (e.g., a transportation terminal or an activity center) that is J km away from that area. Analytic optimization models for these two feeder systems developed in earlier work (*16,17*) are applied. These models provide optimized solutions in closed form with perfectly inelastic (fixed) demand, whereas in this paper the two bus systems are designed to operate at break-even and unequal equilibrium demands because of their different service attributes. Route structures and operating characteristics for the two systems are briefly described as follows.

Fixed-Route System

For fixed route systems, the service area is divided into N zones with a route spacing $r = W/N$, as shown in Figure 1a. A vehicle round-trip consists of (*a*) a line-haul distance J traveled at express speed yV from the major terminal to the service area; (*b*) a delivery route L km long traveled at local speed V along the centerline of the zone, stopping for passengers every s km, with an average delay of hours for each stop; and (*c*) reversal of the previous two phases to collect passengers and carry them to the terminal.

Flexible-Route System

The route structure for the flexible-route subscription service is shown in Figure 1b. The service area is divided into N equal zones, each of which has an area $A = LW/N$. This service zone structure is more flexible than that for fixed-route service. Basically, feeder buses travel from the terminal a line-

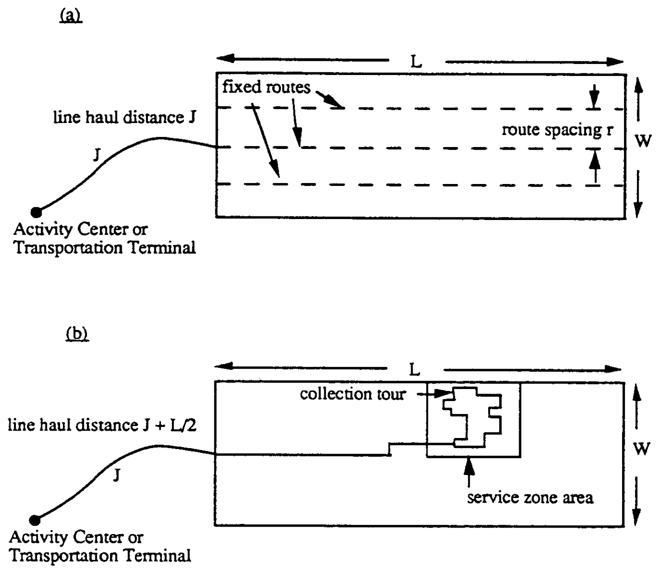


FIGURE 1 **(*a*) Fixed- and (*b*) flexible-route feeder bus systems.**

haul distance J and an average distance $L/2$ km at express speed yV to the center of each zone. They collect passengers at their doorsteps through a tour of n stops and length D_c at local speed V. The values of n and D_c are determined using Stein's formula (*18,19*). To return to their starting point, the buses retrace an average of $L/2$ plus J km at yV km/hr. It is assumed that buses operate on preset schedules with variable routing designed to minimize the tour distance D_c, while the tours are routed on a rectangular grid street network. Tour departure headways are assumed to be equal for all zones in the service area. For both service types the average wait time equals a constant factor z_w times the headway h. As in the fixed-route services, vehicle layover time and external costs of bus services are assumed to be negligible.

On the basis of the assumptions that n points are randomly and independently dispersed over an area A and that an optimal traveling salesman tour has been designed to cover these n points, the collection distance D_c in an optimized zone may be approximated by the following result of Stein (*18,19*):

$$D_c = \phi(nA)^{1/2} \tag{1}$$

TABLE 1 Variable Definitions

Symbol	Definition	Baseline value
a	ratio of wait time and headway for flexible-route bus	0.5
A	service zone area (sq. km) = LW/N'	-
B	bus operating cost ($/veh hr)	40.0
C_o	total operator cost ($/hr)	-
D	equivalent avg. round trip distance for fixed-route bus (km) = 2J/Y + 2L/Y	
D_c	distance of one collection tour for flexible-route bus (km)	-
D_L	equivalent line haul distance for flexible-route bus (km) = (L+W)/z+2J/y	
e_p	demand elasticity parameter for fare	0.07
e_v	demand elasticity parameter for in-vehicle time	0.35
e_w	demand elasticity parameter for wait time	0.7
e_x	demand elasticity parameter for access time	0.7
f	fare ($/trip)	-
F	fleet size (vehicles)	-
g	avg. access speed (km/hr)	4.0
G	consumer surplus ($/hr)	-
h	headway (hrs/veh)	-
J	line haul distance (km)	12.8
k	constant in the demand function	-
L	length of service area (km)	6.4
M	avg. in-vehicle travel time (hr)	-
n	number of pickup points in one collection tour	-
N	number of zones	-
q	potential demand density (trips/sq. km/hr)	39.0
Q	demand density function	-
r	route spacing (km) =L/N	-
R	revenue ($/hr)	-
s	bus stop spacing (km)	0.4
u	avg. number of passengers per pickup point	1.2
V	local service speed (km/hr); fixed-route bus=32, flexible-route bus=28	
v	value of in-vehicle time ($/passenger hr)	5.0
w	value of wait time at bus stop ($/passenger hr)	10.0
W	width of service area (km)	4.8
x	value of access time ($/passenger hr)	10.0
y	express speed/local speed ratio	2.0
Y	social welfare ($/hr)	-
z_w	ratio of wait time and headway for fixed-route bus	0.5
z_x	geometric factor for access distance	0.25
ϕ	circuit factor in collection tour	1.15

In Equation 1, ϕ can be considered as the circuit factor and has been estimated to be 0.765 for a Euclidean metric (*18,19*). With a simple strategy to formulate a good traveling salesman tour in zones of irregular shapes, Daganzo (*20*) has also shown that the value of ϕ can be approximated as 0.9 for a Euclidean metric and as 1.15 for a grid network. For a grid network, this circuit factor, which is 1.15, can be directly derived from the value 0.9 for Euclidean metric by an adjusted factor, which reflects the geometric structure of the street network (*20*). Larson and Odoni (*21*) have also discussed applications of Equation 1.

The demand is also assumed to be deterministic and uniformly distributed over time during each specified period. It is also assumed to be uniformly distributed over space within each specified service area. The demand density can be assumed to be obtained from empirical distributions of demand over time, as analyzed in other related works (*16,17,22*). However, in this paper we simply assume a single period with an average demand density for the analysis.

COMPARISON FOR EQUAL DEMAND

The analytic results for the optimal route structures and service headways for the two bus systems have been derived by Chang and Schonfeld (*16*) for perfectly inelastic demand conditions by minimizing the total cost, which includes user cost and operator cost. The closed-form solutions for route spacing, headway, and service zone can be found in related works (*16,23*), and are shown later in Equations 7 to 10 for the flexible-route bus system.

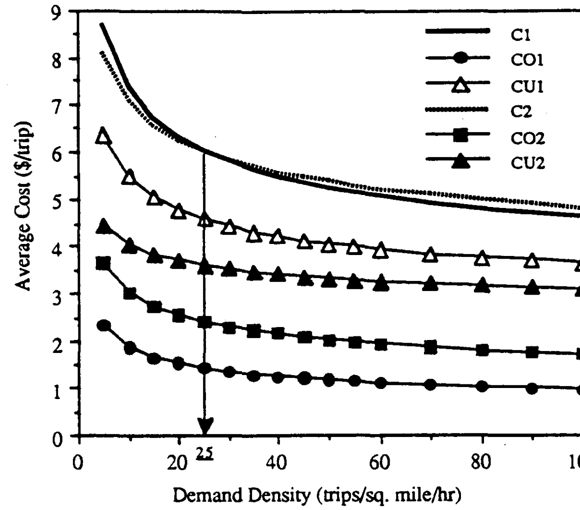
With the analytic average cost functions and the given parameter values for the two systems, we can identify which system is preferable in specific circumstances. For example, the two cost functions in Figure 2 can be used to determine that the flexible-route system is preferable for demand densities below 25 trips per square miles per hour (i.e., 9.8 trips per square kilometer per hour) for the given demand pattern and other assumptions (*16*). Although a comparison for unequal demand has been proposed on the basis of analytic results for inelastic demand conditions, a model with demand elasticity is still needed for comparing route structures, fares, and net social benefits of the two systems at their specific equilibrium demands that might be generated by their different service attributes.

COMPARISON FOR EQUILIBRIUM DEMAND

Objective Function

Various objective functions have been considered appropriate for optimizing bus transit systems (*24*). To compare the two bus systems, maximum social welfare, also known as the net social benefit, is used as the objective function together with a break-even constraint. Denoting G as the consumer surplus, R as the revenue, and C_0 as the operator cost, the break-even problem can be stated as follows:

$$\text{Maximize } Y = G + R - C_0 \qquad \text{subject to } C_0 - R \leq 0$$



Note:
C1 = average cost for fixed-route bus system; = CO1 + CU1
CO1 = average operator cost for fixed-route bus system
CU1 = average user cost for fixed-route bus system
C2 = average cost for flexible-route bus system; = CO2 + CU2
CO2 = average operator cost for flexible-route bus system
CU2 = average user cost for flexible-route bus system

FIGURE 2 System comparison for inelastic demand condition (*16*).

A break-even solution would not exist if the demand function were always below the average operator cost function. That situation would always imply a negative profit. The profitability conditions, which have been evaluated by other studies (*25,26*), are not discussed in this paper. Therefore, it is assumed in the following analysis that the travel demand is sufficient to yield a positive profit in some circumstances for the bus operation considered.

The Lagrange multipliers method is used here for constrained optimization, and the Lagrangian α is formulated as

$$\alpha = G + R - C_0 - \lambda(C_0 - R) \tag{2}$$

where λ is the Lagrange multiplier associated with the break-even constraint. Equation 2 can be rewritten as

$$\alpha = G - (1 + \lambda)(C_0 - R) \tag{2a}$$

which means that solving the problem of maximizing social welfare ($G + R - C_0$) subject to a break-even constraint ($C_0 = R$) is equivalent to solving the problem of maximizing consumer surplus (G) subject to a break-even constraint by defining $1 + \lambda$ as a new Lagrange multiplier.

Linear Demand Function

With a linear demand function in which the demand density is sensitive to various travel time components and fare, analytic results are obtained for fixed-route system under various due to their specific service attributes and that the optimized fare for flexible-route systems is generally higher than for fixed-route systems. Flexible-route bus systems have higher average operator cost (i.e., fare) and lower user costs than fixed-route systems.

The optimality condition that the fare, the average wait cost, and the average access cost are all identical for the fixed-route system at the equilibrium break-even condition does not apply to the flexible-route system, in which the fare (i.e., the average operator cost per trip) is higher than the average wait cost. Sensitivity analyses indicate that the relative advantage of the flexible-route bus system generally increases with lower demand densities, larger service areas, and higher local travel speeds.

In this analysis the two systems are assumed to be mutually exclusive for providing feeder service. Further studies may analyze a system in which both the fixed- and the flexible-route bus services are available and where competition between the two services is allowable. The integration of such systems during various time periods and for different service areas based on their specific characteristics is also worth exploring.

ACKNOWLEDGMENT

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able assumed values from Table 1. The flexible-route results are obtained by the solution procedures developed above, whereas the fixed-route results are obtained directly by the closed-form solutions given in Table 2. In Figure 3 the two welfare functions intersect at a lone-haul distance of 7 km, where the welfare is $4,375/hr. Hence, for the given condition implied by the assumed parameter values, a flexible-route bus system is preferable for line-haul distances below 7 km.

This threshold analysis can be designed for other system parameters, such as value of time, travel speed, and service area. The effect of parameter values on the results of threshold analysis is also worth evaluating. In Figure 4, for example, the effects of potential demand density on the threshold values are shown. Figure 4 shows that potential demand density has little influence on threshold values. The threshold line-haul distances are 7, 5, and 4 km for the potential demand densities 39, 98, and 195 trips/sq. km/hr, respectively. Figure 4 also shows that the welfare functions of the two systems become very similar when the potential demand densities decrease.

The threshold analysis has also been applied to determine which system is preferable for various service areas and travel speeds. Figure 5 shows two welfare functions over a range of service areas. These two functions intersect at a service area of 52.5 km^2, where the welfare is about $6,100/hr. Therefore, given the assumptions implied by the specific parameter values, Figure 5 indicates that the flexible-route bus system is preferable for service areas larger than 52.5 km^2. Since the two welfare functions intersect at such sharp angles, the threshold values of service area are quite sensitive to system parameters. It can also been observed from Figure 4 that the threshold line-haul distances will become more sensitive to system parameters for lower potential demand densities, since the intersection angles tend to sharpen as the potential demand densities become smaller.

Figure 6 shows two welfare functions over a range of local speeds, the bus speed V within the service area. The two functions intersect at a local speed of 33 kph, where the welfare is $4,100/hr. Therefore, given the parameter values and the implied assumptions, we can say that the flexible-route bus systems are preferable for local speeds above 33 kph.

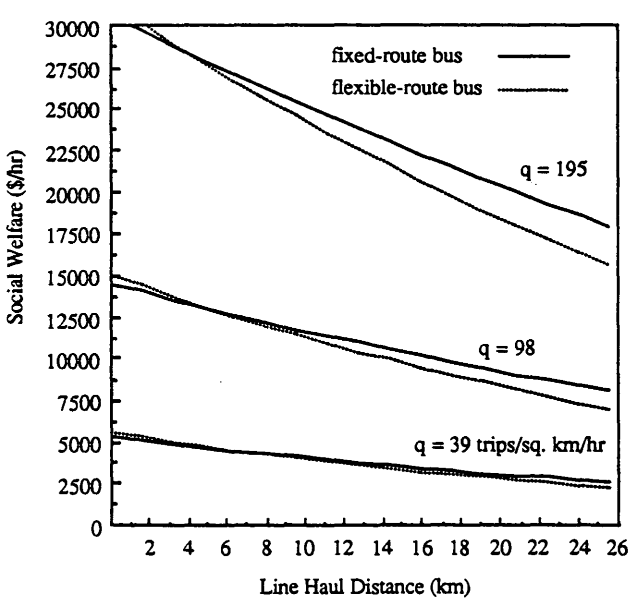

FIGURE 4 Effects of potential demand on threshold line-haul distance.

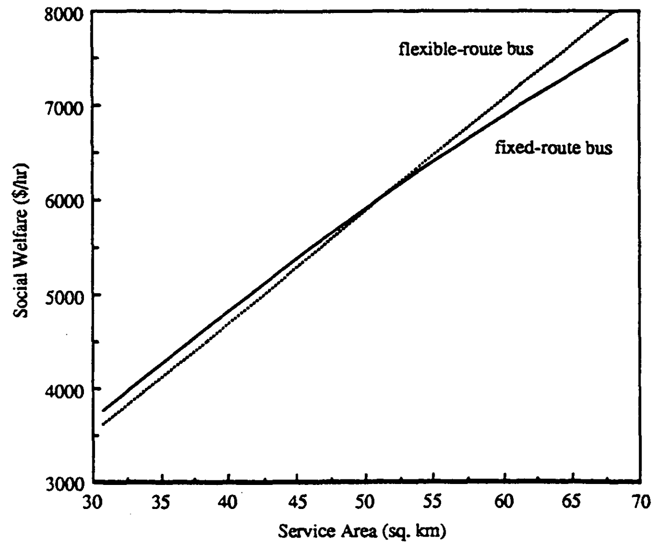

FIGURE 5 Effects of service area on welfare.

CONCLUSION

Welfare relations and results for fixed-route conventional bus and flexible-route subscription bus systems are compared in this paper. Optimization models are formulated for the two feeder bus systems for maximum welfare objective, subject to break-even constraint. The models presented here may be applied in selecting fixed-route or flexible-route bus systems for providing feeder services.

It is shown at the break-even operation that the equilibrium demands and welfare values for the two systems are different

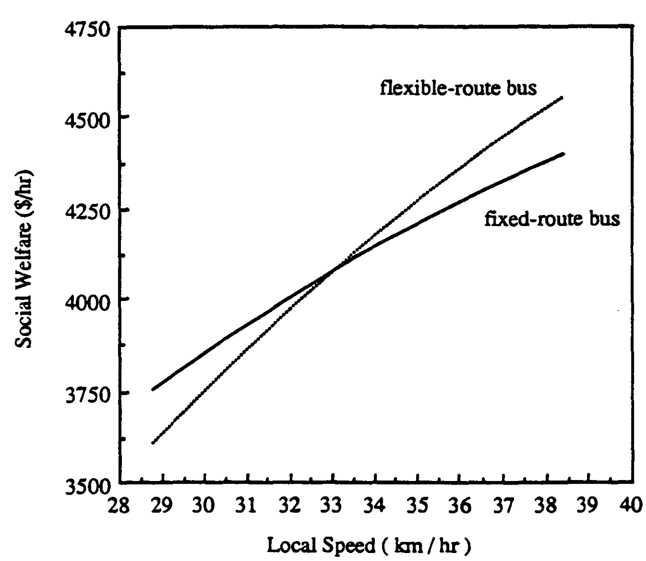

FIGURE 6 Effects of local travel speed on welfare.

where c_o^* is the optimized operator cost per trip for inelastic demand condition and n is the number of stops in one collection tour, approximated as

$$n = \left(\frac{4DB^4L^3Q_i}{awV\phi^2v^3u^5}\right)^{1/6} \tag{10}$$

Step 2. Recalculate demand density Q_i with the demand function:

$$\begin{aligned} Q_i &= F_Q(h_i, M_i, f_i, \cdot) \\ &= q(k - e_w a h_i - e_p f_i - e_v M_i) \end{aligned} \tag{11}$$

Step 3. Set $i = i + 1$ and recalculate the headway, in-vehicle travel time, fare, and demand density using Equations 7 to 11:

$$h_i = F_h(Q_{i-1}, \cdot)$$

$$M_i = F_M(Q_{i-1}, \cdot)$$

$$f_i = F_f(Q_{i-1}, \cdot)$$

$$Q_i = F_Q(h_i, M_i, f_i, \cdot)$$

Step 4. If a stopping rule is satisfied (e.g., $Q_i - Q_{i-1} < \varepsilon$, where ε is a tolerable deviation) STOP, ELSE go to Step 3.

With this solution procedure, the optimal fare f^*, service headway h^*, and the equilibrium in-vehicle travel time M^* for the flexible-route bus system may be obtained. In addition, the equilibrium results of demand Q^*, operator cost C_o^*, revenue R^*, consumer surplus G^*, and social welfare Y^* may be obtained. Comparisons of costs and social welfares for the fixed- and flexible-route bus systems can be conducted accordingly.

NUMERICAL RESULTS

The numerical results for the two bus systems at equilibrium break-even conditions are presented in Table 3 on the basis of the parameter values in Table 1. The optimized fares are \$0.81 per trip and \$1.67 per trip for the fixed- and flexible-route bus systems, respectively. Since the optimized fares for the two systems are obtained at the break-even condition, they are identical to their average operator costs. The two systems differ in their equilibrium demands and social welfare because of their different service attributes. Both the equilibrium demand and the welfare are higher for the flexible-route system than for the fixed-route system. Table 3 shows that the equilibrium demands are 854 and 867 trips per hour for fixed- and flexible-route systems, respectively, with a potential demand of 1,200 trips per hour. The respective welfare values are \$4,458/hr and \$4,470/hr.

The flexible-route bus system has the higher operator cost (i.e., fare) and the lower user cost, which includes wait cost, access cost, and in-vehicle travel cost. At the equilibrium break-even condition the fare, the average wait cost, and the average access cost are all identical for the fixed-route system, but not for the flexible-route system, in which the fare is higher than the average wait cost. These optimality conditions are verified by the numerical results given in Table 3.

TABLE 3 Numerical Results for Equilibrium Break-Even Conditions

Systems	Fixed Route	Flexible Route
Route Spacing (km)	1.30	-
Zone Area (sq. km)	-	4.27
Headway (minutes)	9.6	6.6
Fare (\$/trip)	0.81	1.67
Fleet Size (vehicles)	17	44
Operator Cost (\$/trip)	0.81	1.67
User Cost (\$/trip)	3.56	2.42
Maximum Load (passengers)	37	14
Social Welfare (\$/hr)	4,458	4,470
Equilibrium Demand (trips/hr)	854	867
Potential Demand (trips/hr)	1,200	1,200

The maximum passenger load of 14 for the flexible-route system is significantly different from that of 37 for the fixed-route system. This is due to specific features of flexible-route services, in which passengers pickups in one collection tour should be limited to a certain level. Otherwise, the advantage of this door-to-door service would be reduced by long in-vehicle travel time cost. This analysis suggests that for the flexible-route system it is preferable to use small vehicles to provide door-to-door service. Table 3 also shows that the fleet sizes for the fixed- and flexible-route systems are 17 and 44 vehicles, respectively.

The optimized welfares for the fixed- and flexible-route bus systems may be used to determine which system is preferable under given circumstances. Figure 3 shows a welfare comparison between the two systems. Each has been optimized over a range of line-haul distances for maximum welfare objective subject to the break-even constraint using the reason-

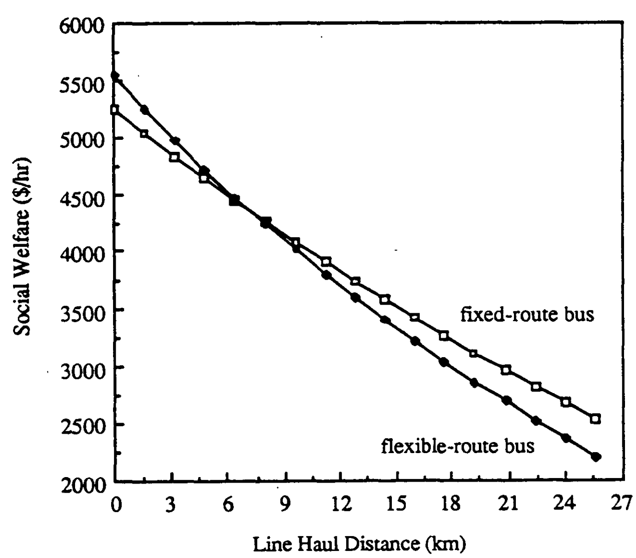

FIGURE 3 Effects of line-haul distance on welfare.

assumptions about the bus route structures (*17,22,27*). This linear demand function is formulated as follows:

$$Q = q(1 - e_w T - e_x X - e_v M - e_p f) \qquad (3)$$

where

- q = potential demand density of the bus service;
- T = wait time, which may be assumed to be a constant factor z_w (usually $z_w = 0.5$ for uniform passenger arrivals at bus stops) multiplied by the headway h;
- X = average access time, which is assumed for the fixed-route system to be $z_w(r + s)/g$, and as defined in Table 1, r is the route spacing, s is the stop spacing, g is the walking speed, and z_x is a geometric access distance factor (usually $z_x = 0.25$ for grid street networks with negligible street spacing) (the average access time is assumed to be zero for the flexible route system, since it provides door-to-door services);
- M = average in-vehicle travel time;
- f = fare, which is uniform for all passengers; and
- e_w, e_x, e_v, and e_p = elasticity factors.

The values of the elasticity factors e_w, e_x, e_v, and e_p are not the actual elasticities in such a linear function. The ratios between the elasticity factors for wait time and fare (e_w/e_p,), for access time and fare (e_x/e_p,), and for in-vehicle time and fare (e_v/e_p) determine the implied values of wait time, access time, and in-vehicle time, respectively.

The analytic results for the optimal route spacing, service headway, fare, consumer surplus, operator cost, and social welfare for the fixed-route systems have been derived by Chang (*17*) and are summarized in Table 2. At equilibrium break-even condition the optimized fare (f^*), the average wait cost ($wz_w h^*$), and the optimized access cost ($xz_x r^*/g$) are all identical for the fixed-route system (17, 23).

For the flexible-route system, the objective function can be stated as follows:

$$\text{Maximize } Y = G + R - C_0 \qquad \text{subject to } R - C_0 \leq 0$$

where G, R, and C_0 are the consumer surplus, the revenue, and the operator cost, respectively, and are defined as follows:

$$G = \frac{LWq}{2e_p}(k - e_w z_w h - e_p f - e_v M)^2 \qquad (4)$$

$$R = fLWq(k - e_w e_w h - e_p f - e_v M) \qquad (5)$$

$$C_0 = \frac{LWB}{VAh}\left(D_L + \phi A\sqrt{q(k - e_w ah - e_p f - e_v M)h/u}\right) \qquad (6)$$

where k is a constant representing a potential demand component insensitive to optimized variables. Obviously, the problem is difficult to solve analytically due to the complexity of Equation 6. A solution algorithm is therefore developed to obtain the equilibrium results. Analytic results for the inelastic demand condition (as shown later in Equations 7 to 10) are used in this algorithm. The solution procedures are stated as follows.

TABLE 2 Analytic Results for Break-Even Fixed-Route Bus Systems

Items	Analytic Results
Route Spacing (r^*)	$U\left(\frac{z_w e_w e_p DBg^2}{qVLz_x^2 e_x^2}\right)^{1/3}$
Service Headway (h^*)	$\frac{z_x e_x r^*}{z_w e_w g}$
Fare (f^*)	$\frac{z_x e_x r^*}{e_p g}$
Revenue (R^*) = Operator Cost (C_o^*)	$\frac{z_w e_w WDBg}{z_x z_x V r^{*2}}$
Consumer Surplus (G^*)	$\frac{LWq}{2e_p}\left(k - \frac{3z_x e_x r^*}{g}\right)^2$
Equilibrium Demand (Q^*)	$LWq(k - 3z_x e_x r^*/g)$

Note: $U = (1 - 3z_x e_x r^*/g)^{-1/3}$

Algorithm

Initialization: Set a demand function $Q_i = F_Q(q, h_i, M_i, f_i)$, where the demand density Q_i is a function of the potential demand q, headway h_i, fare f_i, and in-vehicle travel time M_i. The linear demand function shown in Equation 3, for example, is the demand function used in this analysis, although other nonlinear demand functions may also be considered.

Step 1. Set $i = 0$ and $Q_i = q$, use the following analytic results for inelastic demand condition to obtain h_i and M_i.

$$h_i = F_h(Q_i, \cdot) = \left(\frac{\phi(2B + vun)}{4wa\, Vu^{1/2}Q_i^{1/2}}\right)^{2/3} \qquad (7)$$

$$M_i = F_M(Q_i, \cdot) = \frac{D_L}{2V} + \left(\frac{wa\phi^2 u^2}{2(2B + vun)V^2 Q_i}\right)^{1/3} \qquad (8)$$

It is implied that all potential trips are captive and thus the bus system is designed on the basis of its total potential demand $Q_i = q$.

Since a break-even constraint is considered, the fare f_i can be obtained using

$$f_i = F_f(Q_i, \cdot) = c_o^* = \frac{BD_L}{Vun} + B\left(\frac{4wa\phi^2}{Q_i V^2 u(2B + vun)}\right)^{1/3} \qquad (9)$$