Welfare Maximization with Financial Constraints for Bus Transit Systems

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A bus system with time-dependent demand and supply characteristics is analytically optimized to maximize a welfare objective, subject to financial constraints. With some approximations, equations for optimal route spacing, headways in various periods, and fares are obtained for unconstrained, break-even, and subsidy cases. The relationships between the optimized decision variables and system parameters are thus identified analytically. A numerical example is given for a bus transit system with three service periods. In the vicinity of the maximum welfare solution, the welfare is found to be relatively flat with respect to subsidies. Since subsidy increments yield disproportionately smaller welfare increments, break-even or constrained subsidy solutions may be preferable to pure welfare maximization. A minimum allowable ratio of welfare change to subsidy change is suggested as a criterion for optimizing individual bus systems and efficiently allocating resources among alternatives.

Multiple-period analytic optimization models have been developed for analyzing bus systems (*1*). To extend that work, this paper presents analytic models for optimizing bus systems using a maximum welfare objective subject to constraints on allowable subsidies, including break-even constraints.

Studies on analytic optimization models have been extensively reviewed by Chang and Schonfeld (*1,2*). Such models for public transportation system optimization have most often assumed a perfectly inelastic demand (*2–12*). This assumption may be reasonable for some systems and may simplify models to the point at which analytic solutions may be obtained. However, this assumption may preclude the model from analyzing pricing policy and subsidy issues or from optimizing the system for objectives, such as maximum net social benefit or profit.

Kocur and Hendrickson (*13*) have analyzed bus services with demand elasticity and developed closed-form solutions for optimal route spacing, headway, and fare with various objective functions. This was accomplished with some approximations, most notably with a linear approximation of a logit mode split model. Nash (*14*) assessed alternative objectives for bus transit service in terms of fares, service levels, and financial results. Frankena (*15*) investigated the conditions under which a maximum ridership objective would be economically efficient and analyzed the effects of subsidy on system efficiency. Else (*16*) analyzed optimal fares and subsidies while considering various externalities. Bly and Oldfield (*17*) investigated analytically the effects of subsidy on bus operation. An analytic model considering demand elasticity, financial constraints, and congestion effects also has been developed by Oldfield and Bly (*18*) to determine the optimal vehicle size for urban bus systems. None of these studies considered time-dependent supply and demand characteristics or tradeoffs between subsidies and a welfare objective.

Many studies have used numerical instead of analytic methods to optimize public transportation systems and to investigate issues covered in this paper (*19–22*). However, an analytic approach is used here to find closed-form solutions for the decision variables and objective function and to identify some related optimality conditions.

In this paper, the analytic models for bus systems assumed by Chang and Schonfeld (*1*) are applied to analyze unconstrained, break-even, and subsidy cases. Three decision variables—namely, route spacing, headway, and fare—are optimized jointly in those models. The system assumptions are briefly reviewed in the next section, and then the welfare objective function is formulated. The following section presents the analytic results for the various cases and discusses relations and implications of the analytic results. The section after that presents a numerical case. The final section presents conclusions.

SYSTEM ASSUMPTIONS

The assumed bus system, which can represent a variety of transit operations, is taken from Chang and Schonfeld (*1*). A brief description of the assumed system follows.

Bus System Characteristics

In this analysis, a branched-zone bus system is assumed to provide service for a rectangular area with dimensions $L \times W$, from which trip ends are assumed to access a single point, such as a mass transit station or activity center. Figure 1 illustrates this bus system. The variables and their typical baseline values are defined in Table 1.

The service area has N zones, each of length L and width $r = W/N$. A vehicle round trip to zone j during period t consists of (a) a line haul distance J traveled at express speed yV_t from the starting point to a corner of service area; (b) a distance of W_j km traveled at local nonstop speed bV_t from the corner to the assigned zone; (c) a collection route L km traveled at local speed V_t along the middle of the zone stopping for passengers every d km; and (d) a retracing in reverse order of the first three stages.

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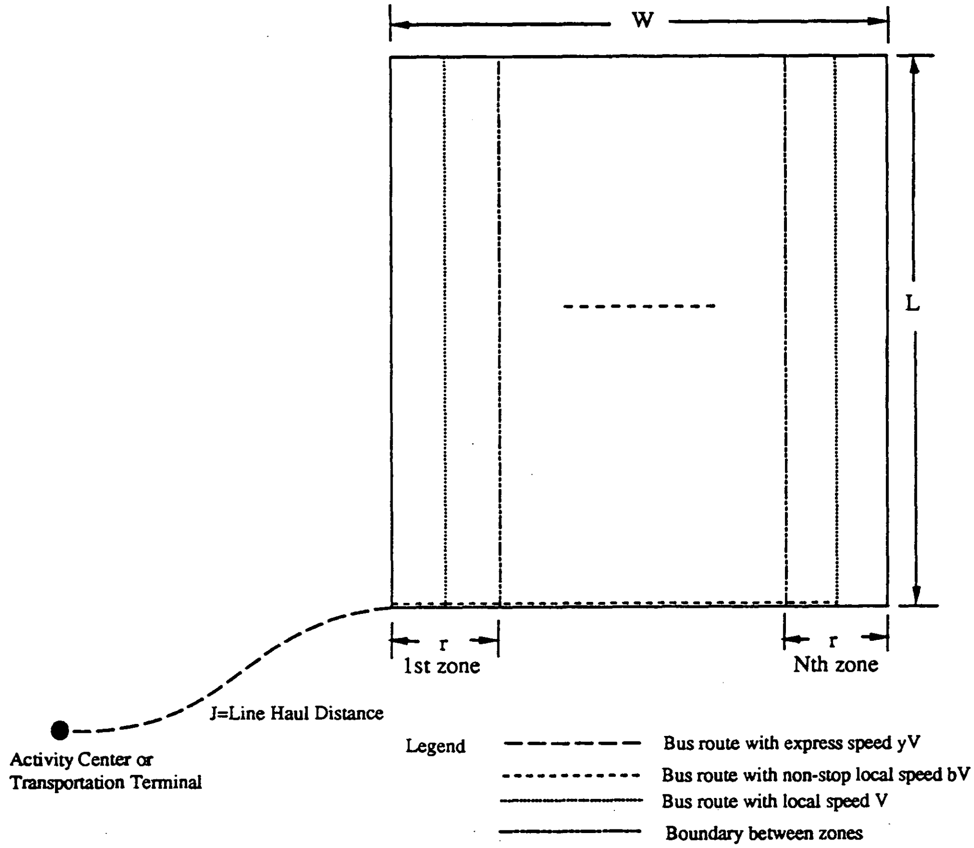

FIGURE 1 Bus system configuration.

Passengers are assumed to walk at speed g between their trip ends and the nearest bus stop along a rectangular street network (parallel and perpendicular to the feeder route) with negligible street spacing. This assumption implies an access distance of $(r + d)/4$ and an access time of $(r + d)/4g$.

The bus network assumed in this paper has been previously analyzed for maximum profit and welfare objectives without financial constraints (*1*). The bus route structure and model formulations may be used to analyze a wide variety of bus service types, including feeder services to and from transfer stations, zone structure services, and radial services to activity centers. They may also be used in some cases to analyze bus systems with many-to-many demand patterns if the system can be separated into subsystems in which many-through-one analysis is applicable [as discussed by Newell (*9*)].

Demand Functions

A linear demand function in which the demand density is sensitive to various travel time components and the fare is formulated as

$$Q_t = q_t\left[1 - e_w z_1 h_t - e_x z_2\left(\frac{r + s}{g}\right) - e_v M_t - e_p f\right] \quad (1)$$

where

q_t = potential demand density of the bus service during each period;
$z_1 h_t$ = wait time, which is assumed to be constant factor z_1 (usually $z_1 = 0.5$ for uniform passenger arrivals at bus stops) multiplied by headway h_t;
$z_2[(r + s)/g]$ = average access time (usually $z_2 = 0.25$ for grid street networks with negligible street spacing);
M_t = average in-vehicle travel time;
f = fare; and
e_w, e_x, e_v, e_p = elasticity factors.

The optimizable decision variables are the headway (h_t) for each period, the route spacing (r), and the fare (f). This implies that the optimized bus route structure and fare are assumed to be the same in all periods, whereas the headways are optimized separately for each period.

The values of elasticity factors e_w, e_x, e_v, and e_p are not the actual elasticities in such a linear function. In addition, the ratios between the elasticity factors for wait time and fare (e_w/e_p), for access time and fare (e_x/e_p), and for in-vehicle time and fare (e_v/e_p) determine the implied values of wait time, access time, and in-vehicle time, respectively.

It is assumed that the potential demand q_t in each period could be determined from the time distribution of demand shown in Figure 2. A step demand distribution (Figure 2*b*) relating monotonically volume levels and their durations can be directly obtained from the empirical demand distribution (Figure 2*a*). Although only three periods are used as an example of the demand distribution in Figure 2*e*, the number

TABLE 1 Variable Definitions

Variable	Definition	Baseline Value
B_t	bus operating cost in period t; B_1, B_2, and B_3 = 50, 25, and 25 dollars/veh. hr.	
b	non-stop ratio = non-stop speed/local speed	2
C	system total cost ($/period)	-
C_o	total operator cost ($/period)	-
D_t	bus avg. round trip time during period t (hrs.) $=2L/V_t+W/bV_t+2J/yV_t$	
e_p	demand elasticity parameter for fare	0.07
e_v	demand elasticity parameter for in-vehicle time	0.35
e_w	demand elasticity parameter for wait time	0.7
e_x	demand elasticity parameter for access time	0.7
F_t	fleet size in period t (vehicles)	-
f	fare ($)	-
G	consumer surplus ($/period)	-
g	average walk speed (km/hour)	4.0
h_t	headway in period t (hrs/vehicle)	-
I_1	the lagrange multiplier associated with the break-even constraint	-
I_2	the lagrange multiplier associated with the subsidy constraint	-
I_s	$(I_2+1)/(2I_2+1)$	-
J	line haul distance (km)	6.4
K	subsidy ($/period)	-
k_t	invariant components of the demand function $=1-e_x d/4g-e_v M_t$	-
L	length of service area (km)	4.8
M_t	passenger avg. trip time during period t (hrs.) $= L/2V_t+W/2bV_t+J/yV_t$	
m	number of time periods	3
P	profit ($/period)	-
Q_t	demand function for period t (Eq. 1)	-
q_t	potential demand density in period t	-
	q_1, q_2, and q_3 = 50, 20, and 5 trips/sq. km/hour, respectively.	
R	total revenue ($/period)	-
r	route spacing (km)	-
s	stop spacing (km)	0.4
T_t	service time in period i (hours); T_1, T_2, and T_3 = 3, 3, 4 hours, respectively.	
V_t	local speed during period t (km/hour)	24.0
W	width of service area (km)	3.2
X	composite variable $= \Sigma_t T_t(D_t B_t q_t)^{1/2}/\Sigma_t T_t q_t$	-
Y	welfare ($/period)	-
y	express ratio = express speed/local speed	2.0
z_1	ratio of wait time/headway	0.5
z_2	geometric factor for determining access time	0.25

and duration of periods are unlimited in the models and may be selected to represent variations over time with whatever precision is desired. Other models (*8,18*) have used smoothed functions (e.g., Figure 2*c*) to represent demand variation over time. However, such smoothing is not necessary to formulate objective functions that are twice differentiable and hence appears to be an unnecessary complication (*23*). The approach used in this work relies on step functions for demand, costs, speeds, and other variables that are obtained directly from the empirical data.

Operator Costs

The operator costs per analysis period (e.g., per day) is the fleet size multiplied by the hourly operating cost and total daily service time. The fleet size is the bus round-trip time divided by the headway. The bus round-trip times D_t are assumed for various periods because different traffic conditions are represented by different speeds:

$$D_t = \frac{2L}{V_t} + \frac{W}{bV_t} + \frac{2L}{yV_t} \tag{2}$$

The hourly operating costs B_t are assumed for different periods. The operator costs per day are

$$C_o = \sum_{t=1}^{m} F_t B_t T_t = \sum_{t=1}^{m} \frac{W D_t B_t T_t}{r\, h_t} \tag{3}$$

MAXIMUM WELFARE OBJECTIVE

Various objective functions have been considered appropriate for optimizing bus transit systems (*24*). In this paper maximum social welfare, also known as the net social benefit or simply welfare, is used as the objective function.

The welfare Y is the sum of the consumer surplus G and producer surplus P:

$$Y = G + P \tag{4}$$

The producer surplus, also known as profit, is the total revenue R minus the operator cost C_o:

$$P = R - C_o \tag{5}$$

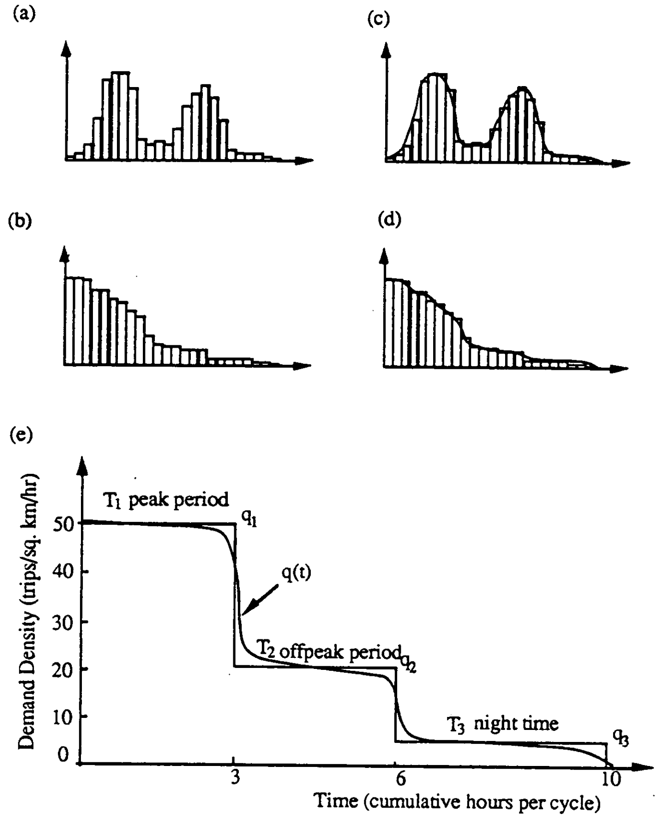

FIGURE 2 Distributions of demand over time: *a*, empirical demand distribution; *b*, transformed demand distribution; *c*, smoothed demand function; *d*, smoothed and transformed demand distribution; *e*, demand distribution assumed in numerical example.

The operator cost has been defined in Equation 3. The total revenue is the fare multiplied by the total demand:

$$R = \sum_{t=1}^{m} fLWT_tQ_t \tag{6}$$

Therefore the total profit can be formulated as

$$P = \sum_{t=1}^{m} fLWT_tq_t(k_t - e_wz_1h_t - e_xz_2r/g - e_pf) - \sum_{t=1}^{m} \frac{WE_tT_t}{r\,h_t} \tag{7}$$

where k_t is a constant representing a component of the potential demand that is insensitive to the decision variables optimized here:

$$k_t = 1 - e_xz_2s/g - e_vM_t \tag{8}$$

By inverting the demand function shown in Equation 1 to find fare as a function of demand and by integrating the inverted function over the demand, the total social benefit can be obtained. Then, the consumer surplus G can be derived as the total social benefit minus the total cost that the users actually pay (*13,25*):

$$G = \left(\frac{LW}{2e_p}\right) \sum_{t=1}^{m} T_tq_t(k_t - e_wz_1h_t - e_xz_2r/g - e_pf)^2 \tag{9}$$

Therefore the welfare can be formulated by adding Equations 7 and 9. In the following sections the major analytic results for the various objectives are presented.

ANALYTIC RESULTS

The objective here is to maximize the welfare:
Maximize

$$Y = G + R - C_o \tag{10}$$

In solving this problem, a deficit constraint is considered that can be generally expressed as follows:

$$C_o \leq R + K \tag{11}$$

It means that the operator cost C_o should not be larger than the sum of the revenue R and subsidy K.

The problem is solved for three cases with different assumptions about the constraint and subsidy: (a) without the constraint; (b) with the constraint $K = 0$, that is, a break-even requirement; and (c) with the constraint $K \neq 0$. These three cases are presented separately.

Unconstrained Results

In the first case, the first-order conditions for an optimum are

$$\partial Y/\partial r = 0 \tag{12}$$

$$\partial Y/\partial h_t = 0 \qquad t = 1, 2, \ldots, m \tag{13}$$

$$\partial Y/\partial f = 0 \tag{14}$$

The optimized fare can be immediately obtained from Equation 14:

$$f^* = 0 \tag{15}$$

The zero fare result is not surprising because the marginal operator cost is 0 in the bus systems considered here. Similar zero-fare results have been discussed previously (*13,24*). However, the marginal costs would become positive if a capacity constraint were binding (*23*) or if certain congestion effects were modeled.

By substituting this result into Equations 12 and 13, the relation between the optimal headway h_t^* and optimal route spacing r^* is found to be

$$h_t^* \cong \left(\frac{E_t}{q_t}\right)^{1/2} \left(\frac{z_2e_xr^*}{Xgz_1e_w}\right) \tag{16}$$

The approximation errors in Equation 18 and some further results below are slight (*2*). Using this optimality result in

Equation 16 and the optimized fare obtained, the optimized route spacing is found to be

$$r^* \cong \left(\frac{X^2 g^2 z_1 e_w e_p}{z_2^2 e_x^2 Lk}\right)^{1/3} \tag{17}$$

The optimized headway for each period can then be obtained by substituting Equation 17 into Equation 16:

$$h_t^* \cong \left(\frac{E_t}{q_t}\right)^{1/2} \left(\frac{z_2 e_x e_p}{XgLz_1^2 e_w^2 k}\right)^{1/3} \tag{18}$$

Since the optimal fare in this case is 0, the optimized welfare is simply the consumer surplus minus the operator cost.

Results with a Break-Even Constraint

In the previous case, the optimized fare is 0, which implies a deficit for the operator. Therefore, in the second case, a break-even constraint $C_o \leq R$ is considered.

The problem can be stated as follows:

Maximize

$$Y = G + R - C_o$$

Subject to

$$C_o - R \leq 0$$

A break-even solution would not exist if the demand function were always below the average operator cost function. That situation would imply a negative profit. Therefore, it is assumed in the following analysis that the travel demand is sufficient to yield a positive profit in some circumstances for the bus operation considered.

The Lagrange multipliers method is used here for constrained optimization, and the Lagrangian α is formulated as follows:

$$\alpha = G + R - C_o - I_1(C_o - R) \tag{19}$$

where I_1 is the Lagrange multiplier associated with the break-even constraint.

Solving the first-order conditions, as shown by Chang and Schonfeld (*2*), the following approximate relationship between headway and route spacing can be obtained:

$$h_t^* \cong \left(\frac{E_t}{q_t}\right)^{1/2} \left(\frac{z_2 e_x r^*}{Xgz_1 e_w}\right) \tag{20}$$

This relationship is identical to that obtained for the previous case.

The following results for optimized route spacing and headway can also be derived (*2*):

$$r^* \cong 1.12\left(\frac{X^2 g^2 z_1 e_w e_p}{z_2^2 e_x^2 Lk}\right)^{1/3} \tag{21}$$

$$h_t^* \cong 1.12\left(\frac{E_t}{q_t}\right)^{1/2} \left(\frac{z_2 e_x e_p}{XgLz_1^2 e_w^2 k}\right)^{1/3} \tag{22}$$

However, the optimized fare is no longer 0. It is found to be

$$f^* = \left(\frac{e_x}{e_p}\right)\left(\frac{z_2 r^*}{g}\right) \tag{23}$$

Since the ratio (e_x/e_p) represents the value of access time and $z_2 r^*/g$ is the average lateral access distance, the result indicates that the fare is identical to the lateral access cost in the optimized system.

The optimal revenue R^* and operator cost C_o^* are equal at break even, and their solutions are shown by Chang and Schonfeld (*2*).

Results with Subsidy

To maximize welfare subject to a subsidy constraint, the problem can be stated as follows:
Maximize

$$Y = G + R - C_o$$

Subject to

$$C_o - R \leq K$$

This problem is also solved using the Lagrange multipliers method. The Lagrangian α is formulated as

$$\alpha = G + R - C_o - I_2(C_o - R - K) \tag{24}$$

in which I_2 is the Lagrange multiplier associated with the subsidy constraint, and K is subsidy or maximum allowable deficit. Solving the first-order conditions, as shown by Chang and Schonfeld (*2*), the relationship between the optimal headway and route spacing can be obtained:

$$h_t^* \cong \left(\frac{E_t}{q_t}\right)^{1/2} \left(\frac{z_2 e_x r^*}{Xgz_1 e_w}\right) \tag{25}$$

The relations among the fare, the route spacing, and the Lagrange multiplier are also derived:

$$f = \left(\frac{I_2}{2I_2 + 1}\right)\left(\frac{k}{e_p} - \frac{2z_2 e_x r}{ge_p}\right) \tag{26}$$

$$r = \left(\frac{X^2 g^2 z_1 e_w e_p (2I_2 + 1)}{z_2^2 e_x^2 Lk(1 - 3z_2 e_x r/gk)(I_2 + 1)}\right)^{1/3} \tag{27}$$

Equation 25 shows that the proportionality relation between the optimized headway and route spacing can still be obtained. Solving with an approximation (*2*), the Lagrange multiplier I_2 is found to be

$$I_2^* = \frac{-\mu_2 + [\mu_2^2 - 4\mu_3(\mu_2 - \mu_1)]^{1/2}}{2(\mu_2 - \mu_1)} \tag{28}$$

in which μ_1, μ_2, and μ_3 are defined as follows:

$$\mu_1 = \left(\frac{X^2 z_1 z_2 e_w e_x}{gkLe_p^2}\right)^{1/3} \tag{29}$$

$$\mu_2 = \frac{4K}{kQ_T} + \left(\frac{k}{e_p}\right)[1 - 2e_p\mu_1]^2 - 3\mu_1 \quad (30)$$

$$\mu_3 = \frac{K}{kQ_T} - \mu_1 \quad (31)$$

The positive root in Equation 28 is the solution of the quadratic equation whenever the constraint is binding, because the negative root yields negative values of shadow price I_2.

Therefore, the optimized route spacing can be obtained by substituting Equation 28 into Equation 27. The optimized headway and fare can then be obtained by substituting the optimized route spacing into Equations 25 and 26, respectively. The optimal operator cost C_o^*, revenue R^*, and consumer surplus G^* can also be obtained (*2*). The welfare can be derived as the consumer surplus minus the subsidy: $Y^* = G^* - K$.

Discussion of Analytic Results

Analytic results concerning the design variables, including route spacing, headway, or fare, or all of these, are summarized in Table 2. These results show more clearly than any numerical results what the relationships among variables should be for optimized bus services under various objectives.

To a large extent the sensitivity of optimized design variables to the various system parameters can be determined by visually inspecting the functions rather than by numerical analysis. Taking, for example, systems optimized for maximum welfare subject to a break-even constraint, we can observe that the headway is proportional to the ½ power of the bus operating cost and round trip time ($E_t = B_tD_t$) and to the ⅓ power of the access time and fare elasticity factors. Similar sensitivity relations for the optimal route spacing and fare also can be observed (*2*).

For unconstrained welfare maximization, the optimized fare is 0, whereas for break-even welfare maximization the optimized fare is equal to the lateral access cost. When marginal costs are 0, such zero-fare results are expected and are similar to those obtained in previous works (*1,13,24*).

The results also show the optimality of a constant ratio between route spacing and headway. Such results are very similar to findings in several previous transit system optimization studies (*8,10,13*), except that a time-dependent factor $(E_t/q_t)^{1/2}$ can now be incorporated. This means that the following relation for feeder systems always holds for all periods:

$$h_t^*\left(\frac{q_t}{D_tB_t}\right)^{1/2} = \left(\frac{z_2e_x}{Xgz_1e_w}\right)r^* \qquad t = 1, 2, \ldots, m \quad (32)$$

Equation 32 was rewritten from Equations 16, 20, and 25; time-dependent headways and parameters, such as q_t, B_t, and D_t, are combined on the left side of Equation 32.

The analytic models presented here have not considered a vehicle capacity constraint. Vehicles may be overloaded in some cases unless a vehicle capacity constraint is applied. Chang and Schonfeld (*2*) show how these results can be modified to satisfy vehicle capacity or load factor constraints.

NUMERICAL EVALUATION

Numerical examples for various cases under different objectives are presented and compared to illustrate the applicability of the models developed. The baseline parameter values shown in Table 1 are used in numerical examples. The numerical results are presented in Table 3. These results are computed for a 6.4- × 4.8-km rectangular service area with a three-period demand pattern in which potential demand densities are 50, 20, and 5 trips/km²/hr during service periods of 3, 3, and 4 hr, respectively. These conditions have been shown in

TABLE 2 Analytically Optimized Decision Variables

unconstrained case	break-even case	subsidy case
$r^* = \left(\frac{X^2g^2z_1e_we_p}{z_2^2e_x^2Lk}\right)^{\frac{1}{3}}$	$r^* \cong 1.12\left(\frac{X^2g^2z_1e_we_p}{z_2^2e_x^2Lk}\right)^{\frac{1}{3}}$	$r^* \cong \left(\frac{X^2g^2z_1e_we_p}{z_2^2e_x^2LkI_s^*}\right)^{\frac{1}{3}}$
$h_t^* = \left(\frac{E_t}{q_t}\right)^{\frac{1}{2}}\left(\frac{z_2e_xe_p}{XgLz_1^2e_w^2k}\right)^{\frac{1}{3}}$	$h_t^* \cong 1.12\left(\frac{E_t}{q_t}\right)^{\frac{1}{2}}\left(\frac{z_2e_xe_p}{XgLz_1^2e_w^2k}\right)^{\frac{1}{3}}$	$h_t^* \cong \left(\frac{E_t}{q_t}\right)^{\frac{1}{2}}\left(\frac{z_2e_xe_p}{XgLz_1^2e_w^2kI_s^*}\right)^{\frac{1}{3}}$
$h_t^* = \left(\frac{E_t}{q_t}\right)^{\frac{1}{2}}\left(\frac{z_2e_xr^*}{Xgz_1e_w}\right)O_t$	$h_t^* = \left(\frac{E_t}{q_t}\right)^{\frac{1}{2}}\left(\frac{z_2e_xr^*}{Xgz_1e_w}\right)\Psi_t$	$h_t^* = \left(\frac{E_t}{q_t}\right)^{\frac{1}{2}}\left(\frac{z_2e_xr^*}{Xgz_1e_w}\right)\Psi_t$
$f^* = 0$	$f^* = \frac{z_2e_xr^*}{ge_p}$	$f^* = \left(\frac{I_2^*}{2I_2^*+1}\right)\left(\frac{k}{e_p} - \frac{2z_2e_xr^*}{ge_p}\right)$

TABLE 3 Numerical Results for Bus Systems

	unconstrained		break-even	subsidy [a] 1,500	subsidy [a] 3,000
Route Spacing (km)	1.715	(1.715)[b]	1.933	1.904	1.808
Fare ($)	0		1.21	0.82	0.51
Headway (hours)	0.199	(0.152)	0.225	0.220	0.116
	0.203	(0.203)	0.260	0.226	0.209
	0.364	(0.364)	0.411	0.403	0.324
Fleet Size (no. of vehicles)	23	(30)	18	19	21
	19	(19)	13	15	17
	9	(9)	7	7	8
Demand Density (trips/sq. km/hour)	29.26	(30.04)	24.00	25.83	27.19
	12.31	(12.31)	10.19	10.96	11.50
	2.91	(2.91)	2.46	2.61	2.76
Avg. Cost ($/trip)	6.538		7.123	6.973	6.758
Avg. User Cost ($/trip)	5.213		5.915	5.754	5.414
Avg. Operator Cost ($/trip)	1.325		1.208	1.219	1.284
Avg. Wait Cost ($/trip)	1.072		1.208	1.190	1.130
Operator Cost ($/day)	5,565	(6,627)	4,250	4,523	5,016
Revenue ($/day)	0		4,243	3,043	1,992
Profit ($/day)	-5,565		-7	-1,480	-3,024
Consumer Surplus ($/day)	19,025	(19,684)	13,297	14,861	16,461
Welfare ($/day)	13,464	(13,057)	13,290	13,381	13,437
Bus Load (passengers/veh.)	64		67	67	66
	25		33	30	29
	12		13	13	13
Shadow Price	-		0.19	0.07	0.11

a. Results for subsidies of 1,500/3,000 dollars per day, respectively.
b. Results with vehicle capacity constraint.

Figure 2*e*. The bus operating costs during these three periods—peak, offpeak, and night—are assumed to be $50, $25, and $25/vehicle-hr, respectively.

1. The optimal fare is 0 in the unconstrained case, $1.21 in the break-even case, and $0.82 if a subsidy of $1,500/day is provided. That subsidy represents about 30 percent of the daily operator cost. When the subsidy increases to $3,000/day, the optimal fare becomes $0.51.
2. For break-even welfare maximization, the consumer surplus and welfare are $13,297 and $13,290/day, respectively. Deficits for the constrained cases have small deviations from the theoretical results. For example, deficit (or profit) for the break-even case should be 0, whereas we obtain $7/day; when subsidy is constrained to $3,000 the profit should be −$3,000, whereas we obtain −$3,024. These deviations are the result of minor approximations in the analytic solutions.
3. In comparing unconstrained and break-even cases, we find that when the break-even constraint is removed, the deficit increases from 0 to $5,565, whereas the welfare rises by $174 (1.3 percent of $13,290). Thus, at least for the typical parameter values used in this analysis, the financial and political advantages of a break-even policy are quite strong.
4. When the subsidies increase, the optimal route spacing and headway decrease, and the fleet size increases. The equilibrium demand, consumer surplus, and welfare also increase.
5. The shadow price associated with the break-even constraint is $0.19, indicating that welfare would be increased by $0.19 if the deficit were increased from 0 to $1. The shadow prices associated with the subsidy constraints are $0.11 and $0.07 for subsidies of $1,500 and $3,000, respectively. These indicate that the welfare would be increased by $0.11 and $0.07 if the subsidies of $1,500 and $3,000 were increased to $1,501 and $3,001, respectively.

Basically, in the vicinity of the unconstrained welfare maximization solution, that is, as the subsidy approaches $5,565, the shadow price approaches 0. These relationships can be shown conceptually in Figure 3. It is shown that the optimal welfare Y^* is obtained with a subsidy of K^*. In the numerical results (Table 3) the optimal welfare of $13,464 is obtained with a deficit (and subsidy) of $5,565 in the unconstrained case. In the break-even case, the subsidy is 0, and the welfare is Y_B, which in the numerical results is $13,290. In Figure 3 the welfare becomes Y_1 when the subsidy is K_1. The subsidy constraint is not binding whenever it is to the right of the maximum social welfare point (e.g., $K_2 > K^*$). As long as a subsidy K_1 is binding, the slope of the curve $\Delta Y/\Delta K$ at $K = K_1$ may be interpreted as the shadow price of the subsidy

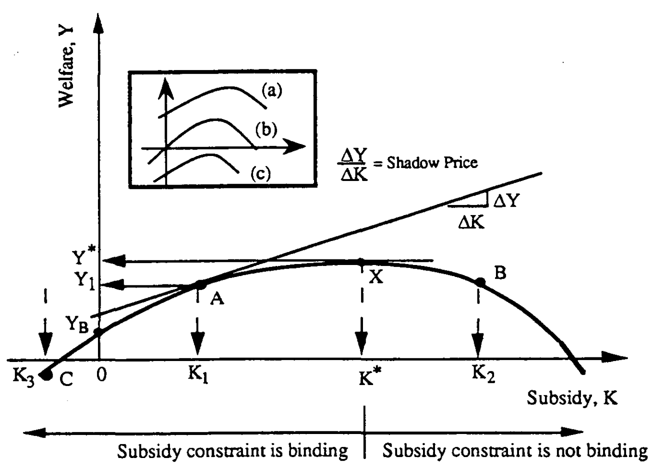

FIGURE 3 Conceptual relation between subsidy and welfare.

constraint. This shadow price $\Delta Y/\Delta K$ indicates how much opportunity there is to increase the welfare Y per increment of subsidy ΔK. Given some information about society's willingness to subsidize, which may also be expressed as a $\Delta Y/\Delta K$ ratio (i.e., a minimum acceptable welfare increment per unit of subsidy, or a minimum acceptable rate of return to subsidy), this indicator may be used as an allocation criterion to determine how much to subsidize a particular activity. Thus for our bus system the subsidy K would be increased as long as the slope $\Delta Y/\Delta K$ in Figure 3 exceeded a minimum acceptable $\Delta Y/\Delta K$ rate. This approach allows an efficient allocation of resources among various transportation and nontransportation activities. Figure 3 (Curve *a* in the inset) shows that this approach may sometimes imply a negative subsidy (i.e., a profit) in cases in which the welfare Y is positive at negative values of the subsidy K.

Figure 3 (Curve *a* in the inset) also indicates that if the social welfare function would shift downward to reflect higher operator cost functions or lower demand functions (e.g., from *a* to *c* in Figure 3), the maximum possible solutions for profit (Point *C*), break-even welfare (Y_B), subsidy-constrained welfare (A), and unconstrained welfare (X) would gradually become negative, in that order.

The results in Table 3 suggest that the break-even case may well be preferable to the unconstrained welfare maximization case, because by removing the break-even constraint the welfare rises by only $174 (1.3 percent), whereas the deficit increases from 0 to $5,565. It is desirable to examine the sensitivity of this result to the elasticity factors used in the demand function.

The peak-period busloads in Table 3 exceed the capacity of standard buses. The corresponding capacity constrained results can be obtained with the analytic models developed by Chang and Schonfeld (*2*) and are also presented for pure welfare maximization cases in Table 3. These results eliminate the overload problem in the peak period.

Table 4 shows the welfare results for the two cases with several fare elasticity factors. The comparison shows that the welfare rises by only 1.3 and 1.6 percent, whereas the deficits rise from 0 to $5,565 and $6,971 for elasticity factors of 0.07 and 0.05, respectively. These results suggest that in such cases, in which large increases in subsidies are required for such smaller increases in welfare, operators (and taxpayers) may find the break-even objective preferable.

Table 4 also indicates the effects of the fare elasticity factor on the optimal results. As expected from analytic results for route spacings, when the fare elasticity factors decrease from 0.07 to 0.05, the optimal route spacings decrease from 1.715 to 1.534 km for the unconstrained case, and from 1.933 to 1.718 km for the break-even case. The optimal fares are all 0 for the unconstrained case and increase from $1.21 to $1.50 when the fare elasticity factors decrease from 0.07 to 0.05.

It is also worth presenting the effects of subsidy on the optimized results, because these effects may not be visually perceived from the analytic results in the subsidy case. Figure 4 shows that in maximizing welfare the optimal fare varies inversely with the subsidy. Figure 5 shows the effects of subsidy on consumer surplus and welfare. The consumer surplus increases with the subsidy, whereas the producer surplus (profit) is the subsidy with a negative sign. These relationships have the net effect that the optimal welfare is very flat over a wide range of subsidy values. These results indicate that the break-even objective in welfare maximization may be quite acceptable, because it yields a zero deficit and only slightly less welfare than the unconstrained case. The discussion of Figure 3 also suggests how other solutions, with less than maximum welfare, corresponding to smaller levels of subsidy, or even profits, may be found preferable on the basis of the minimum

TABLE 4 Effects of Fare Elasticity Factors

(a) Case 1: Unconstrained Welfare Maximization

fare elast. factor	route spacing	consumer surplus	profit	operator cost	welfare	fare
0.05	1.534	27,989	-6,971	6,971	21,018	0
0.06	1.630	22,724	-6,137	6,137	16,587	0
0.07	1.715	19,029	-5,565	5,565	13,464	0

(b) Case 2: Break-Even Welfare Maximization

fare elast. factor	route spacing	consumer surplus	profit	operator cost	welfare	fare
0.05	1.718	20,682	0	5,557	20,682	1.50
0.06	1.826	16,375	0	4,921	16,375	1.33
0.07	1.933	13,297	-7	4,243	13,290	1.21

(c) Comparison

fare elasticity factor	unconstrained case		break-even case		change in welfare
	welfare (1)	operator profit (2)	welfare (3)	operator profit (4)	(1) - (3)
0.05	21,018	-6,971	20,682	0	336(1.6%)
0.06	16,587	-6,137	16,375	0	212(1.3%)
0.07	13,464	-5,565	13,290	0	174(1.3%)

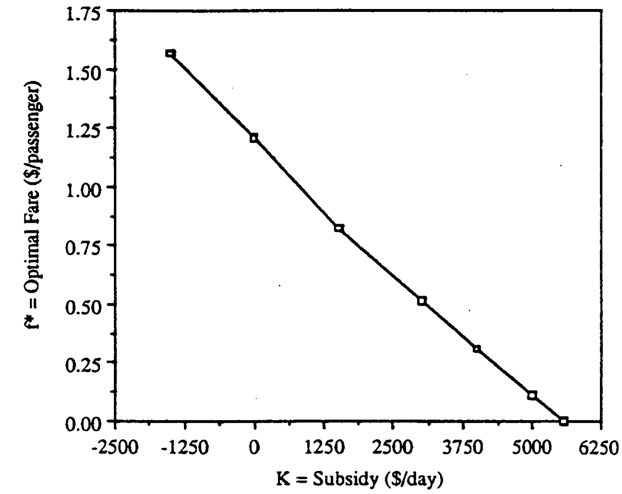

FIGURE 4 Effects of subsidy on optimal fare.

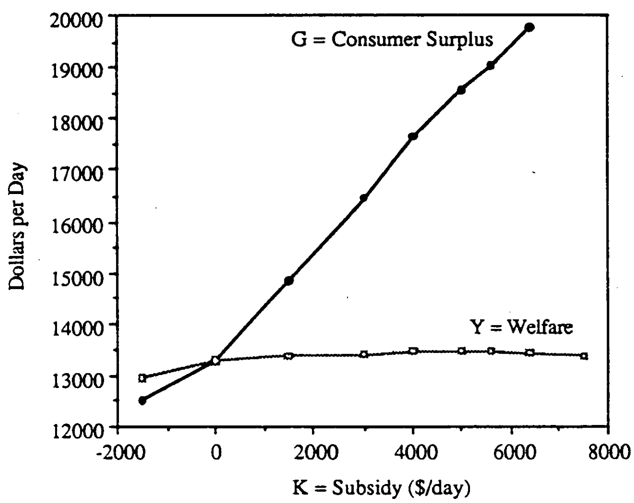

FIGURE 5 Effects of subsidy on consumer surplus and welfare.

acceptable $\Delta Y/\Delta K$ criterion. It would then be desirable to optimize the welfare subject to a constraint on the ratio of Δ welfare/Δ subsidy, as discussed earlier.

CONCLUSIONS

A fairly general bus system is analytically optimized to maximize welfare with various financial constraints. Closed-form solutions are derived for the optimal design variables (e.g., route spacing, headway, and fare), as summarized in Table 1. Interrelationships among the optimized design variables, the objective functions, and the system parameters are identified for various cases.

The optimality of a constant ratio between route spacing and headway, which has been found in previous studies for various bus network and demand conditions, is also found to be maintained with a multiperiod adjustment factor for all cases considered. It is not surprising that the optimal fare for welfare maximization is 0 in systems in which, in the absence of vehicle size constraints and congestion effects, the marginal operator cost is 0.

The effects of subsidy on the optimized results are presented. In maximizing welfare the optimal fare varies inversely with the subsidy. The consumer surplus increases with the subsidy, whereas the profit is the subsidy with a negative sign. These relationships have the net effect that the optimal welfare is very flat over a wide range of subsidy values. The effects of the bus operating costs on the optimized welfare, route spacing, fare, and fleet size are also evaluated (*2*).

The most interesting finding of this study is that the welfare-versus-subsidy function is very flat over a wide range in the vicinity of the optimum, as suggested in Figure 3 and shown numerically in Figure 5. Furthermore, our sensitivity analysis and preliminary results for very different kinds of transit system, such as flexible route paratransit (*26*), suggest that this is not an isolated case based on an accidental combination of parameters but a typical situation. This implies that subsidies can be greatly reduced below those required for maximum welfare, with only slight sacrifices in welfare. However, the relative funding burden would shift from subsidies (i.e., taxpayers) to fares (users). The changes in optimized systems associated with subsidy reductions, such as the changes in ridership, fares, network headways, and service levels can be analyzed with the analytic models presented here.

Comparisons of the various cases considered indicate that the effects of subsidy on welfare are very small over a wide range around the maximum welfare case. Thus, welfare maximization with a break-even constraint yields a zero deficit with very small reductions in welfare, compared with the subsidized cases (including unconstrained welfare maximization). Such results imply that the break-even objective might be preferable to subsidized cases whenever break-even solutions exist on a relatively flat part of a welfare function. In other words, where large increases in subsidies are required for much smaller increases in social welfare, operators (and taxpayers) may find the break-even objective preferable.

More generally, the conceptual discussion of Figure 3 suggests how a minimum acceptable $\Delta Y/\Delta K$ (i.e., ratio of welfare change to subsidy change) criterion may be used to determine the proper amount of subsidy or profit, and to efficiently allocate resources among various activities, including the bus systems modeled here.

REFERENCES

1. S. K. Chang and P. M. Schonfeld. Multiple Period Optimization of Bus Transit Systems. *Transportation Research*, Vol. 25B, No. 6, 1991, pp. 453–478.
2. S. K. Chang and P. M. Schonfeld. *Welfare Maximization with Financial Constraints for Bus Transit Systems*. TSC Report 92-24. Department of Civil Engineering, University of Maryland, College Park, 1992.
3. A. Black. *The Design of Urban Mass Transit Systems: A General Model*. Ph.D. dissertation. Cornell University, Ithaca, N.Y., 1975.
4. B. Byrne. Public Transportation Line Position and Headways for Minimum System and User Cost in A Radial Case. *Transportation Research*, Vol. 9, 1975, pp. 97–102.
5. G. C. Clarens and V. F. Hurdle. An Operating Strategy for a Commuter Bus System. *Transportation Science*, Vol. 9, No. 1, 1975, pp. 1–20.

6. N. S. A. Ghoneim and S. C. Wirasinghe. Optimum Zone Configuration for Planned Urban Commuter Rail Lines. *Transportation Science*, Vol. 21, No. 2, 1987, pp. 106–114.
7. E. M. Holroyd. The Optimal Bus Service: A Theoretical Model for A Large Uniform Urban Area. *Proc., 3rd International Symposium on the Theory of Traffic Flow*, New York, N.Y., 1965, pp. 308–328.
8. V. F. Hurdle. Minimum Cost Locations for Parallel Public Transit Routes. *Transportation Science*, Vol. 7, No. 4, 1973, pp. 340–350.
9. G. F. Newell. Dispatching Policies for a Transportation Route. *Transportation Science*, Vol. 5, No. 1, 1971, pp. 91–105.
10. S. Tsao and P. M. Schonfeld. Branched Transit Services: An Analysis. *Journal of Transportation Engineering*, ASCE, Vol. 110, No. 1, 1984, pp. 112–128.
11. R. Vaughan. Optimum Polar Networks for an Urban System with a Many-to-Many Travel Demand. *Transportation Research*, Vol. 20B, No. 3, 1986, pp. 215–224.
12. V. Vuchic and G. Newell. Rapid Transit Interstation Spacing for Minimum Travel Time. *Transportation Science*, Vol. 2, No. 4, 1968, pp. 359–374.
13. G. Kocur and C. Hendrickson. Design of Local Bus Service with Demand Equilibrium. *Transportation Science*, Vol. 16, No. 2, 1982, pp. 149–170.
14. C. A. Nash. Management Objectives, Fares and Service Levels in Bus Transport. *Journal of Transport Economics and Policy*, Vol. 12, No. 1, 1978, pp. 70–85.
15. M. W. Frankena. The Efficiency of Public Transport Objectives and Subsidy Formulas. *Journal of Transport Economics and Policy*, Vol. 17, No. 1, 1983, pp. 67–76.
16. P. K. Else. Optimal Pricing and Subsidies for Scheduled Transport Services. *Journal of Transport Economics and Policy*, Vol. 19, No. 3, 1985, pp. 263–279.
17. P. H. Bly and R. H. Oldfield. An Analytic Assessment of Subsidies to Bus Services. *Transportation Science*, Vol. 20, No. 3, 1986, pp. 200–212.
18. R. H. Oldfield and P. H. Bly. An Analytic Investigation of Optimal Bus Size. *Transportation Research*, Vol. 22B, No. 5, 1988, pp. 319–337.
19. P. G. Furth and N. H. M. Wilson. Setting Frequencies on Bus Routes: Theory and Practice. In *Transportation Research Record 818*, TRB, National Research Council, Washington, D.C., 1981, pp. 1–7.
20. E. K. Morlok and P. A. Viton. Feasibility of Profitable Transit Service in Radial Urban Corridors. In *Transportation Research Record 980*, TRB, National Research Council, Washington, D.C., 1984, pp. 46–54.
21. P. A. Viton. Privately-Provided Urban Transport Services. *Journal of Transport Economics and Policy*, Vol. 16, No. 1, 1982, pp. 85–94.
22. P. A. Viton. Pareto-Optimal Urban Transportation Equilibria. In *Research in Transportation Economics*, Vol. 1, Jai Press, Inc., Greenwich, Conn., 1983, pp. 75–101.
23. S. K. Chang. *Analytic Optimization of Bus Systems in Heterogeneous Environments*. Ph.D. dissertation. UMCP-TSC-DS-90-2. Department of Civil Engineering, University of Maryland, College Park, 1990.
24. C. A. Nash. *Economics of Public Transport*. Longman Inc., New York, N.Y., 1982.
25. M. Wohl and C. Hendrickson. *Transportation Investment and Pricing Principles*. John Wiley and Sons, Inc., New York, N.Y., 1984.
26. S. K. Chang and C. J. Lee. Welfare Comparison of Fixed- and Flexible-Route Bus Systems. In *Transportation Research Record 1390*, TRB, National Research Council, Washington, D.C., 1993.

Publication of this paper sponsored by Committee on Transportation Economics.