An l_p-Norm Origin-Destination Estimation Method That Minimizes Site-Specific Data Requirements

YUPO CHAN AND M. YUNUS RAHI

An efficient l_p-approximation algorithm to estimate a likely origin-destination (O-D) matrix while minimizing site-specific data-collection effort is presented. It was found that the trip-distribution curve is a useful supplement to site-specific link counts since it can be borrowed from a similar community, or that an outdated local curve can be employed without significant loss of accuracy. Imbedding such a generic trip-distribution curve within the algorithmic procedure gives a more accurate O-D estimation and link-count reproduction in general, although the number of iterations is increased. Test results from a medium-sized city show that the extra computational effort is a small price to pay for the improvement in O-D accuracy. The l_p-approximation algorithm is shown to be theoretically related to familiar O-D estimation techniques such as entropy maximization, information minimization, and generalized inverse, yet it is more robust and theoretically more satisfying.

The fundamental and indispensable data required to operationalize origin-destination (O-D) estimation algorithms are link counts. Additional data requirements differ depending on the specific methodology. Some require an old O-D matrix, often referred to as a base (or target) O-D matrix, whereas others require a control total on productions and attractions, but they can be labeled as site-specific information. Consistent with the resource-saving objective of this class of O-D estimation techniques, the intention here is to minimize the collection of site-specific data, and to the extent possible, use generic information transferable from other cities of similar size and urban structure (*1*). The authors propose to use trip-distribution curves, also known as trip-length frequency curves, to supplement basic site-specific data, such as link counts, given the invariant nature of these curves, which has been attributed to travel-time budget theories (*2,3*).

BASIC THEORIES

The O-D estimation problem can be thought of as solving the linear equation set

$$A\mathbf{F} = (a_i^k)\mathbf{F} = \mathbf{V} \qquad (1.1)$$

where A is an mxn' assignment matrix consisting of entries a_i^k (*4,5*). Let us say that the entry a_i^k assumes a value 1 for single-path assignment when an O-D pair k uses a particular link i, and 0 otherwise. **F** is a variable vector of n' entries, each of which is F^k, where F^k is the kth O-D demand to be estimated. **V** is a vector of link counts, consisting of m observations in the sample, each of which is V_i, where V_i is the ith link count. In the more general case of multipath assignment, a_i^k assumes a fractional value or 0. Thus for q-path traffic-assignment procedures, **F** will assume $qn' = n$ entries, with each F^k replicated q times for the q copies of the traffic assignment. Likewise, the A matrix is expanded to m by n in dimension, with each column replicated q times corresponding to the percentage of the O-D demand that follows a particular path.

Viewed in this light, the estimation of O-D demands becomes a matrix-inversion problem: $\mathbf{F} = A^+\mathbf{V}$, where A^+ is the generalized inverse of A. For an $mxnA$ matrix ($m < n$) of rank m the generalized inverse A^+ is simply $A^T(AA^T)^{-1}$, where both $(\mathbf{V} - A\mathbf{F})^T(\mathbf{V} - A\mathbf{F})$ and $\mathbf{F}^T\mathbf{F}$ are minimized. Here, the first dot-product is the deviation between observed and estimated link counts, following a typical least-square approach (*6*). The second is the sum of squares of F^k. For a fixed sum of F^k's (or F, the total number of trips in the study area), the minimization of $\mathbf{F}^TF$ yields $F^k = F/n'$ or an equalized set of O-Ds, which does not necessarily minimize the first dot product.

It is interesting to note that the entropy-maximization formulation of the O-D estimation problem (*7*), namely

$$\max W = \frac{F!}{\prod_{k=1}^{n} (F^k)!} \qquad (1.2)$$

also yields the same solution for a given F. Both give rise to an equalized set of O-D demands (*8*). This is an interesting finding inasmuch as the two approaches are among the most common methods of O-D estimation.

Between the generalized-inverse and entropy-maximization formulations, there has been some debates as to the best way to estimate O-Ds. Even though the matrix-inversion method appears simple, it was found through extensive experimentation that round-off errors during the computational process can be large (*9,10*). Moreover, generalized inversion sometimes yields negative solutions, which have no physical meaning in the context of the problem discussed here (*8*). The entropy-maximization method, on the other hand, has the conceptual appeal of obtaining the most likely O-D pattern, yet the process to operationalize Equation 1.2, and the results

Y. Chan, Department of Operational Sciences, Graduate School of Engineering, Air Force Institute of Technology, Wright-Patterson Air Force Base, Ohio 45433. M. Y. Rahi, Associated Traffic Consultants, 99 S. Chester Avenue, Suite 200, Pasadena, Calif. 91106.

are far from perfect. For example, it has the tendency to "lock up" in the slightest presence of data inconsistency, which introduces infeasibility into the mathematical program. Chan et al. (*4*) and Xu and Chan (*11,12*) pointed out a more robust and more accurate algorithm for solving large scale O-D estimation problems. Maher (*13*) confirmed previous findings that both the maximum entropy model and its cousin, the information-minimization model, produce counterintuitive results. Hamerslag and Immers (*14*) pointed out some severe limitations of both entropy maximization and information minimization models. Yang et al. (*15*) showed that a constrained least-square algorithm consistently yielded a more accurate and more reliable O-D estimate.

In choosing among O-D estimation methods, it is important to keep in mind the versatility consideration. For example, can the technique be easily applied to a number of cities with minimal data collection beyond link counts? In this context, universal parameters embedded in the algorithm that are transferable between cities are sought. Extensive-experimentation with graph-theoretic parameters suggests that there are few commonalities among the network taxonomy (as represented in assignment matrices A) between cities (*8*). A linear network representing a transportation corridor, for example, appears to yield more consistent total number of trips for external-external, internal-external, and external-internal movements as compared with internal-internal movements. However, no satisfactory explanation can be found to account for this. As another example, eigenvalues of the assignment matrix A have the strong appeal of characterizing the natural frequency of the network structure. However, when the matrix is not square in dimension, which is the rule rather than the exception, eigenvalues are often not available.

In view of travel-time budget theories, the authors identify the trip-distribution curve as one of the few transferable parameters among cities of similar size and urban structure (*2,3*). If an O-D estimation algorithm can take full advantage of this transferable parameter, it is a more serviceable technique inasmuch as it requires minimal site-specific data collection.

In reviewing Equations 1.1 and 1.2, one can see that there is no obvious relationship between the trip-distribution curve and generalized inverse of the assignment matrix because matrix inversion is simple an algebraic computational procedure. On the other hand, previous research by Chan et al. (*4*) indicates that there may be links between entropy maximization and trip-distribution curves. Trip-distribution curves—either borrowed or locally collected—can serve as another set of input data for this estimation approach. The payoff for imbedding the trip-distribution curve within entropy maximization appears high.

ENTROPY MAXIMIZATION APPROACH

If W is the entropy function as shown in Equation 1.2, it is typical to take its logarithm W' as a first step of maximization:

$$W' = \log F! - \sum_{k=1}^{n} \log F^k! \tag{2.1}$$

Using Stirling's approximation and after simplification, the maximum entropy formulation can be expressed as

$$\max Z = - \sum_{k=1}^{n} (F^k \log F^k - F^k) \tag{2.2}$$

subject to the link counts V_i:

$$\sum_{k=1}^{n} a_i^k F^k = V_i \qquad \forall\, i \tag{2.3}$$

and the trip-frequence distributions F_c:

$$\sum_{k=1}^{n} P_c^k F^k = P^k(C)F = F_c \qquad \forall\, c \tag{2.4}$$

where

$$P_c^k = \begin{cases} 1 \text{ If } F^k \text{ is of duration } C\ (C - \Delta C \le C \le C + \Delta C) \\ 0 \text{ otherwise} \end{cases} \tag{2.5}$$

F_c represents the total trips of duration C; and $P^k(C)$ is the probability of a trip k being C min long.

In this light, $(P_c^k) = P$ can be thought of as a *pxn* matrix similar to the *mxnA* matrix, with p being the number of travel-cost intervals defined for the trip-distribution curve. To all these is added the nonnegativity constraint, $F^k \ge 0 \quad \forall\, k$. Notice the given F in Equation 2.5 can either be supplied exogenously (collected locally) or generated endogenously (from local link counts and borrowed trip distribution). This will be discussed further when the algorithm is explained in detail.

The Lagrangian for this constrained optimization problem is

$$L = - \sum_{k=1}^{n} (F^k \log F^k - F^k) - \sum_{i=1}^{n} \lambda_i \left(\sum_{k=1}^{n} a_i^k \mathrm{F}^k - \mathrm{V}_i \right) - \sum_{c=1}^{p} \lambda_c \left(\sum_{k=1}^{n} P_c^k F^k - F_c \right) \tag{2.6}$$

The symmetry between the second and third terms above clearly shows the suitability of trip-distribution data as supplement to link counts. A typical calculus solution to this Lagrangian yields.

$$F^k = \exp\left(- \sum_{c \in H} \lambda_c \right) \exp\left(- \sum_{i \in K} \lambda_i \right) \tag{2.7}$$

where the summation is carried over all links i that carry flow between the O-D pair k, denoted here as the set K and all trip durations C that pertain to O-D pair k, H.

Now setting

$$\frac{\exp(-\lambda_i)}{V_i} = v_i \qquad \forall\, i \in K$$

$$\frac{\exp(-\lambda_c)}{F_c} = u_c \qquad \forall\, c \in H \tag{2.8}$$

results in

$$F^k = \left(\prod_{i \in K} v_i V_i\right) \prod_{c \in H} u_c F_c \tag{2.9}$$

Obviously, in the case of all-or-nothing single-path assignment, the set H has a membership of one. This method of estimating O-D demands yields the conventional multiproportional product-form solution, except that the explanatory variables include trip-frequency parameters—specifically the Lagrange multiplier λ_c defined for each trip duration C.

To arrive at a satisfactory solution to F^k is no easy task, as many researchers have labored continuously during the last 2 decades on this problem. First, the objective function of the mathematical program as formulated in Equations 2.1–2.5 is nonlinear, and it is not strictly concave in F^k. Therefore, it does not necessarily have a unique solution in terms of the O-D variables F^k's (*16*). Besides, multiple optima in terms of nonunique O-D specific link volumes and path routings exist. This nonuniqueness is well known among researchers because the underlying problem, that of finding an O-D matrix that produces the observed link flows and trip distribution, is underspecified. In other words, numerous O-D matrices can produce a given set of observed link counts and a specified trip-distribution curve. The choice between these alternative matrices has to be based on additional criteria. Finally, numerical intricacies are involved in solving a nonlinear programming problem as formulated by Equations 2.1–2.5. Most hill-climbing algorithms are sensitive to the redundancies and inconsistencies within and between constraints shown in Equations 2.3 and 2.4. (*14*).

MATHEMATICAL PROGRAM BASED ON l_p-APPROXIMATION

To overcome the shortcomings of traditional approaches such as entropy-maximization, the comprehensive set of criteria that the estimated O-D solution is to satisfy, including the additional criteria that may guarantee convergence and solution uniqueness, are reviewed. The authors have already mentioned that the O-D estimation problem has to minimize the deviations between the observed link volumes and the estimated values. This can be related to the l_p-approximation methods ($p = 1, 2, \ldots \infty$), in which deviations (between observed and estimated values) are minimized according to some predefined representation of norm vectors. l_p-norms represent one of the most general ways to measure deviation of the estimates from the observed values. For example, the l_1-approximation will minimize the sum of the absolute deviations (*16,17*):

$$l_1\text{: } \min\|\mathbf{V} - \mathbf{AF}\|_1 = \min \sum_{i=1}^{m} \left| V_i - \sum_{k=1}^{n} a_i^k F^k \right| \tag{3.1}$$

The l_2-approximation, on the other hand, minimizes the absolute value of the sum of the squares of the deviation (i.e., the traditional least square solution):

$$l_2\text{: } \min\|\mathbf{V} - A\mathbf{F}\|_2 = \min\left[\sum_{i=1}^{m} \left(V_i - \sum_{k=1}^{n} a_i^k F^k\right)^2\right]^{1/2} \tag{3.2}$$

Finally, the l_∞-approximation, also known as the Chebyshev approximation, minimizes the maximum of the absolute deviations (*18*):

$$l_\infty\text{: } \min\|\mathbf{V} - A\mathbf{F}\|_\infty = \min \max_{1 \le i \le m} \left| V_i - \sum_{k=1}^{n} a_i^k F^k \right| \tag{3.3}$$

Any one of the l_p-norms may be viewed as an objective function. By setting the maximum allowable deviation for link-volume replication, bounds are placed on the accuracy of the estimation—a desirable feature of a solution algorithm of this kind.

A Chebyshev approximation similar to Equation 3.3 may be written for trip-distribution replication:

$$l'_\infty\text{: } \min\|\mathbf{F}_c - P\mathbf{F}\|_\infty = \min \max_{1 \le c \le p} \left| F_c - \sum_{k=1}^{n} P_c^k F^k \right| \tag{3.4}$$

The same applies to l_1 and l_2 norms as well. The properties of Chebyshev approximation, are demonstrated in the following simple nonnetwork case:

$$l''_\infty\text{: } \min \max|F^k - F_a^k|$$

$$s.t. \sum_{k=1}^{n'} F^k = F$$

$$F^k \ge 0 \qquad \forall\, k \tag{3.5}$$

All F_a^ks are further assumed to be equal. It can be shown quite easily that the solution is $F^k = F/n'$ (i.e., all estimates are equalized as observed in both the entropy model and the generalized inverse model). It can be seen, therefore, that the l_∞-approximation plays a similar role as entropy and inverse models, but it does much more. Suppose F_a^ks are not equal. The estimates $F^k = F_a^k$ as long as $\Sigma_k F_a^k = F$.

On the other hand, if $\Sigma_k F_a^k \neq F$

$$F^k = |F_a^k - \Delta F| \tag{3.6}$$

where $\Delta F = (1/n')|F_a - F|$. It can be seen, therefore, that the l_∞-approximation tracks the observed O-D values instead of simply equalizing the grand sum F, which is a highly desirable property.

At this point, it appears desirable to use Equation 3.5 as a criterion for measuring the performance of the model. However, the base O-D's $F^k{}_a$s are often not available. Even if they are, it is not clear whether the base O-D should be mimicked. For these reasons, it is not an operational objective function. A more practicable approach is to look at link and trip-distribution reproduction. Thus with Equations 3.3 and 3.4 as the major solution criteria, an optimization model can be set up with this additional constraint beyond Equations 2.3 and 2.4 if desired:

$$\sum_{k=1}^{n} |F^k - F_a^k| \le X \tag{3.7}$$

This constraint ensures that the estimated O-Ds are not very different from the base O-Ds, F_a^k. Specifically, one limits the maximum total deviation to be X. Similarly, a second constraint can be set up to limit the deviation between the user-optimizing link travel-cost at V_i and the estimated total travel costs to Y.

$$\sum_{i=1}^{m} \int_0^{V_i} c_i(x)\, dx - \sum_{k=1}^{n} C^k F^k/q \leq Y \tag{3.8}$$

Notice that in this constraint, $c_i(x)$ stands for the link travel-cost function and C^k stands for the estimated multipath travel-cost between the O-D pair k (*19*). In order to operationalize this constraint, link counts will have to be collected for the entire network on all m links.

Constraint Equation 3.7 may or may not be effective depending on whether a base O-D trip demand matrix is available. Equation 3.8, while more readily enforceable and well correlated to O-D reproduction (*4*), is often too aggregate a measure of solution accuracy in the judgment of the authors, since many different O-D matrices can give rise to the same deviation Y. For all practical purposes, link count reproduction accuracy (or l_∞ in Equation 3.3) and trip-frequency replication (Equation 3.4) are the only tangible measures of algorithmic convergence.

A new mathematical program is proposed with the objective functions 3.3 and 3.4. The program's first order conditions—which require (among others, such as Equations 3.7 and 3.8) that the link counts and trip-distribution curves be reproduced–are to be met within some convergence tolerance, rather than exactly. Thus, Equations 2.3 and 2.4 in the entropy maximization formulation effectively turned into objective functions. Furthermore, an iterative descent-gradient method is proposed for the solution of these objective functions—rather than an ascent method for 2.1.

To show how the solution to the surrogate mathematical program actually yields a solution to the original O-D estimation problem is no easy task. For that matter, researchers have been struggling with this problem, including those who work with the traditional entropy approach. Many of the arguments would have to be less than rigorous. First, to the extent that some of the widely disseminated formulations such as matrix inversion yields a least square solution $(\mathbf{V} - A\mathbf{F})^T(\mathbf{V} - A\mathbf{F})$, objective function 3.2 and its generalization 3.3 are plausible surrogates, following the arguments made in Equations 3.5 and 3.6. It is a simple extension to cover the minimization of $(\mathbf{F}_c - P\mathbf{F})^T(\mathbf{F}_c - P\mathbf{F})$ as well. If desirable, one can view this as a disaggregation of the entropy objective function shown as Equation 2.6, wherein the second and third terms are taken as the two objective functions to be minimized; the first term may be taken care of implicitly by constraint 3.7 and the general properties of l_∞-norm as shown in Equation 3.5.

Second, the multiproportional product solution of Equation 2.7 strongly suggests gradient algorithms, in which the Language multipliers λ_i and λ_c serve as weights placed on the relative importance of link count reproduction or trip frequency reproduction during optimization in Equation 2.6. Lagrange multipliers are interpreted in this case as the extent to which Equations 2.3 and 2.4 are satisfied, just as the multiobjective optimization algorithm involving Equations 3.3 and 3.4 tries to trace the noninferior solutions.

In summary, the mathematical program proposed to solve includes Equations 3.3, 3.4 and 1.1 of the original problem. To the extent that the entropy-maximization and matrix-inversion paradigm is a widely disseminated description of the original O-D estimation problem, the authors have tried to show the relationship between their formulation, the matrix inversion, and the entropy formulations. No attempt has been made to show that the formulation will yield a solution such as Equation 2.9 which by itself is an approximation. The authors' approach is much more fundamental, in that the original O-D estimation problem is stated in terms of l_p-approximation, where the quantifiables such as link counts and frequency distributions are to be replicated. After some lengthy discussion, the authors finally recommended $p = \infty$, which echoes the intuitive requirement to minimize the worst deviations from the observed and the most likely estimate interpretation of entropy models, subject to the network geometry constraint on flow (Equation 1.1). In the following section, it will be shown that aside from a regular multi-objective linear programming package, a more efficient gradient algorithm can be readily put forth to solve this minimax programming problem consisting of two objective functions (3.3 and 3.4) and a linear constraint (1.1). Also the optimization criteria in the algorithm are equivalent to and more encompassing than the generalized inversion and entropy approaches.

ALGORITHM

Learning from the computational experiences of existing solution algorithms (*4,11,12,14,20–22*), the following iterative gradient algorithm is suggested for the l_∞-norm minimization model:

Initialization

The iterative algorithm can be started by setting the iteration counter s to zero ($s = 0$). Then the following is defined.

$$F^k(0) = F^k_a \tag{4.1}$$

where the base O-Ds (such as an old O-D matrix) are available. Alternatively,

$$F^k(0) = \frac{\sum_{i=1}^{m} a_i^k F_i^k}{\sum_{i=1}^{m} a_i^k} \tag{4.2}$$

for the situation where link counts are the only information available. Here $F_i^k = V_i / \Sigma_{k=1}^{m} a_i^k \quad \forall\, i \in K$. Finally,

$$F^k(0) = \frac{\sum_{i=1}^{m} a_i^k F_i^k}{\sum_{i=1}^{m} a_i^k} = \frac{\sum_{i=1}^{m} a_i^k P_i^k V_i}{\sum_{i=1}^{m} a_i^k} \tag{4.3}$$

where the trip-distribution curve F_c is available in addition to link counts. In Equation 4.3, $P_i^k = a_i^k P^k(C) / \Sigma_{k=1}^{n} a_i^k P^k(C)$, and $P^k(C)$ stands for the probability that trip k has the same travel cost C as read from the trip-frequency distribution F_c. Extensive computations by Chan et al. (4) and Xu and Chan (11,12) show that of all three initialization procedures, the last one (Equation 4.3) is the most effective.

This result is not surprising since Equation 4.2 is nothing more than the inverse of a regular assignment according to Equation 1.1. Its fundamental structure is related to entropy maximization, which yields $F^k = F/n'$ when $a_i^k = 1$ for all i and k. In other words, imagine a network in which sampled links carry flows from every O-D pair, then Equation 4.2 reduces to $F^k = F/n^1$. This applies, for example, to F traffic counts on a freeway section, from which the pertinent entrance and exit ramps of the traffic are to be inferred. The result is an equal amount of traffic for each entrance and exit ramps. Another way of saying this is that when network geometry is totally set aside, equal O-Ds would be the most likely inference from entropy maximization. When network geometry is taken into account, Equation 4.2 will result.

In Equation 4.3, O-D inference is assisted by the knowledge of the trip-distribution curve. Thus, not only does network information get used, but trip-distribution information is taken into account as well. To see this more clearly, consider the close cousin of entropy maximization: the information minimizing model (14):

$$l_p\text{: } \min \Sigma_k F^k \ln(F^k/F_a^k) \tag{4.4}$$

such that

$$\Sigma_k b_j^k F^k = d_j \tag{4.5}$$

where b_j^k can assume the form of a_i^k or P_c^k, and d_j can assume the form of V_i or F_c, as shown in Equations 2.3 and 2.4. (Setting $F^k = 1$ for all k, or when there is no prior information, results in the familiar entropy maximization model.)

Solution of this model yields

$$F^k = F_a^k R_0 \prod_j R_j \tag{4.6}$$

where $R_0 = \exp(-1)$ and

$$R_j = \exp(\lambda_j \Sigma_k b_j^k) \tag{4.7}$$

where

$$\exp(\lambda_j \Sigma_k b_j^k) = \begin{cases} \exp(\lambda_c) \\ \exp\left(\sum_{j \in K} \lambda_j\right) \end{cases} \tag{4.8}$$

Notice this is the same as Equation 2.7 in the case of single-path assignment, except for the sign which simply reflects the difference between information minimization and entropy maximization. Most important, rearranging the multiproduct form of Equation 4.6 into $F^k = F_a^k R_0 R_c R_i$ shows that just like a base O-D matrix F_a^k (Equation 4.1), R_c is simply another piece of prior information that can assist in more accurate determination of F^k. Although links between models are established, this formulation further accentuates some of the shortcomings of information/entropy formulations. First, Equation 4.5 is not defined for $F_a^k = 0$. Second, inconsistencies in specifying Equation 4.5 will "derail" any solution algorithms for the nonlinear program, as mentioned previously. This will be demonstrated in an example. The l_∞-approximation algorithm advanced here is rid of these problems.

Iteration

After initialization, algorithmic steps can be written for the remaining iterations of the algorithm. In the following steps, the iteration index s is set to one to start the gradient algorithm.

Step 1. The various link volume estimates $V_i(s)$ are determined from a traffic assignment in accordance with Equation 2.3 (or 1.1 of the original formulation):

$$\sum_{k=1}^{n} a_i^k F^k(s) = V_i(s) \qquad i = 1, 2, \ldots, m$$

Step 2. Modify trip-probability Equation 2.4 to compute the total number of trips of duration C, $F_c(s)$, from a given trip distribution:

$$\sum_{k=1}^{n} P_c^k F^k(s) = P^k(C)F(s) = F_c(s) \qquad c = 1, 2, \ldots, p$$

where $F(s)$ is the sum of estimated trips during the current sth iteration, and $F_c(s)$ is the sum of all $F^k(s)$ that belong to interval c. Instead of an extogenously determined F in Equations 1.2 and 3.5, this algorithm has the option to make F self-adjusting. The result is that F^k has as much a tendency to "track" the F_a^k as to equalize among themselves.

Step 3. The link volume estimates $V_i(s)$ are compared with the observed volumes in the form of an error ratio:

$$R_i^k(s) = \frac{V_i}{V_i(s)} a_i^k \qquad i = 1, 2, \ldots, m \tag{4.9}$$

Likewise, for the trip distribution $F_c(s)$.

$$R_c^k(s) = \frac{F_c}{F_c(s)} P_c^k \qquad c = 1, 2, \ldots, p \tag{4.10}$$

A single composite error ratio can be obtained for all links carrying flows between O-D pair k if desired:

$$Q^k(s) = \frac{\left[\sum_i a_i^k R_i^k(s) + \sum_c P_c^k R_c^k(s)\right]}{\left(\sum_i a_i^k + \sum_c P_c^k\right)} \qquad \forall\, k \tag{4.11}$$

Step 4. The composite error ratio is then used as an adjustment factor to the pertinent O-D estimate F_i^k from the previous iteration:

$$F_i^k(s + 1) \leftarrow Q^k(s)F_i^k(s) \qquad \forall\, i, k \tag{4.12}$$

where the iteration index s is now incremented by 1.

These four steps are applied to all links with an observed flow and repeated successively for convergence. The iterative steps yield a new set of estimates each time not only for the link volumes, but also for the corresponding O-D estimates, $F^k(s)$, according to Equation 4.3 (when $s > 0$). Convergence is obtained when the O-Ds, on assignment on the network, reproduce the observed link volumes and trip-distribution curve faithfully (among other measures specified here).

Step 5. Algorithmic convergence is gauged by observing whether an error limit is kept, as indicated by a subset of Equations 3.3 through 3.8, whichever apply. Representative of such limits is the operational measure of "a specified number of links and trip-probability equations that exceed the maximum error of 5 percent." Ten percent of the links or trip frequency in violation is a typical specification. If a sufficiently small error limit is specified for both link count and trip distribution, the min-max objective functions of Equations 3.3 and 3.4 are realized. Although used here, an alternative algorithm is to rewrite Equation 4.11 "in series" instead of "in parallel:"

$$Q^k(s) = R_i^k(s)R_c^k(s) \qquad \forall\, k \qquad i = 1, 2, \ldots, m; \quad c = 1, 2, \ldots, p \tag{4.13}$$

which, aside from the symmetry, has a product form consistent with the analytical solution shown as Equation 2.9. This version of the algorithm accentuates the multiproportional information update emphasis. In a disaggregate fashion, for each link i, each trip-duration interval c, and each O-D pair k, an adjustment based on the most recent information is applied, as shown in Equations 4.9 and 4.10, similar to Equations 2.7 and 2.9. This update is performed in such a way that it involves the prior-information of Equation 4.3 in each iteration as well.

Both the "parallel" and "series" algorithms are simple (with Equation 4.13 being disaggregate and more "elegant"). It will be shown that the computational simplicity also results in a fairly efficient algorithm, except in situations in which data are inconsistent. The computational complexity of this algorithm is $O\ (mnp)$, which is far more efficient than regular multiobjective linear programming codes. Furthermore, it is robust enough to converge in spite of any inconsistencies that may be in Equations 2.3 and 2.4—something that cannot be claimed by other mathematical programming packages. In effect, this algorithm is specifically designed to exploit the sparsity of the A matrix of 0-1s, which is prevalent in the authors' model. All the summations over i and c in Equations 4.3 through 4.11 have very few nonzero entries, thus affording a compact data structure and efficient calculations.

Of the two ways to handle the multiple objectives of minimizing link-error and trip-distribution-error, "in-series" algorithms give sequential weights in each iteration to both "link error" and "trip-distribution error," whereas "in-parallel" algorithms apply an aggregate weight to each O-D reflecting the relative number of observations in link counts vis-a-vis trip-distribution frequencies. For example, a balanced gradient is applied in the special case when $a_i^k = P_c^k = 1$ for all i, c, and k. This is the reason for the choice of "in parallel" over the "in series" version, namely in its capacity to adaptively adjust the gradient. The weights among the two objective functions as shown in Equations 4.11 and 4.13 can be related to the Lagrange multipliers of Equation 2.6. Perhaps the best way to see this is through a comparison between Equations 2.7 and 4.12, in which the O-D's are shaped incrementally over all observation of V_i and F_c.

In summary, a gradient algorithm to solve the multiobjective minimax program has been outlined. It represents, in the opinion of the authors, a modest step forward. Not only is the O-D estimation problem viewed in a different light by a unifying l_p-approximation framework, but an operational algorithm is designed to perform the computation required of such a multicriteria optimization problem. Such an algorithm is versatile enough to examine the whole family of l_p-approximations of more than one figure-of-merit, from the familiar $p = 1$ and 2 cases to the intuitively satisfying $p = \infty$ case. Recently, Schneider and Zenios (*23*) related an O-D estimation algorithm such as the one above to the general problem of "matrix balancing" and elaborated on the efficiency of the algorithmic variety employed here.

EXAMPLE

An illustration of the algorithm using a hypothetical five-zone network (Figure 1 and Table 1) is helpful. Without loss of generality, let us say that all of the 16 links in the figure are bidirectional and uncapacitated ($q = 1$). The probability method of initialization (Equation 4.3) was used. A target O-D matrix, the minimum-time paths, the trip-frequency probabilities, the seven observed link counts, and two turning movements are shown. There are 10 O-D pairs (one-way) and 9 observed-flow data (not all of which are independent, notably the two turning movements are the same as link flows. For that reason, the turning movements are simply redundant information.) As an illustration only, this first case is a "determinate" system where $m = n$. The initial step of the algorithm is the conversion of various trip-duration probabilities into normalized O-D share allocations for each link volume (Equation 4.3). This is conducted in Table 2, where the initial allocation

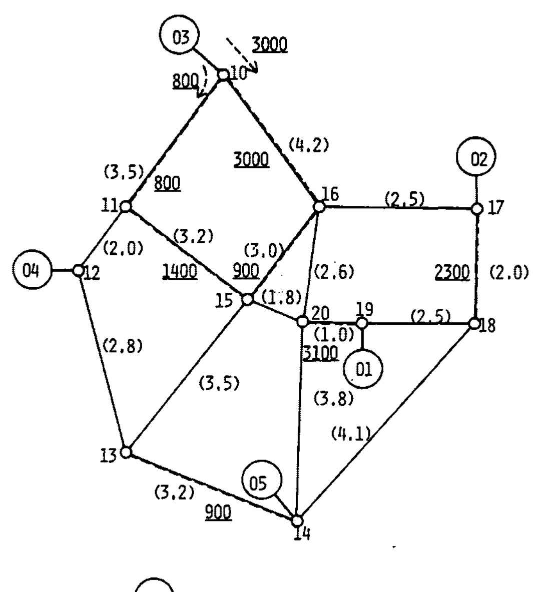

FIGURE 1 Five-zone example network.

TABLE 1 Data for Five-Zone Network

Zone Pair k	O-D Zones*	O-D Path k	Skim Tree C^k	Trip Freq. P^k%	Target O-D, F^k	Final Est. O-D, $F^k(7)$
1	1 - 2	19,18,17	5	27	1,100	1,139
2	1 - 3	19,20,16,10	8	15	1,000	848
3	1 - 4	19,20,15,11, 12	8	15	500	653
4	1 - 5	19,20,14	5	27	1,600	1,575
5	2 - 3	17,16,10	7	15	1,500	1,488
6	2 - 4	17,16,15,11, 12	11	14	900	804
7	2 - 5	17,18,14	6	29	1,200	1,179
8	3 - 4	10,11,12	6	29	800	806
9	3 - 5	10,16,20,14	11	14	500	674
10	4 - 5	12,13,14	6	29	900	907

* A symmetric O-D matrix is assumed. Thus an O-D pair p-q (p<q) stands for both zone pairs p-q and q-p.

of link volumes is made to arrive at F_i^k. Notice that there is more than one estimate of each O-D volume, as pointed out earlier in the description of the algorithm. According to Equation 4.3, these different estimates from the different link volumes are averaged, yielding $F^k(0)$. The algorithm proceeds to the iteration phase in which the five-step procedure is executed. Such a procedure is shown in Table 3, where the allocated link volumes $F^k(s)$ are revised according to both the link-error ratio (Equation 4.9) and the trip-distribution error ratio (Equation 4.10). The adjustment using the error ratios results in a revised set of O-D allocations from link counts, hence revised average O-D estimates $F^k(s)$, in each iteration. When the 5 percent error/10 percent violation convergence criteria are met (Step 5 of the algorithm), the average O-D

TABLE 2 l_∞-Norm Algorithm

									Iteration One				Iteration Two		
Link i	Link in Fig. 1[c]	Obs. Vol. V_i	Zonal Pair k	Trip Freq. P^k%	Normal P_i^k %	Vol. Alloc. $F_i^k(0)$	Avg. O-D $F^k(0)$	Est. Trips of Dur. C, $F_c(0)$	Adj. ratio $R_i^k(1)$	Adj. ratio $R_c^k(1)$	Avg. est. $F_i^k(1)$	$F_c(1)$	Adj. ratio $R_i^k(2)$	Adj. ratio $R_c^k(2)$	Avg. est. $F_i^k(2)$
1	10-11[a]	800	8(3-4)	29	100.0	800	800	2876	1.000	0.995	798	2880	1.002	0.999	799
2	10-16[b]	3000	2(1-3)	15	34.1	1023	919	1488	1.036	0.881	869	1490	0.994	0.929	849
			5(2-3)	15	34.1	1023	1023	1488		1.455	1274	1490		1.170	1378
			9(3-5)	14	31.8	954	954	1389		0.797	874	1390		0.874	817
3	11-15	1400	3(1-4)	15	51.7	724	770	1488	0.899	0.881	701	1490	0.988	0.949	684
			6(2-4)	14	48.3	676	788	1388		0.797	716	1390		0.874	715
4	13-14	900	10(4-5)	29	100.0	900	900	2876	1.000	0.995	898	2880	1.002	0.999	898
5	15-16	900	6(2-4)	14	100.0	900	788	1388	1.142	0.797	716	1390	1.256	0.874	715
6	17-18	2300	1(1-2)	27	48.2	1109	1109	2678	1.000	1.039	1130	2682	0.992	1.026	1140
			7(2-5)	29	51.8	1191	1191	2876		0.995	1188	2880		0.999	1183
7	19-20	3100	2(1-3)	15	26.3	816	919	1488	0.982	0.881	869	1490	1.015	0.949	849
			3(1-4)	15	26.3	816	770	1488		0.881	701	1490		0.949	684
			4(1-5)	27	47.4	1469	1469	2678		1.039	1484	2682		1.026	1515
							9923				9932				9978

[a] same as turning movement from 3-10 to 10-11
[b] same as turning movement from 3-10 to 10-16
[c] Link (i,j), where i<j, is bi-directional. It stands for both (i,j) and (j,i).

TABLE 3 Results for 42-District York Network

		Borrowed Trip Probabilities			Site-specific Trip Probabilities		
Violation Limit	Criteria	Trip-probability equation not used*	Trip-probability equation used O-D sum known*	O-D sum unknown*	Trip-probability equation not used*	Trip-probability equation used O-D sum known*	O-D sum unknown*
20% link violations	Z_1	0.707 (0.577)	0.661 (0.668)	0.577 (0.671)	0.629 (0.607)	0.555 (0.560)	0.539 (0.543)
	Z_2	0.027 (0.026)	0.011 (0.021)	0.019 (0.018)	0.023 (0.032)	0.018 (0.018)	0.017 (0.017)
	Z_3	0.066 (0.128)	0.004 (0.182)	0.091 (0.175)	0.092 (0.106)	0.122 (0.109)	0.139 (0.134)
	D	-0.123 (-0.040)	-0.021 (0.019)	-0.109 (0.028)	-0.078 (-0.091)	-0.013 (-0.026)	-0.034 (-0.057)
	Std. dev.**	166 (222)	188 (316)	168 (320)	185 (192)	243 (223)	237 (214)
	No. of iterations	6 (3)	33 (12)	10 (11)	5 (3)	10 (9)	11 (9)
10% link violations	Z_1	0.708 (0.577)	0.658 (0.678)	0.584 (0.690)	0.630 (0.606)	0.573 (0.590)	0.549 (0.551)
	Z_2	0.019 (0.026)	0.005 (0.015)	0.006 (0.007)	0.015 (0.018)	0.004 (0.005)	0.006 (0.007)
	Z_3	0.065 (0.128)	0.020 (0.189)	0.084 (0.173)	0.092 (0.104)	0.108 (0.089)	0.137 (0.130)
	D	-0.117 (-0.040)	-0.008 (0.014)	-0.109 (0.033)	-0.072 (-0.085)	-0.035 (-0.008)	-0.036 (-0.053)
	Std. dev.**	170 (222)	191 (315)	168 (322)	190 (199)	243 (223)	235 (215)
	No. of iterations	8 (3)	51 (27)	38 (27)	7 (5)	49 (41)	56 (27)

* The first entry values correspond to smaller-city data set (curve borrowed from slightly larger city); the values in parentheses correspond to larger-city data set (curve borrowed from slightly smaller city).
** Standard deviation for observed O-D's is 284 (305).

estimates from that iteration are the final O-Ds, as shown in the last column of Table 1.

The same problem was solved by removing some of the link count and trip-frequency observations. Including only the strategically located link flows (17,18), (11,12) and (13,14) provided an underdetermined system in which $m < n$. The l_∞-approximation algorithm converged to similar solutions as the full rank example above (*24,25*).

Using both the full rank and an underdetermined input data, consisting of links (10,11), (11,15), (13,14), (15,16), (17,18) and selected trip-frequency distributions, the entropy formulation was again solved by a regular nonlinear programming package. The solution was consistent with the previous one by l_∞-approximation (*24,26*), although full-rank input and extraneous constraints tend to cause convergence problems (*24,25*). In view of these convergence problems, the objective function of the nonlinear programming problem was linearized and both separable programming (*25*) and the Frank-Wolfe algorithm (*27*) were used to solve the problem. In the case of separable programming, the algorithm was robust enough to yield fairly consistent solutions for both an underdetermined and full-rank input. In the case of Frank-Wolfe, including only seven link-count information yielded the same solution as previous algorithms. Adding trip-distribution information tended to cause nonconvergence, apparently due to inconsistency with the link-count equations.

These computational experiences, conducted in a control environment in a small network, confirm previous findings regarding the fragility of the entropy/information-based models, particularly with regard to input data. It further supports the serviceability of the l_p-approximation (particularly l_∞-approximation) algorithm in terms of its robustness and efficiency. This, together with similar findings elsewhere (*4,14,15,23*), point toward the focus of this paper: the role of additional trip-distribution information and the l_p-approximation algorithm. A set of experiments using a large-scale data set were carefully designed to address this in further detail.

EXPERIMENTS

The above algorithm was used to conduct a set of experiments. The experiments were intended to resolve three computational issues:

1. Between the use of the link-count-adjustment factor (R_l^k) and the trip-distribution-adjustment factor (R_c^k), does the latter enhance solution accuracy and algorithmic efficiency?
2. Between an outdated or borrowed trip distribution curve and a locally collected one, which will perform better?
3. How much is compromised should one minimize site-specific data collection?

Experimental Design

To answer these questions, a complete evaluation and sensitivity analysis was performed on a real-world network of York, Pennsylvania. These controlled experiments were scientifically designed to evaluate the performance of the algorithm, particularly its ability to compute an entire family of l_p-norms over several criteria. One of the evaluation measures is the normalized deviation between the observed and estimated O-Ds following the constraint defined in Equation 3.7, where the observed O-Ds are available (since it is a controlled experiment):

$$Z_1 = \frac{\sum_{k=1}^{n} |F^k - F_a^k|}{\sum_{k=1}^{n} F_a^k} \tag{5.1}$$

Another evaluation measure is the normalized deviation between estimated and observed link volumes, following the objective function defined in Equation 3.1:

$$Z_2 = \frac{\sum_{i=1}^{m} \left| \sum_{k=1}^{n} a_i^k F_i^k - V_i \right|}{\sum_{i=1}^{m} V_i} \tag{5.2}$$

Although not used here, a similar criterion can be defined for the compliance with a local trip-distribution function (but not necessarily with a borrowed curve).

A third measure, related to Equation 3.8, documents the difference between the observed versus estimated total costs (in vehicle-hours of travel). The following is a special single-path case of constraint 3.8 when a_i^k assumes 0-1 values (instead of fractional values).

$$Z_3 = \frac{\sum_{i=1}^{m} c_i V_i - \sum_{k=1}^{n} C^k F^k}{\sum_{i=1}^{m} c_i V_i} \tag{5.3}$$

The controlled nature of the experiment allows Z_3 to be computed even though not all link counts are used for O-D estimation.

The fourth is a measure of the difference between sums of the estimated O-Ds and the observed O-Ds. This allows one to assess whether the algorithm overestimates or underestimates the total number of O-Ds:

$$D = \frac{\sum_{k=1}^{n} F^k - \sum_{k=1}^{n} F_a^k}{\sum_{k=1}^{n} F_a^k} \tag{5.4}$$

The spread of the estimated O-Ds is compared with the observed via the standard deviation (σ) statistic. This allows one to gauge the uniformity of the O-D estimates inasmuch as both the generalized-inverse and entropy-maximization procedures tend to equalize F^k's. A small value of σ, for example, shows uniformity among O-D estimates for the l_p-approximation algorithm and vice versa.

Finally, all experiments are evaluated by the rate of convergence, defined as the number of iterations required to reach a specific error limit. An example of such a limit is the percentage of links that are outside the error tolerable for link volume estimates. We will recall that this termination criterion realizes the l_∞-approximation as shown in Equations 3.3 and 3.4—particularly the two first order conditions of this gradient algorithm.

As mentioned previously, each experiment is designed to compare the proposed algorithm with the version where the trip-probability equation is not used. When the trip-probability equations are used, two cases need to be tested: either the sum of the O-Ds is known or that it is not. In the former case, the equalizing property, as shown in Equation 3.5, is tested. In the latter case, the absence of such property is expected—all through the use of the standard deviation (σ) statistic. In the case of a site-specific trip-distribution curve being available, it is more likely than not that the O-D sum is also available. On the other hand, when a borrowed trip distribution curve is used, it is unlikely that such a sum is known.

The above experimental design is best illustrated by the five-zone example, where a locally collected trip-distribution curve is assumed available. To make the example interesting, it is assumed also that the total number of trips is not known a priori. The algorithm is iterated until no more than 10 percent of the link volumes and trip-probability equations exceed the 5 percent error. The thrust of comprehensive tests were performed in the York network, which consists of 42 districts, 101 nodes, and 861 symmetrical O-D pairs—a considerably large network for such experimentation. Although the violation limit is 10 percent for the five-zone example, both 10 percent and 20 percent are tested for the 42-district network.

To support the theme of the research, the authors experimented with trip-distribution transferability. Two curves are identified in York, the first representing an outdated distribution, the second, the current distribution:

$$P^k(C) = 15.82\, C \exp(-0.379C) \qquad R^2 = 0.975 \tag{5.5}$$

$$P^k(C) = 12.85\, C \exp(-0.353C) \qquad R^2 = 0.927 \tag{5.6}$$

Since the York metropolitan area has grown in population and development during the last 2 decades, the authors refer to the outdated curve as from the "smaller city data set," whereas the current curve is from the "larger city data set." Experiments were then performed on the current data set consisting of network geometry, base matrix, and sample counts using an outdated trip-distribution curve. Conversely, experiments were performed on the outdated data set (or smaller-city data), borrowing the current trip-distribution curve. Although the former set of experiments represents the common practice, the latter is also valid from an experimental design standpoint, in that both cases represent borrowing a distribution curve from a "similar" city.

Notice a trip-distribution curve is involved in the initialization phase (Equation 4.3), even though it may not be included in the iterative phase. For this reason, there are two columns again under the heading "trip probability equation not used" in Table 3, corresponding to the smaller-city and larger-city curve, respectively, being used to initialize the algorithm.

Empirical Results

Notice in Table 3 that the proposed algorithm consistently gives an equally accurate and often a more accurate O-D and link-count reproduction (Z_1 and Z_2) when the O-D sum is not known a priori. This is gratifying in that the objective of minimizing site-specific data requirement is achieved, where the additional local information on the total-number-of-trips is not necessary to obtain quality algorithmic performance. Not only is the information superfluous, but its absence gives rise to more accurate O-D estimation than when it is collected. Instead of merely equalizing the estimated values (as in the case of matrix inversion and entropy maximization), the estimated O-Ds are now allowed to approximate the variability of the target O-Ds better.

Along this line, the results from experiments where a borrowed curve is used (Table 3) are comparable in accuracy to those where a curve is available locally. As long as a trip-distribution curve is employed, a 33 percent link-sampling rate as used in the experiments in Table 3 does not significantly compromise the O-D estimation accuracies when compared with the 100 percent sample. The 100 percent sample is not included here due to space limitations. The interested reader may consult work by Rahi (*8*) for this information. As it turns out, the algorithm becomes more efficient and converges faster with the 33 percent sampling rate because there are fewer inconsistencies to resolve. This finding reinforces the computer runs on the five-zone example and further supports the authors' claim that although the algorithm is robust enough to handle redundant data, it is much less data-hungry for the same degree of accuracy.

As suggested previously there is little advantage, if any, to gathering site-specific data, such as the total number of O-D trips. First, it introduces inaccuracy to the solution by equalizing O-Ds, as mentioned previously. Also, it tends to prolong the number of iterations before convergence is obtained in all cases. This is again a gratifying finding, saying that collecting irrelevant data does not only waste resources, it also harms the technical performance of the algorithm.

Because the O-D estimates are required to conform to a prescribed trip distribution, more prior information is imposed on the estimation process than other traditional algorithms and hence results in more heterogeneous O-D estimates that better approximate the base O-Ds. This is illustrated by Equation 4.3 and most particularly by Equation 4.5. The authors' claim, however, is highly predicated upon the shape of the trip-distribution curve. For example, a more peaked distribution curve from a smaller city tends to result in a much less uniform set of O-Ds (Equations 5.5 and 5.6, and Table 3). A more peaked curve also tends to result in a large O-D sum in general.

One point about the use of trip-distribution curves is quite clear. Should it be employed in O-D estimation, an accurate specification of the probability values is advisable for better overall algorithmic performance. This is true for all cases—whether the trip-distribution curve is borrowed, and irrespective of the violation limit set in the convergence criteria. Numerical round-odd errors in trip-distribution input tend to prolong algorithmic convergence because there are more inconsistencies to reconcile.

For the same reason, including trip-probability constraints typically prolongs the number of iterations required when compared with using link counts alone. It was found that the lower the error limit set, or the minimax objective functions are to be better achieved, the larger the number of iterations required to resolve these inconsistencies—as one would expect. For example, lowering from a 20 percent link violation rate to 10 percent dramatically increases the number of iterations by a factor of five. This is the price one pays for saving site-specific data-collection efforts. Irrespective of the increase, computation time is no more than a few minutes on an Amdahl V-816 because the computational complexity of such an algorithm is polynomial.

SUMMARY AND CONCLUSION

On the basis of the plethora of research on origin-destination estimation during the last 2 decades, the authors synthesize here an improved theory and algorithm that is a general version of entropy maximization, information minimization and matrix-inverse models. The objective is to estimate O-Ds with the least amount of site-specific data collection. Beyond the site-specific link counts, the authors wish to rely exclusively on generic data (i.e., data that can be borrowed from other communities of similar size and development structure or from data collected for the same community in a previous survey). Specifically, the trip-frequency or trip-length distribution curve is identified as the most promising piece of "transferable" information to supplement site-specific link counts.

An l_∞-approximation algorithm is synthesized on the basis of experiences with the widely disseminated generalized-inversion and entropy-maximization theories. The authors' approach takes advantage of their strengths, such as the analytical property of entropy-maximization, which readily allows for the inclusion of generic information such as trip-frequency curves in a multiproduct form. l_p-approximation methods $p = 1, 2, \ldots, \infty$ represent a more fundamental approach to modeling the original O-D estimation problem than the least-square assumption ($p = 2$) of generalized inverse. The result is a flexible, successive-approximation algorithm, assuming the multiproportional product form. In this multiobjective optimization model, adjustments to O-D estimates are made not only through link-count reproduction, but also trip-frequency reproduction. The latter represents the unique feature of the algorithm presented in this paper.

Care was exercised in the design of experiments, where the algorithm was compared with a version in which the trip-frequency information was not fully used. On the basis of testing of the 42-district York, Pennsylvania, network, it was found that the algorithm generally gives more accurate O-D and link-count reproductions. Furthermore, the use of borrowed trip-distribution curves yields equally accurate estimates as when a site-specific curve is available. Although one pays for this in terms of computer time, it is a gratifying result because site-specific data-collection effort, judged to by far be the much more expensive and onerous task, is in fact minimized. Inclusion of trip-distribution information in the O-D estimation algorithm and the relaxed requirement on O-D sum also tend to ameliorate commonly observed tendency for many algorithms to equalize the estimated O-Ds.

In formulating the l_p-approximation problem as a multiobjective optimization algorithm in which the link-counts and trip-distribution are to be replicated, the relative weights placed among these two objective functions are shown to be related to the Lagrange multipliers of the entropy formulation. Thus both the weight or Lagrange multiplier reflect the extent to which replication has been achieved. The l_∞-approximation algorithm was also shown to have similar optimality conditions as the familiar entropy-maximization and information-minimization models in that equalized O-Ds constitute the most likely estimates for a given O-D sum. In designing the experiments here, however, comparison with an entropy-maximization algorithm was considered in illustrative computation only. Extensive experimentation was performed and published in an earlier phase of this research effort (*4*), in which the serviceability of the present approach (with only link counts as input data) has been established. Also shortcomings of the entropy, information and inverse models—such as the tendency for the algorithm to "lock up" at the slightest trace of data inconsistency—have been adequately reported elsewhere in the literature.

It is obvious that more empirical work can be performed to fine tune the results reported here. The l_∞-approximation techniques should be further investigated as a way to solve the O-D estimation problem because the theoretical structure of such an approach is related to general multiobjective programming, with its many analytical properties. Furthermore, the authors' network problem will invariably result in a sparse tableau consisting of 0-1 entries. One should be prepared to exploit this data structure by clever solution algorithms, of which the one presented here may be a modest beginning.

ACKNOWLEDGMENTS

The authors would like to acknowledge the valuable inputs from colleagues and friends at Washington State University, Air Force Institute of Technology, and around the world, including but not limited to K. Ariyawansa, J. Khisty, C. Millham, and W. C. Xu. The authors benefitted from the helpful suggestions of the referees in several earlier drafts of this paper. The authors alone are responsible for the statements made in this paper.

REFERENCES

1. Ben-Akiva, M. Methods to Combine Different Data Sources and Estimate Origin-Destination Matrices. *Transportation and Traffic Theory* (N. Gartner and N. Wilson, eds.) Elsevier Science Publishing, New York, N.Y., 1987, pp. 459–481.
2. Chan, Y. and Ou, F. Equilibration Procedure to Forecast Areawide Travel, *Journal of Transportation Engineering*, American Society of Civil Engineers, Vol. 112, No. 6, 1986.
3. Zahavi, Y. The UMOT Project. Report DOT-RSPA-DPB-20-79-3. U.S. Department of Transportation, 1979.
4. Chan, Y., E. J. Regan, and W.-M. Pan. Inferring on Origin-Destination Matrix Directly from Network Flow Sampling, *Transportation Planning and Technology*, Vol. 11, Gordon and Breach, Science Publishers, Inc., United Kingdom, 1986, pp. 27–46.
5. Beil, D. *Matrixmodelle zur Ermittlung von Teilstromen aus Querschnz, Hsmessungen.* Doctoral dissertation. Universitat Karlsruhe, Karlsruhe, West Germany, 1979.
6. McNeil, S., and C. Hendrickson. A Regression Formulation of the Matrix Estimation Problem. *Transportation Science*, Vol. 19, No. 3, 1985, pp. 278–292.
7. Van Vliet, D., and L. Willumsen. Validation of the ME2 Model for Estimating Trip Matrices from Traffic Counts. *Proc., 8th International Symposium on Transportation and Traffic Theory*, University of Toronto, Canada, 1981.
8. Rahi, M. Y. *Estimation of Origin-Destination Trip Matrix from Link Traffic Information*. Doctoral Dissertation. Dept. of Civil and Environmental Engineering, Washington State University, Pullman, 1987.
9. Faddeev, D., and V. Faddeeva. *Computational Methods of Linear Algebra,* W. H. Freeman and Company, San Francisco and London, 1963, p. 159.
10. Noble, B. *Applied Linear Algebra*. Prentice Hall, Inc., Englewood Cliffs, N.J., 1969.
11. Xu, W. and Y. Chan. Estimating an Origin-Destination Matrix with Fuzzy Weights—Part I: Methodology, *Transportation Planning and Technology*, Vol. 17, 1993, pp. 127–144.
12. Xu, W. and Y. Chan. Estimating an Origin-Destination Matrix with Fuzzy Weights—Part II: Case Studies. *Transportation Planning and Technology*, Vol. 17, 1993, pp. 145–163.
13. Maher, M. J. Bias in the Estimation of O-D Flows from Link Counts, *Traffic Engineering and Control*, Dec. 1987, pp. 624–627.
14. Hamerslag, R. and L. H. Immers. Estimation of Trip Matrices: Shortcomings and Possibilities for Improvements. In *Transportation Research Record* 1203, TRB, National Research Council, Washington, D.C., 1989, pp. 27–38.
15. Yang, H., Y. Iida, and T. Sasaki. An Analysis of the Reliability of an Origin-Destination Trip Matrix Estimated from Traffic Counts, *Transportation Research*, Vol. 25B, No. 5, 1991, pp. 351–363.
16. Abdelmalek, N. Linear l_1 Approximation for a Discrete Point Set and l_1 Solutions of Over-determined Linear Equations. *Journal of Assoc. Comput. Mach.*, Vol. 18, 1971.
17. Barrodale, I., and F. Roberts. An Improved Algorithm for l_1 Linear Approximation, *SIAM Journal on Numerical Analysis*, Vol. 10, No. 5, 1973.
18. Barrodale, I., and C. Phillips. An Improved Algorithm for Discrete Chebyshev Linear Approximation. *Proc., 4th Manitoba Conference on Numerical Mathematics*, Manitoba, Canada, 1974.
19. O'Neill, W. A. A Micro Computer Model for Estimating Link Use Proportions for Trip Table Estimation Models. Presented at 69th Annual Meeting of the Transportation Research Board, Washington, D.C., 1990.
20. Gur, Y. J. Trip Table Estimation Based on Partial Volume Counts. Presented at 64th Annual Meeting of the Transportation Research Board, Washington, D.C., 1985.
21. Han, A., and E. Sullivan. Trip Table Synthesis for CBD Networks: Evaluation of the LINKOD Model. In *Transportation Research Record 944*, TRB, National Research Council, Washington, D.C., 1983, pp. 106–112.
22. Van Zuylen, H., and L. Willumsen. The Most Likely Trip Matrix Estimated from Traffic Counts. *Transportation Research,* 14B, 1980, pp. 473–476.
23. Schneider, M. H., and S. A. Zenios. A Comparative Study of Algorithms for Matrix Balancing, *Operations Research*, Vol. 38, No. 3, 1990, pp. 439–455.
24. Isensee, E. K., and Y. Chan. Optimizing Scout Locations in Forward Areas Along a Transportation Network. Working paper. Department of Operational Sciences, Air Force Institute of Technology, Wright-Patterson Air Force Base, Ohio, 1990.
25. Newhall, C., M. Wilmer, and Y. Chan. Determining Sensor Locations to Estimate Origin-Destination Movements. Working paper. Department of Operational Sciences, Air Force Institute of Technology, Wright-Patterson Air Force Base, Ohio, 1990.
26. Davidson, P. A., and Y. Chan. Locating Scouts on a Transportation Network. Working paper. Department of Operational Sciences, Air Force Institute of Technology, Wright-Patterson Air Force Base, Ohio, 1989.
27. Munoz, J., G. Starmack, and Y. Chan. *Locating Sensors at Optimum Points on an Enemy Transportation Network System to Estimate Origin-Destination Movements.* Working paper. Department of Operational Sciences, Air Force Institute of Technology, Wright-Patterson Air Force Base, Ohio, 1989.

Publication of this paper sponsored by Committee on Transportation Supply Analysis.