Distance-Based Model for Estimating a Bus Route Origin-Destination Matrix

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An origin-destination (O-D) matrix is a valuable tool for bus service planning. Unfortunately this trip table is not commonly available to the planner because of survey costs. However, stop-level on-off totals are often available. Past research has concentrated on using these totals along with a small O-D survey as a "seed matrix" to generate the full O-D matrix. Such a seed is subject to bias and sampling error and also incurs the survey cost. A method is described in which the seed matrix is generated using a propensity function that models the propensity of travel as a function of travel distance. The proposed function is a product of a power term and an exponential term, equivalent to a gamma distribution. When applied to one-directional travel, the gamma seed is shown to be reduced to a power function. The power function exponent is estimated by maximum likelihood for data from bus routes in Boston and Miami and is consistently found to be near 1.0. The gamma seed combined with the biproportional method to match origin and destination totals is shown to be effective in generating O-D matrixes for Boston and Miami routes. In a practical design application, design measures were found to be relatively insensitive to changes in the function parameter.

A route-level origin-destination (O-D) matrix is an important tool of the transit analyst. This trip table contains passengers' trip length data that enables the able analyst to test service improvements such as express, limited-stop, and short-turning services, or combining or splitting routes (*1*). Unfortunately, route-level O-D matrixes are not commonly available because of cost restrictions.

On-off counts, which represent row and column totals of the O-D matrix, are often available because they are used for funding and planning purposes. When on (origin) and off (destination) totals are known, the problem of trip distribution is to determine the matrix $\{t_{ij}\}$ that matches the given on and off totals, that is, that satisfies the constraints

$$\sum_{j} t_{ij} = t_{i\cdot} \qquad \text{for all } i \tag{1}$$

$$\sum_{i}^{j} t_{ij} = t.j \qquad \text{for all } j \tag{2}$$

where

t_{ij} = number of trips from i to j,
$t_{i\cdot}$ = boardings at stop i, and
$t_{\cdot j}$ = alightings at stop j.

Many solutions meet these constraints. The estimation problem is to find the complying matrix that best fits a "seed matrix" embodying prior information about the preferences of trip makers. Two main features distinguish trip distribution models. The first is the source of the seed matrix. The literature on trip distribution for general transportation planning emphasizes two sources of the seed: old surveys and, with the gravity model, distance-based models of impedance or its inverse, propensity. The literature on bus route O-D matrix estimation concentrates primarily on using data from a small O-D sample as the seed. This paper follows the example of an SG Associates study done of Cleveland bus routes in using a distance-based propensity (*2*).

The second main feature of a trip distribution model is the criterion of what constitutes a good fit to the seed, and therefore how the seed should be expanded to match the row and column totals. Methods of expanding the seed that have been studied include the biproportional method (*3*), least squares (*4*), and iterative methods based on maximum likelihood (*3,5*), maximum entropy (*6*), and minimum information (*7*). As Ben-Akiva et al. demonstrate, results for transit route O-D matrixes are extremely insensitive to the method of expansion (*3*). They recommend the biproportional method because of its computational advantages. Another advantage of the biproportional method is that it is compatible with the gravity model of trip distribution.

BIPROPORTIONAL METHOD AND GRAVITY MODEL OF TRIP DISTRIBUTION

The biproportional method produces estimates that have the form

$$t_{ij} = s_{ij}A_iB_j \qquad \text{for all } i, j \tag{3}$$

where A_i and B_j are endogenous row and column factors that balance the matrix, that is, enable it to satisfy Equations 1 and 2. There is no general method for solving for these factors in closed form. The most popular way of finding them is through a procedure known variously as iterative proportional fit (*3*) or Bregman's balancing method (*7*), in which all the rows are proportionately factored to match their row totals, all the columns are factored likewise, and the process is repeated until it converges. Convergence is guaranteed, and the resulting matrix solution is unique (*7,8*). If a_i^k is the balancing factor for row i at iteration k, then $A_i = \Pi_k a_i^k$, and similarly for the column factors.

In trip distribution using the gravity model, the seed matrix is a matrix of propensities (reciprocal of impedance or friction) that are primarily a function of distance; that is,

$$s_{ij} = p(d_{ij})\, K_{ij} \tag{4}$$

where

d_{ij} = distance from i to j (km or min),
$p(\,)$ = propensity function, and

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K_{ij} = an empirical adjustment factor, set equal to 1 in the absence of special information.

The doubly constrained gravity model (so called because both origin and destination totals are given) is usually expressed as a share model, sometimes called the interactance model (*9*):

$$t_{ij} = t_{i\cdot} \frac{X_j s_{ij}}{\sum_{j'} X_{j'} s_{ij'}} \quad \text{for all } i, j \tag{5}$$

where X_j is an endogenous factor for column j. There is no general closed-form solution for X_j. The typical solution algorithm begins with $X_j = t_{\cdot j}$, the total attractions at j. Equation 5 is applied to generate a trial matrix. The share form implicitly guarantees that Equation 1 is satisfied, but Equation 2 generally is not. The adjustment is then to multiply each X_j by the ratio (target column j total)/(current column j total). The procedure iterates, repeatedly generating a new trial matrix and adjusting all the column factors until it converges.

Although the doubly constrained gravity algorithm differs from iterative proportional fit, both procedures, in fact, produce identical results (ignoring roundoff error and premature termination). To demonstrate this, one can simply express Equation 5 in the following form:

$$t_{ij} = s_{ij} W_i X_j \tag{6}$$

where $W_i = t_{i\cdot}/(\Sigma_{j'} s_{ij'} X_{j'})$ is a row-specific factor, not dependent on any particular column, that, like X_j, is endogenous to the procedure. Equation 6 is a biproportional form: the product of a cell-specific seed, a row-specific factor, and a column-specific factor. Because the biproportional solution is unique, the doubly constrained gravity model is therefore equivalent to the biproportional model. Each W_i and X_j is equal at convergence to its corresponding iterative proportional fit factor A_i and B_j except for a scalar (the solution will be unchanged if the row factors are all multiplied by a scalar and the column factors divided by the same scalar). Therefore, the biproportional method can be interpreted as a gravity model, in which A_i and B_j are the true "masses," that is, the inherent productiveness and attractiveness of origin i and destination j, and s_{ij} is the inherent propensity of travel from i to j. The interaction of these three factors determines the number of trips from i to j. Unfortunately, because of this interaction, none of the factors can be observed directly.

SOURCES OF SEED MATRIX

In most of the literature on estimating bus route O-D matrixes, the seed matrix is the data from a small-sample O-D survey. This data source has three shortcomings: the survey cost, nonresponse bias, and bias and imprecision due to small sample size. Nonresponse bias occurs when the passengers who do not respond follow different travel patterns than responding passengers. Such a situation arises when response rate is affected by passengers not getting a seat, passengers making short trips, and buses passing through neighborhoods of varying levels of literacy or cooperation. Imprecision is a common problem with small samples. A rule of thumb is that there ought to be at least five counted passengers in an O-D cell for it to be statistically significant. When no passengers are counted in a cell, problems in updating occur—especially when the biproportional method is used in which a cell with a zero seed will remain zero after updating, biasing the results. Although Bayesian methods have been developed for such nonstructural zero cases (*3*), the estimates are still heavily influenced by the empirical seed's patterns. Aggregating to the segment level before updating can also introduce large biases in favor of intrasegment travel (*10*). Historically, problems of nonresponse and small samples have constantly plagued transit O-D surveys, and updating surveys to reliable on-off totals has not eliminated the problem. Resulting O-D surveys still suffer from a lack of believability.

Another possible seed is a "null seed" of equal values (for convenience equal to 1) for all O-D pairs except for O-D pairs that are not valid, whose values are 0. In the bus route problem, an O-D pair is not valid if it represents travel in the wrong direction or if it is on the matrix diagonal. It is also possible to disqualify O-D pairs that represent very short trips if the analyst believes that no one would make a trip that short. Furth and Navick (*10*) show that a null seed with biproportional updating is equivalent to a procedure developed by Tsygalnitsky (*11*), a single-pass recursive algorithm in which all passengers eligible to alight are deemed equally likely to alight at a particular destination. A passenger is eligible to alight if he or she has not yet alighted and has met the minimum distance qualification. Tsygalnitsky's method showed good results at the stop level, even on routes with a significant amount of turnover (*11,12*).

The null seed is plea of ignorance, assigning equal propensity to all valid O-D pairs. However, when on-off totals are given, it is often an effective plea, as it will often outperform a small sample seed. Furth and Navick found that, even without accounting for nonresponse bias, prediction accuracy was better using the null seed than with a small sample seed with a sample size of 100 responses (*10*). Geva et al. also found that it was the absolute sample size and not the sampling ratio that strongly influences estimation accuracy (*5*).

This research, more fully documented by Navick (*13*), was motivated by the desire to develop a more believable and more accurate seed matrix than a null seed without using a small-sample survey. Sometimes there are analysis problems in which an O-D survey cannot be taken and a seed matrix is needed, as in the problem of updating a ride check with multiple point checks (*14*). An analogous development has occurred in modeling O-D flows through intersections. Although various updating methods were developed (the same methods used with transit O-D matrixes), the only options for a seed matrix were either a small sample or a null seed using citywide averages of proportions of vehicles going left, through, and right (*15*), until a model of propensity was developed on the basis of explanatory factors such as intersection angle and competing shortcuts (*16*). Intuitively, the factors that best explain transit trip distribution are a preference for short trips (due to the disutility or travel time), competition with walking for very short trips, price, and effects of competing transit services. Because of the prevalence of flat fares, the price effect has been ignored. The effects of competing services can best be modeled as a modification to an initial framework of an isolated route. The remaining two factors then suggest that propensity be a function of distance, starting off low, increasing as walking loses its appeal, and then decreasing as the trip length disutility begins to overcome the utility of the trip purpose. The Cleveland study found that the trip length distribution followed this pattern (*2*). A gamma

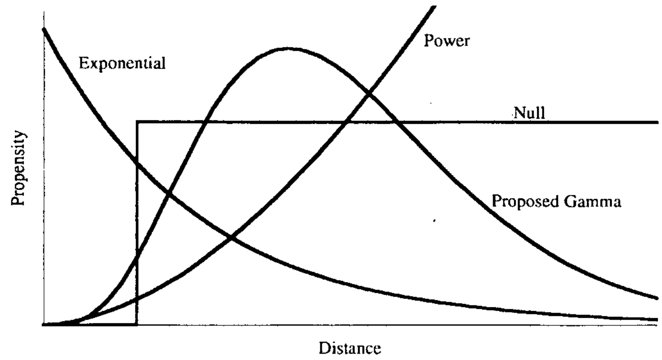

FIGURE 1 Propensity functions.

distribution with $\alpha > 0$ has the desired shape, having the form

$$s_{ij} = d_{ij}^{\alpha} e^{-d_{ij}\beta} \tag{7}$$

Propensity is relative and, unlike a probability distribution, is not required to integrate to 1 because it will only be rescaled in the updating process. Therefore the gamma function scalar needed for a probability distribution may be omitted. The gamma propensity can be thought of as a product of a power function and exponential function. It is illustrated in Figure 1, where it is compared with the null seed, an exponential propensity (if $\alpha = 0$), and a power function propensity (if $\beta = 0$). This propensity function has been used in vehicle trip distribution. Bellomo et al. found the gamma to be a very good fit to automobile trips in Detroit (*17*), and Nihan used it in a gravity model to distribute vehicles along a freeway given ramp on-off volumes (*18*).

"NO QUESTIONS ASKED" SURVEY

The authors' primary source of data for estimating and validating the propensity model was a set of O-D matrixes for three Boston area bus routes. To minimize the effects of nonresponse and sample size bias, a "no questions asked" survey (*11,19*) was conducted on three Massachusetts Bay Transportation Authority (MBTA) routes that have little competition from other transit routes and much passenger turnover. To ensure data quality, the authors were directly involved in data collection and compilation, supervising a team of engineering students.

As passengers boarded, they were handed a card coded with their origin stop number and were asked to simply return the card to a surveyor on leaving the bus. To the authors' knowledge, this is the first application of the "no questions asked" survey at the stop level rather that at the segment level. To get stop-level detail, three surveyors were needed for each bus, two at the front and one at the rear door. At the front door, the first surveyor held a box containing the survey cards, one bunch for each origin stop. Also in the box was the return bunch, consisting initially of specially colored header cards, one per stop (coded by stop number). The first surveyor kept the stop list and made sure that the second surveyor had in hand the bunch of cards for the origin stop being approached. He handed a card to each boarding passenger and collected cards from the alighting passengers. The collected cards were filed by the first surveyor in the return bunch behind the header card of the alighting stop. The rear door surveyor also collected and filed cards from alighting passengers. By being well-prepared and aggressive, the group of surveyors was able to get a response rate of more than 90 percent on every trip, even though two of these routes operate in inner-city areas where typical on-board surveys get a 30 percent response rate. Such a small number of passengers refused to participate that it was possible in most cases to "follow" them and handle their cards for them. The authors' experience with the "no questions asked" survey was very positive, and they enthusiastically recommend it as the best way to directly obtain O-D data when it can be done.

O-D data were also obtained for several Miami bus routes collected using "no questions asked" surveys. In Miami the cards were coded by route segment rather than by stop. Each segment was about 1.6 km (1 mi) long. Table 1 presents the Boston and Miami routes selected for analysis.

NORMALIZED TRIP PROPENSITIES

O-D matrixes obtained from the MBTA surveys were used to investigate the shape of the propensity distribution. These O-D matrixes contain information about propensity and about the popularity of origins and destinations. To uncover the propensity the matrixes had to be normalized, that is, the popularity factor had to be minimized. For example, although it is assumed that propensity to travel eventually decreases as distance increases, a strong attractor such as a mall or rapid transit station at the end of a route may overcome the propensity decay. This attractive power will be reflected by a large number of alightings at the end of the route and should not be mistaken as a desire for longer trips.

Normalizing a matrix usually means updating each row and column total to the same constant, but this is not appropriate for a one-directional, and therefore triangular, matrix in which there are many cells with zero propensity. Therefore each row and column total was normalized to equal the number of valid cells contributing to it, making the average normalized value per cell unity. Any normalized value above 1 implies a greater-than-average travel propensity; values below 1, a smaller-than-average propensity. Matrix cells of equal travel distance were then aggregated within each bus trip and over all bus trips within a route. Because stop spacing does not vary much on the routes studied, travel distance was simply measured in stops. Then aggregating over all the Boston routes, the mean normalized propensity for each travel distance was determined.

A plot of the mean normalized propensity versus travel distance is shown in Figure 2. It supports the assumption of a gamma propensity, showing an increasing propensity for approximately the first seven stops, a leveling off until approximately Stop 27 [about 6.4 km (4 mi)] and then a decay until the end of the route. However, the routes surveyed were only about 8.1 km (5 mi) long, and so further exploration with longer routes is needed to see whether the decay is significant.

ESTIMATION OF PROPENSITY MODEL

Maximum likelihood can be used to estimate the parameters of the gamma propensity function. Each cell of the O-D matrix can be considered an independent Poisson variable T_{ij} with expected

TABLE 1 "No Questions Asked" Survey Data

City	Route	Direction	Period	Trips	Passengers
Boston	1	in	AM	2	176
			PM	1	54
		out	AM	2	129
			PM	1	61
	66	in	AM	4	244
			PM	2	102
		out	AM	4	333
			PM	2	183
	77	in	PM	3	81
		out	PM	3	158
Miami	53	in & out	PREAM	n.a.	255
			AM		377
			MID		573
			PM		0
			EVE		0
			SAT		972
			SUN		195
	54-6	in & out	PREAM	n.a.	896
			AM		1,371
			MID		4,156
			PM		2,630
			EVE		1,902
			SAT		620
			SUN		1,549
	G	in & out	PREAM	n.a.	88
			AM		846
			MID		1,875
			PM		222
			EVE		293
			SAT		1,628
			SUN		1,044

n.a. = data not available

value λ_{ij}. If it is assumed that passengers arrive at stop i in a Poisson process, a typical assumption, and they are "stamped" with their destination stop j with conditional probability $p_{j|i}$, then the number of trips in cell (i, j) will be Poisson distributed. (Alternatively, one could simply assume that passengers arrive in a Poisson process for each O-D pair.) The probability of a realization t_{ij}, given it came from such a distribution, is

$$P[T_{ij} = t_{ij}] = \frac{\lambda^{t_{ij}} e^{-\lambda_{ij}}}{t_{ij}!} \tag{8}$$

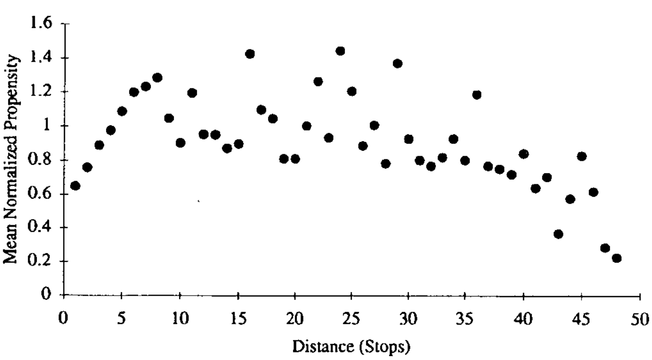

FIGURE 2 Mean normalized propensities for Boston routes.

Each λ_{ij} represents the mean number of trips between origin i and destination j and is assumed, following the gravity model, to be the product of three factors: a productiveness factor A_i, an attractiveness factor B_j, and a distance-based propensity:

$$\lambda_{ij} = s_{ij}(d_{ij})A_iB_j = d_{ij}^{\alpha}\, e^{-d_{ij}\beta}A_iB_j \tag{9}$$

The likelihood function, L, is the probability that the observed matrix is a realization of independent cells that are each Poisson distributed with parameters λ_{ij} that are a function of the parameters A_i, B_j, α, and β:

$$L = \prod_i \prod_j \frac{\left(d_{ij}^{\alpha} e^{-d_{ij}\beta}A_iB_j\right)^{t_{ij}} e^{-(d_{ij}^{\alpha} e^{-d_{ij}\beta}A_iB_j)}}{t_{ij}!} \tag{10}$$

As is common in maximum likelihood estimation, the log likelihood function, LL, is maximized:

$$LL = \sum_i \sum_j \{t_{ij}[\alpha \ln(d_{ij}) - d_{ij}\beta + \ln(A_i) + \ln(B_j)] - d_{ij}^{\alpha}\, e^{-d_{ij}\beta}A_iB_j - \ln(t_{ij}!)\} \tag{11}$$

To maximize the log likelihood function, partial derivatives with respect to α, β, A_i, and B_j are set equal to 0:

$$\frac{\partial LL}{\partial \alpha} = \sum_i \sum_j \ln(d_{ij})(t_{ij} - d_{ij}^{\alpha} e^{-d_{ij}\beta} A_i B_j) = 0 \quad (12)$$

$$\frac{\partial LL}{\partial \beta} \sum_i \sum_j d_{ij}(d_{ij}^{\alpha} e^{-d_{ij}\beta} A_i B_j - t_{ij}) = 0 \quad (13)$$

$$\frac{\partial LL}{\partial A_i} = \sum_j \left[\frac{t_{ij}}{A_i} - d_{ij}^{\alpha} e^{-d_{ij}\beta} B_j\right] = 0 \quad (14)$$

$$\frac{\partial LL}{\partial B_j} = \sum_i \left[\frac{t_{ij}}{B_j} - d_{ij}^{\alpha} e^{-d_{ij}\beta} A_i\right] = 0 \quad (15)$$

Rearranging the partial derivative expressions for A_i and B_j:

$$\sum_j t_{ij} = t_{i.} = A_i \sum_j d_{ij}^{\alpha} e^{-d_{ij}} B_j = \sum_j s_{ij} A_i B_j = \sum_j \lambda_{ij} \quad (16)$$

$$\sum_i t_{ij} = t_{.j} = B_i \sum_i d_{ij}^{\alpha} e^{-d_{ij}} A_i = \sum_i s_{ij} A_i B_j = \sum_i \lambda_{ij} \quad (17)$$

Equations 16 and 17, equivalent to Equations 1 and 2, will be satisfied by updating the seed matrix $\{s_{ij}\}$ using the biproportional method to match the given row and column totals $t_{i.}$ and $t_{.j}$. Notice that this biproportional application arises without explicit constraints that the matrix of estimates $\{\lambda_{ij}\}$ agree with any row or column total.

Investigation of the partial with respect to β reveals that the problem can be further simplified for this one-directional bus route problem. For an upper triangular O-D matrix,

$$\frac{\partial LL}{\partial \beta} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d_{ij}(d_{ij}^{\alpha} e^{-d_{ij}\beta} A_i B_j - t_{ij})$$
$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathrm{d}_{ij}(\lambda_{ij} - t_{ij}) \quad (18)$$

Separating the expression,

$$\frac{\partial LL}{\partial \beta} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d_{ij}\lambda_{ij} - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d_{ij} t_{ij} \quad (19)$$

For the one-directional case, the distance between any O-D pair (i,j) can be expressed as the sum of the distances of stop-to-stop segments:

$$d_{ij} = \sum_{k=i}^{j-1} d_{k,k+1} \quad (20)$$

Substituting

$$\frac{\partial LL}{\partial \beta} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{k=i}^{j-1} d_{k,k+1}\lambda_{ij} - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{k=i}^{j-1} d_{k,k+1} t_{ij} \quad (21)$$

Changing the summation order,

$$\frac{\partial LL}{\partial \beta} = \sum_{k=1}^{n-1} d_{k,k+1} \sum_{i=1}^{k} \sum_{j=k+1}^{n} \lambda_{ij} - \sum_{k=1}^{n-1} d_{k,k+1} \sum_{i=1}^{k} \sum_{j=k+1}^{n} t_{ij} \quad (22)$$

Combining under a single summation,

$$\frac{\partial LL}{\partial \beta} = \sum_{k=1}^{n-1} d_{k,k+1} \left[\sum_{i=1}^{k} \sum_{j=k+1}^{n} (\lambda_{ij} - t_{ij})\right] \quad (23)$$

In this final expression, each k defines a rectangular block of O-D cells representing all of the O-D pairs that cross the segment between stop k and stop $k+1$. Since the sum of any such block is simply the volume on segment $(k,k+1)$, Equation 23 may be rewritten as

$$\frac{\partial LL}{\partial \beta} = \sum_{k=1}^{n-1} d_{k,k+1} \sum_{i=1}^{k} [(\lambda_{i.} - \lambda_{.i}) - (t_{i.} - t_{.i})] = 0 \quad (24)$$

since $\lambda_{i.} = t_{i.}$ and $\lambda_{.i} = t_{.j}$ for all i (Equations 16 and 17). This implies that β and the exponential term of the gamma propensity seed do not affect the likelihood function for the case of one-directional travel. Arbitrarily setting β to 0 allows the seed to be expressed simply as a power function alone:

$$s_{ij}(d_{ij}) = d_{ij}^{\alpha} \quad (25)$$

This result is not an indication that a power seed represents the propensity of travelers—a decay in propensity as distance increases is definitely believed; however, an exponential decay cannot be identified in a one-directional scenario.

The insignificance of the exponential term in the one-directional case can also be proved directly from a property of the biproportional method. A biproportional update $\{\lambda_{ij}\}$ of a seed matrix $\{s_{ij}\}$ will have the following cross-product property (*3*) for cells (i,j) and (u,v) with $s_{ij} > 0$ and $s_{uv} > 0$:

$$\frac{s_{ij}s_{uv}}{s_{uj}S_{iv}} = \frac{\lambda_{ij}\lambda_{uv}}{\lambda_{uj}\lambda_{iv}} \quad (26)$$

Inserting the gamma propensity seed along with the row and column updating factors,

$$\frac{s_{ij}s_{uv}}{s_{uj}s_{iv}} = \frac{(d_{ij}^{\alpha}e^{-\beta d_{ij}}A_iB_j)(d_{uv}^{\alpha}e^{-\beta d_{uv}}A_uB_v)}{(d_{uj}^{\alpha}e^{-\beta d_{uj}}A_uB_j)(d_{iv}^{\alpha}e^{-\beta d_{iv}'}A_iB_v)} \quad (27)$$

Collecting terms and canceling the updating factors,

$$\frac{s_{ij}s_{uv}}{s_{uj}s_{iv}} = \frac{(d_{ij}d_{uv})^{\alpha}e^{-\beta(d_{ij}+d_{uv})}}{(d_{uj}d_{iv})^{\alpha}e^{-\beta(d_{uj}+d_{iv})}} \quad (28)$$

In the one-directional case, the following relationships between stops must hold: $i < j$, $u < v$, $u < j$, and $i < v$; otherwise one or more of the seeds in Equation 26 will be 0. Therefore $i < u < v < j$. By placing this relationship on a number line, it is observed that

$$d_{ij} + d_{uv} = d_{uj} + d_{iv} \quad (29)$$

The exponential terms in Equation 28 will therefore cancel for any values of ß, implying that the value of ß is immaterial for the case of one-directional travel.

EQUIVALENCE OF EXPONENTIAL AND NULL SEEDS

A common propensity function used in gravity models is the exponential function. For example, Sheffi derived the maximum entropy result for the doubly constrained gravity model and found the propensity to be exponential (*20*). If α is set equal to 0, the proposed gamma propensity seed becomes an exponential seed. The foregoing results shows that the β does not affect the matrix estimate for one-directional travel. Now if β is set equal to 0, the seed becomes the null seed. Therefore, in the case of one-directional travel, the null seed, which assumes equal propensity for any travel distance, is equivalent to an exponential seed, which implies equal conditional propensity. That is, the propensity for ending a trip at the next stop, given that it has not yet ended, does not change with distance. This extends the result found by Furth and Navick (*10*) and proves that with one-directional travel, Tsygalnitsky's method, the biproportional/gravity method with a null seed, and the biproportional/gravity method with an exponential seed are all equivalent.

MAXIMUM LIKELIHOOD ESTIMATES OF ALPHA

Using one-directional data from Boston, ß could not be estimated. Using the power seed and the likelihood function given earlier, α was estimated from both the Boston and Miami data. The data were first analyzed on a disaggregate level. For each trip, the log likelihood for values of a ranging from -1.0 to 4.0 with a step size of 0.2 was computed, and the optimal α identified. This enumeration method was chosen to allow for aggregation over routes and time periods.

The Miami data were in a two-directional, segment-level format that called for slight changes in the methodology. To place the matrixes in one-directional triangular form, the diagonal cells were split in half for each direction. Also the seed matrix had to be changed because of the diagonal being included in the analysis. Propensity assignment to each cell of the segment-to-segment matrix was the average of the stop-to-stop propensities included in that cell. Maximum likelihood estimation for α was then applied as in the Boston case.

The results of the maximum likelihood estimation are presented in Table 2. The table is aggregated at the route, city, and two-city levels for various periods. As a broad observation, $\alpha = 1.0$ fits all the combinations reasonably well. An attempt was made to observe varying travel propensities at different times of the day and days of the week. No patterns emerged; parameter values were scattered about 1.0 for a.m., p.m., weekday, and weekend trips. This result differs significantly from the exponent of -1.8 estimated in the Cleveland study (*2*) (the reported exponent is $+1.8$, but that is the exponent for impedance, the reciprocal of propensity). In that study, stop-to-stop "distance" was measured as the sum of travel time and route headway. But, more important, that study did not control for alighting totals, and therefore its propensity function is dominated by the decay in the trip length distribution.

Two hypothesis tests were conducted to investigate the statistical strength of a universal alpha. Two proposed universal alphas—1.0 (propensity increasing linearly with distance, in addition to an unspecified exponential decay) and 0 (the null seed, or merely exponential decay)—were tested for equivalence against the maximum likelihood estimate for each particular case. The likelihood ratio test with 1 degree of freedom and a significance level of 0.05 was used. A rejection of the hypothesis implies a poor fit for the so-called universal alpha. $\alpha = 1.0$ was not rejected in 38 percent of the 42 cases, while the null seed was rejected in all except two cases.

Although the performance of the $\alpha = 1.0$ seed is not staggering, consideration must be given to the likelihood of almost any hypothesized value being rejected when there is a large sample size. For model application, a planner must typically choose a value of α without the benefit of data from which to estimate a locally preferred value. Overall, the results show enough consistency and support for a value near $\alpha = 1.0$ that this value is recommended until and unless analysis of additional data points to a preferred value.

PREDICTION ACCURACY AND SENSITIVITY

Using on-off totals for each MBTA trip surveyed, O-D matrixes were estimated for using various universal alphas and compared with the observed matrix. A planner will typically care more about segment-level accuracy than a stop-level accuracy, since misallocating passengers from one stop to a neighboring stop is usually inconsequential. Therefore the stop-level estimated and observed O-D matrixes were aggregated to the segment level using five-

TABLE 2 Maximum Likelihood Alphas

	Period							
Route	AM	Mid	PM	Sat	Sun	WkDay	WkEnd	Day
Bos 1	1.0	n.a.	1.6	n.a.	n.a.	1.0	n.a.	n.a.
Bos 66	1.8	n.a.	0.8	n.a.	n.a.	1.4	n.a.	n.a.
Bos 77	n.a.	n.a.	1.0	n.a.	n.a.	n.a.	n.a.	n.a.
Mia 53	2.4	1.8	n.a.	2.0	0.4	n.a.	2.0	n.a.
Mia 54-6	0.8	0.6	1.6	-0.8	0.6	1.2	0.4	1.0
Mia G	0.6	0.4	0.2	0.6	0.8	0.4	0.6	0.6
Bos	1.4	n.a.	1.0	n.a.	n.a.	1.2	n.a.	n.a.
Mia	0.8	0.6	1.4	0.4	0.8	0.8	0.6	0.8
Bos-Mia	1.0	n.a.	1.4	n.a.	n.a.	0.8	n.a.	n.a.

n.a. = data not available

TABLE 3 Segment-Level Accuracy

	Relative Root-Mean-Square Error (%)							
	Boston Route 1				Boston Route 66			
Alpha	In/AM	In/PM	Out/AM	Out/PM	In/AM	In/PM	Out/AM	Out/PM
0	1.54	2.15	1.72	1.64	0.80	0.98	1.14	1.36
1	1.72	1.79	1.64	1.52	0.68	0.88	0.92	1.39
2	2.03	1.73	1.83	1.59	0.82	1.06	0.97	1.59
optimal	1.61	1.75	1.70	1.52	0.72	0.86	0.94	1.37

stop segments. The matrixes were then compared using the following relative root-mean-square error (*RRMSE*) formula:

$$RRMSE = \frac{1}{t_{..}} \sqrt{\frac{\sum_{i=1}^{m} \sum_{j=i}^{m} (\lambda_{ij} - t_{ij})^2}{K}} \qquad (30)$$

where

i,j = segments,
m = number of segments,
K = number of valid segment-level O-D pairs, and
$t_{..}$ = total boardings for trip.

This measure describes the difference between the generated segment-level O-D cell value and the observed cell in relation to the total boardings. The results are given in Table 3 for different values of α. The results show little sensitivity to α, with the measure of error varying between 0.68 and 2.15 percent of total boardings. Of several proposed universal alphas including the null seed ($\alpha = 0$), $\alpha = 1.0$ performs the best in all but three of the cases. The *RRMSE*s for $\alpha = 1.0$ do not significantly deviate from those of the optimal values, with an average error of approximately 1.5 percent over all the routes.

To examine the distribution of errors, the relative absolute error for each (segment-level) O-D cell was calculated using the formula

$$(\text{Relative absolute error})_{ij} = \frac{|\lambda_{ij} - t_{ij}|}{t_{..}} \qquad (31)$$

A cumulative distribution of relative absolute errors was constructed over all the route-direction-period combinations. The median relative absolute error was 0.5 percent of total boardings, and the 95th-percentile value was approximately 2.6 percent of total boardings. Both the segment-level and the stop-level errors appear quite reasonable from a planning standpoint.

SENSITIVITY ANALYSIS FOR DESIGN APPLICATION

Another test of a model for generating O-D matrixes is how it will perform under practical design applications. One application in which the O-D matrix is an important tool is the design of limited-stop service to complement local service. A limited-stop route is effective where the route is composed of a few heavily used stops. By combining local service with limited-stop route, where stops are made only at those with the most passenger movements, passenger travel times, and sometimes vehicle operating hours can be reduced.

In our application, those stops in the top 20th percentile by passenger movements were designated as limited stops. All passengers whose origins and destinations were both within one stop of a designated stop were assumed to use the limited stop; the others remained on the local route. In this way, the O-D matrix is split into two. The key design measures for a limited-stop route are peak volume (which governs the cost of the route) and total boardings (which governs the benefit of the route).

Both the observed and estimated O-D matrixes were analyzed for limited-stop service on MBTA Routes 1 and 66 for both directions and in various periods. The results show no practical differences in the design measures for α = 0.0, 1.0, 2.0, and the optimal alpha. Compared with the design measures obtained using the observed O-D matrix, the greatest errors occurred on RT 66/OUT/AM; these errors were 13 boardings per hour (5 percent) and 18 passengers per hour (11 percent) in peak volume. RT 66/OUT/PM had a discrepancy of 14 passengers per hour (7 percent) in peak volume. For the other six route/direction/periods examined, the errors were considerably smaller.

CONCLUSIONS

Through the separation of observed O-D matrixes into two components, propensity and popularity, the gamma distribution was found to be a good representation of passenger trip length. Normalized propensities increased for approximately 1.6 km (1 mi), leveled off for the next 4.8 km (3 mi), and decreased to the end of the route. This model of propensity coupled with the belief in the gravity model enables an O-D matrix to be generated effectively to match on-off counts. The updating method used was the iterative proportional fit, which was shown to be the equivalent of the gravity (share) model.

The gamma propensity function is reduced to a power function in the one-directional case because of the properties of the biproportional method. The exponential term is unobservable in this case, although still a propensity component. Following from this finding, the null seed and the exponential seed, both based on assumptions of equal propensity, were shown to be equivalent in the one-directional case.

The power function's parameter, α, was estimated using the O-D data from both the Boston and Miami routes. In general, α = 1.0 was observed to fit all combinations of routes, days, and times reasonably well. Statistically, a universal α = 1.0 performed better than the null seed ($\alpha = 0$) as measured by the likelihood ratio tests. A test of segment-level accuracy revealed that $\alpha = 1.0$ performed the best, yielding an *RRMSE* in the estimate of a segment-to-segment O-D pair of approximately 1.5 percent of total boardings. For an individual stop-level O-D cell, the median absolute relative error for $\alpha = 1.0$ was approximately 0.5 percent of total boardings. These error magnitudes appear reasonable for planning purposes. In a practical application, limited-stop route design, the design measures were insensitive to the choice of α.

The one-directional nature of this problem limits the degrees of freedom and allows for little variability in final matrix estimation. However, a planner that has on-off totals can now confidently investigate potential route changes in the office using a generated O-D matrix from a power seed with $\alpha = 1.0$ (propensity increasing linearly with distance) rather from an expensive O-D survey. However, care must be taken in applying this method when there is significant competition between routes, as mentioned elsewhere (*10*); competition will lower the propensity to travel between common stop pairs on both routes.

Finally, a methodology has been developed to extend this work to the generation of O-D matrixes for transit networks. In this two-directional problem, the exponential term of the gamma propensity seed will contribute significantly and must be estimated. The two-directional propensity's power term should not vary significantly from the findings in this study.

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