Cost Versus Time Equilibrium over a Network

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Most traffic assignment models assume that the generalized cost experienced by a traveler making a given trip on a network results from a combination of time and monetary expenses that is the same for everybody. To represent disaggregate trade-offs between time and monetary expenses, a model that differentiates travelers by means of an attribute, value of time, was designed. It was assumed that this attribute is continuously distributed across the population of trip-makers. After giving the economic foundation of the cost-versus-time model with continuous values of time, variable demand, and congestion effects on travel times, it is mathematically characterized as a solution of a convex minimization program. Then existence and uniqueness results as well as a convenient algorithm that avoids path storage are provided. Finally a small numerical example that demonstrates the relevance of considering continuously distributed values of time when evaluating toll highway projects is presented.

Traffic assignment is an important part of the transportation planning process, enabling one to simulate the trips made by people faced with a given transportation network. The models used to design new network facilities or to test new policies generally assume that all people experience the same generalized time on a given route, making a uniform trade-off between cost and time expenses.

For the evaluation of toll road projects that have mushroomed in France's largest towns, the differentiation of people according to their value of time (VOT; an attribute used to convert time into money) has proved an important advancement. Explicit modeling of the trade-offs between cost and time provides a more realistic way of simulating the users' responses to toll charges.

A first approach is to use a stochastic assignment model by having the random part of the utility account for the dispersion of trip-makers' VOTs. Both the logit model (*1*) and the probit model (*2*) can be adapted to that purpose. However if it is recognized that the VOT and its dispersion have a sound behavioral basis then a modeler should try to account for them analytically.

A second line of attack also consists of differentiating several classes of motorists, each one with a given VOT. The theoretical framework for the multiple user classes model has been worked out by Dafermos (*3*) in the deterministic case and Daganzo (*4*) in the stochastic case. An implementation is available in the SATURN package (*5*).

In France most interurban mode choice models are related to the second methodology, with the only difference being that the VOT is assumed to be continuously distributed across the trip-makers (*6–9*). Such models are known as *cost-versus-time models* (*modèles prix-temps* in French). Before adapting those models to urban path choice, congestion effects on travel times should be considered. There have been some attempts (*10,11*) to develop equilibrium assignment models able to compute a cost-versus-time equilibrium with travel times that depend on traffic flows. The theoretical background as well as the algorithms are heuristic.

To end the state-of-the-art review, a paper by Dial (*12*) should be mentioned. The paper presents a cost-versus-time model with a view to addressing both mode and route choices, but congestion is not taken into account.

The purpose of this paper is to introduce a cost-versus-time equilibrium model with variable demand, continuously distributed VOT, and flow-dependent travel time functions. This model can be used to study the potential traffic on urban toll roads and to assess middle- and long-run predictions owing to the variability of demand in the medium and long terms.

The remainder of the paper comprises four parts. First, the economic background of the cost-versus-time model is set. Second, the mathematical framework required to ensure the consistency of the model and to derive existence and uniqueness results is given. A convex programming characterization of a cost-versus-time equilibrium is provided. This section may be skipped by readers who are not interested in technical issues. Third, an algorithm to compute the cost-versus-time equilibrium is designed. It is convenient because it avoids path storage and enumeration. Finally a short example of an evaluation of a toll highway project is provided; it shows that an aggregate (single VOT) model gives results (specifically for the optimal toll and toll revenues) that are substantially different from those of the true, disaggregate cost-versus-time model.

ECONOMIC ISSUES

Modeling Disaggregate Cost-Versus-Time Trade-offs

If i is a trip-maker with VOT v_i and k is a path with travel time T_k and travel cost (price) P_k, the generalized travel time $G_k(i)$ experienced by the ith traveler on path k results from a combination of time and money expenses:

$$G_k(i) = T_k + \frac{P_k}{v_i} \tag{1}$$

A utility-maximizing trip-maker will travel on the path that exhibits the minimum generalized travel time to his or her own point of view.

If there are only two alternative paths, the first one cheaper but slower and the second one faster but more expensive, people with high VOTs would choose the second path, whereas people with low VOTs would be satisfied with the first one. Taking a French

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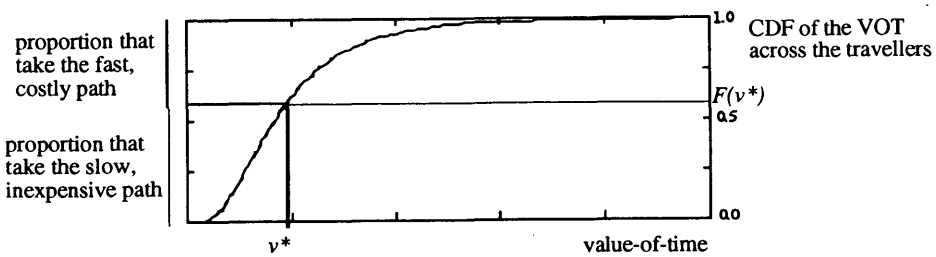

FIGURE 1 Market shares of two alternative paths.

interurban mode choice example, the first path may be thought of as a train and the second path as an airplane.

The frontier VOT v^* between the two paths is such that it equalizes their generalized times:

$$T_1 + \frac{P_1}{v^*} = T_2 + \frac{P_2}{v^*} \tag{2}$$

hence

$$v^* = \frac{P_2 - P_1}{T_1 - T_2} \tag{3}$$

Travelers with VOTs of $v_i \leq v^*$ choose the slow, inexpensive Path 1, whereas travelers with VOT of $v_i > v^*$ choose the fast, costly Path 2.

Given the statistical distribution of VOT across the trip-makers' population from its cumulative probability density function (CDF)

$$F(v) = \int_0^v h(x)dx \tag{4}$$

where $h(x)$ is the probability density function of VOT, and the proportion of people with VOT between x and $x + dx$ is $h(x)dx$, then (Figure 1)

- The market share of the first slow but inexpensive path is $S_1 = F(v^*) = \int_0^{v^*} h(x)dx$, and
- The market share of the second fast but costly path is $S_2 = 1 - S_1 = \int_{v^*}^{+\infty} h(x)dx$.

A way to infer the VOT statistical distribution is to derive it from the income distribution, which is in general well fitted by a log-normal probability density function (PDF) (*6,7*). Figure 2 depicts such a distribution. Another suggestion (*12*) is to consider a gamma distribution, which leads to a similar shape.

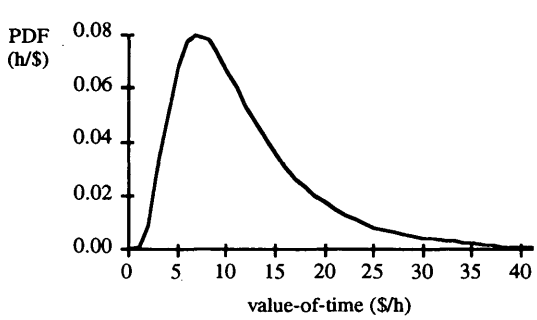

FIGURE 2 Log-normal distribution of VOT.

Efficient Paths

Call *efficient* a path such that there exists some positive VOT for which the path ensures a minimum generalized travel time. In the previous example there are only two paths, both of which are efficient. In most cases, however, there are numerous paths, among which only a few are efficient. If all paths are represented in a cost-versus-time diagram (where a path k is given coordinates T_k and P_k), the efficient paths are those with no alternative that would be both quicker and cheaper (Figure 3). In the cost-versus-time model only efficient paths may be assigned positive flows. If M efficient paths are ranked with respect to increasing prices, then the mth efficient path is traveled on by trip-makers with a VOT of v_i belonging to $[v^*_{m-1}; v^*_m]$, where v^*_m is the frontier VOT between efficient paths m and $m + 1$, defined as in Equation 3 as

$$v^*_m = \frac{P_{m+1} - P_m}{T_m - T_{m+1}} \tag{5}$$

Assuming a total trip rate of q, the mth efficient path is assigned a flow equal to

$$q \int_{v^*_{m-1}}^{v^*_m} h(v)dv = q[F(v^*_m) - F(v^*_{m-1})] \tag{6}$$

Note that for consistency with respect to the first and last efficient alternatives, in the latter case the upper bound must be $+\infty$, and in the former case the lower bound must be 0. See Figure 4 for an illustration.

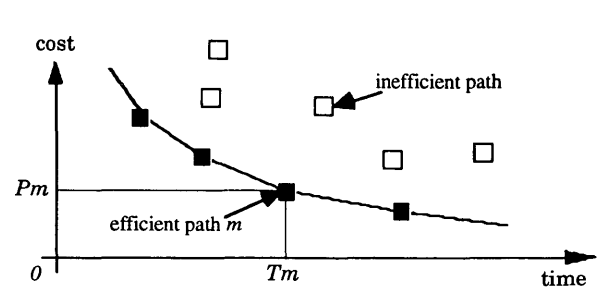

FIGURE 3 Efficient paths in a cost-versus-time diagram.

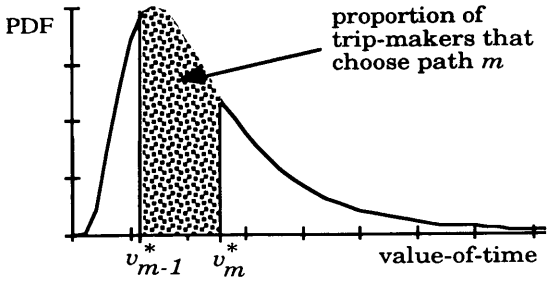

FIGURE 4 Market share of *m*th efficient path.

Contributions

The previous subsections have introduced the rule of sharing the traffic between the paths that underlies previous cost-versus-time models (*6–8,12*). To apply the rule those models have assumed that the prices and the travel times of the paths are fixed in advance.

However especially in urban road networks congestion effects may change the travel times of the paths and the definition of the set of efficient paths as well. Heuristic adaptations of the cost-versus-time sharing rule (*10,11*) have lacked a consistent theoretical framework.

The first contribution is also aimed at providing tools to take congestion into account within the cost-versus-time framework. The second contribution allows the volume of demand (the origin-destination traffic flow) to depend on the level of service.

Congestion Effects

The more vehicles there are on a road the more delay each of them experiences. When modeling urban road networks it is necessary to allow for increasing travel time with respect to flow (*13*). Thus it is assumed that for each network arc a there is a travel time function $t_a = t_a(x_a)$ that relates travel time t_a to vehicular flow x_a. Defining the travel time of a path k as the sum of the travel times of the arcs a that make up path k, it thus depends on traffic flows.

Allowing for Elastic Demand

Elastic demand is the economic tool used to model the fact that a change in supply entails either more people making a trip if the change is an improvement or some people relinquishing a trip in the case of a decrease in quality (to relinquish may be to choose another mode or another time of day).

It is also assumed that the actual trip rate q is a decreasing function D with respect to the mean generalized travel time $\mathbf{G}$:

$$q = D(\mathbf{G}) \tag{8}$$

In the case of the cost-versus-time model we assume an aggregate measure of the mean generalized travel time; denoting by $G(v)$ the minimum generalized travel time experienced by a trip-maker with VOT v, then

$$\mathbf{G} = \int_0^{+\infty} G(v)h(v)dv \tag{9}$$

Recall that $h(v)dv$ is the proportion of trip-makers with VOT between v and $v + dv$.

The elasticity of demand could also be modeled in a disaggregate way (*14*), but it would involve a mathematical framework with Hilbertian spaces of infinite dimension. Even more sophisticated is the Matisse model (*9*), which alllows for cross-elasticities between segments of demand.

MATHEMATICAL DEVELOPMENTS

This section is rather technical. First, some notation is introduced. Second, *monetary expense classes of paths* that aggregate paths with the same price and that are the rigorous tool of dealing with the efficient paths are defined. Third, conditions that characterize a cost-versus-time equilibrium are set up. Fourth, a mathematical convex minimization program is presented; in that program the Kuhn-Tucker conditions are equivalent to the definition of a cost-versus-time equilibrium. Lastly existence and uniqueness are asserted without proof. For detailed proofs of the mathematical results, the reader is referred to previous reports (*15,16*).

Basic Notation

Demand Side

The demand is a set of couples $[D_{rs}(t), h_{rs}(v)]_{rs}$, where r-s is an origin-destination (O-D) pair, $D_{rs}(t)$ is the demand function for trips between r and s (it is assumed to be a continuous and monotically decreasing function with respect to the travel time t), v is a VOT, a number that belongs to a subset Ω of $\mathbf{R}^+$, and $h_{rs}(v)$ is the probability density function of the random variable VOT among the travelers on O-D pair r-s; it is assumed to be a continuous and bounded function that remains nonnegative on the interior of its support. The cumulative density function associated with h_{rs} is as follows:

$$F_{rs}(v) = \int_0^v h_{rs}(\theta)\, d\theta$$

It is a continuous function that increases with respect to v. Its inverse function is

$$F_{rs}^{-1}(t) = MAX[v;\ F_{rs}(v) < t],$$

and is increasing and continuous on the right with respect to t.

A primitive form of $1/F_{rs}^{-1}$ is

$$E_{rs}(x) = \int_0^x 1/F_{rs}^{-1}(\theta)\, d\theta$$

The inverse demand function

$$D_{rs}^{-1}(q) = MAX[t;\ D_{rs}(t) > q]$$

is decreasing and continuous on the right with respect to q.

q_{rs} is the trip rate from origin r to destination s.

Supply Side

a is an arc. x_a is the flow on arc a. $t_a(x_a)$ is the travel time function on arc a; it is assumed to be a positive, continuous, and increasing function with respect to x_a.

Evaluation of Alternatives

k is a path from origin r to destination s, and it is assumed that it does not comprise any given arc more than once, f_{rs}^k is the flow on path k connecting O-D pair r-s, δ_{rs}^{ak} is the indicator variable ($\delta_{rs}^{ak} = 1$ if arc a is on route k between r and s and 0 otherwise), T_{rs}^k is the travel time proper on path k from r to s [$T_{rs}^k = \Sigma_a\ \delta_{rs}^{ak}\ t_a(x_a)$], P_{rs}^k is the monetary expense of path k from r to s (assumed to be nonnegative), and $G_{rs}^k(v)$ is the generalized travel time on path k from r to s as experienced by a trip-maker with VOT v of

$$G_{rs}^k(v) = T_{rs}^k + \frac{P_{rs}^k}{v} \tag{10}$$

Feasible Flow Pattern and Monetary Expense Classes of Paths

Definition 1: Feasible Flow Pattern

A feasible flow pattern is defined as a path flow vector $\boldsymbol{f} = (f_{rs}^k)$ such that

$$- \ \forall\ r, s, k \quad f_{rs}^k \geq 0$$

$$- \ \forall\ r, s \quad q_{rs} = \sum_k f_{rs}^k \leq q_{rso}$$

where q_{rso} is some positive constant (any O-D flow is physically bounded).

The basic principle is to aggregate paths that share the same monetary cost. To that end the so-called monetary expense classes of paths is used. It will help to characterize efficient paths.

Definition 2: Monetary Expense Classes of Paths

For every O-D pair r-s the paths between the equivalency classes of the equivalency relationship R_{rs} [$(k\ R_{rs}\ l)$ if $P_{rs}^k = P_{rs}^l$] are shared. Those classes are called *monetary expense* (ME) classes of routes. They are indexed with respect to increasing prices, with indexes from 1 to K_{rs}.

$I_{rs}(k)$ is defined as the class index of the path (r-s-)k and Δ_{rs}^{ki} as an indicator variable: Δ_{rs}^{ki} is equal to 1 if i is equal to $I_{rs}(k)$ and Δ_{rs}^{ki} is equal to 0 otherwise.

Additional Notation Related to ME Classes of Paths

q_{rs}^m is equal to $\Sigma_k\ \Delta_{rs}^{km} f_{rs}^k$ is the traffic flow on the paths of the mth ME class from r to s, and Q_{rs}^m is equal to $\Sigma_{i \leq m}\ q_{rs}^i$ the traffic flow from r to s on the paths whose prices are less than or equal to the price on the paths of the mth class. It also holds that q_{rs} is equal to $\Sigma_m\ q_{rs}^m$, which is equal to $Q_{rs}^{K_{rs}}$. Q_{rs}^0 is defined as 0 for ease of writing. $\mathbf{P}_{rs}^m$ denotes the monetary cost on the paths of the mth class for O-D pair r-s, and $\mathbf{T}_{rs}^m$ is the minimum travel time proper across the paths of this class.

The minimum generalized travel time experienced by a traveler with VOT v on the paths of the mth monetary expense class is

$$\mathbf{T}_{rs}^m + \mathbf{P}_{rs}^m / v \tag{11}$$

Cost-Versus-Time Equilibrium Conditions

Definition 3: Cost-Versus-Time Equilibrium Conditions

The feasible flow pattern $\boldsymbol{f}$ is a cost-versus-time equilibrium if and only if the following conditions (C1 to C3) are satisfied:

$$\text{C1: } \forall\ r, s, k \quad f_{rs}^k > 0 \Rightarrow T_{rs}^k = \mathbf{T}_{rs}^{I_{rs}(k)} \tag{12}$$

For every O-D pair r-s, for two monetary expenses classes m and n that are utilized ($q_{rs}^m > 0$ and $q_{rs}^n > 0$), it holds that

$$\text{C2: } \mathbf{T}_{rs}^m + \sum_{j=m}^{K_{rs}-1} \frac{\mathbf{P}_{rs}^j - \mathbf{P}_{rs}^{j+1}}{F_{rs}^{-1}(Q_{rs}^j/q_{rs})} = \mathbf{T}_{rs}^l + \sum_{j=n}^{K_{rs}-1} \frac{\mathbf{P}_{rs}^j - \mathbf{P}_{rs}^{j+1}}{F_{rs}^{-1}(Q_{rs}^j/q_{rs})} \tag{13}$$

In the variable-demand case for every O-D pair r-s such that q_{rs} is greater than 0, it holds that

$$\text{C3: } D_{rs}^{-1}(q_{rs}) = \sum_{m=1}^{K_{rs}} \left[\frac{q_{rs}^m}{q_{rs}} \mathbf{T}_{rs}^m + \mathbf{P}_{rs}^m \left(E_{rs}\left(\frac{Q_{rs}^m}{q_{rs}}\right) - E_{rs}\left(\frac{Q_{rs}^{m-1}}{q_{rs}}\right) \right) \right] \tag{14}$$

Economic Interpretation

The equilibrium conditions may be compared with the definitional conditions of a Wardropian user equilibrium (W1 and W2):

$$\text{W1: } \forall\ r, s, k \quad f_{rs}^k > 0 \Rightarrow T_{rs}^k = MIN_{k'} T_{rs}^{k'}$$

that is, a path that is traveled on must present a minimum travel time, and in the variable-demand case, for every O-D pair r-s such that q_{rs} is greater than 0, it holds that

$$\text{W2: } D_{rs}^{-1}(q_{rs}) = MIN_k\ T_{rs}^k$$

C1 corresponds to W1 restricted to the paths that belong to the same ME class; in the cost-versus-time model, the equilibration of flows owing to congestion effects prevails only *inside* each of the ME classes of paths.

C3 is analogous to W2 since it relates the volume of demand to a mean minimum generalized travel time. To see that the term on the right side of C3 stands for the definition of the generalized travel time presented in Equation 8, one must change the variables under the integration symbol:

$$\sum_{i=1}^{K_{rs}} \mathbf{T}^i_{rs} \frac{q^i_{rs}}{q_{rs}} + \mathbf{P}^i_{rs}\left[E_{rs}\left(\frac{Q^j_{rs}}{q_{rs}}\right) - E_{rs}\left(\frac{Q^{i-1}_{rs}}{q_{rs}}\right)\right] = \sum_i \int_{F^{-1}_{rs}(Q^{i-1}_{rs}/q_{rs})}^{F^{-1}_{rs}(Q^i_{rs}/q_{rs})} (\mathbf{T}^i_{rs} + \mathbf{P}^i_{rs}/v)h_{rs}(v)dv$$

$$\geq \sum_i \int_{F^{-1}_{rs}(Q^{i-1}_{rs}/q_{rs})}^{F^{-1}_{rs}(Q^i_{rs}/q_{rs})} MIN_j(\mathbf{T}^j_{rs} + \mathbf{P}^j_{rs}/v)h_{rs}(v)dv = \int_\Omega MIN_j(\mathbf{T}^j_{rs} + \mathbf{P}^j_{rs}/v)h_{rs}(v)dv$$

with equality being the case when C1 and C2 are satisfied.

C2 is specific to the cost-versus-time model. It determines the market share of each ME class of paths. If the ME classes i and $(n = i + p)$ are utilized when the classes $i + j$ for j in $[1; p - 1]$ are not (that is, $Q^{i+j}_{rs} = Q^i_{rs}$), then C2 reduces to

$$F^{-1}_{rs}(Q^i_{rs}/q_{rs}) = (\mathbf{P}^{j+p}_{rs} - \mathbf{P}^i_{rs})/(\mathbf{T}^i_{rs} - \mathbf{T}^{i+p}_{rs}) \tag{15}$$

$F^{-1}_{rs}(Q^i_{rs}/q_{rs})$ is the frontier VOT between the alternatives i and $i + p$; when C2 holds the paths of the ith class are used by the travelers whose VOT belongs to $[v^*_{rs,i-1}; v^*_{rs,i}]$ because these paths enable them to minimize $\mathbf{T} + \mathbf{P}/v$ (compare Equation 15 with Equation 5).

Extreme Characterization of a Cost-Versus-Time Equilibrium

Theorem 1 is the convex program for the cost-versus-time equilibrium. The feasible flow pattern f is a cost-versus-time equilibrium if and only if it solves the extremal convex problem MIN $J(f)$ on the set of all feasible flow patterns, where function J is defined as

$$J(f) = \sum_a \int_0^{x_a} t_a(x)dx + \sum_{rs}\left(\left\{q_{rs}\sum_{j=1}^{K_{rs}} \mathbf{P}^j_{rs}\left[E_{rs}\left(\frac{Q^j_{rs}}{q_{rs}}\right) - E_{rs}\left(\frac{Q^{j-1}_{rs}}{q_{rs}}\right)\right]\right\} - \int_0^{q_{rs}} D^{-1}_{rs}(q)dq\right) \tag{16}$$

subject to the definitional constraints

$$x_a = \sum_{rsk} \delta^{ak}_{rs} f^k_{rs} \tag{16a}$$

$$q^i_{rs} = \sum_k \Delta^{ki}_{rs} f^k_{rs} \tag{16b}$$

$$Q^i_{rs} = \sum_{j \leq i} q^j_{rs} \tag{16c}$$

$$q_{rs} = \sum_{j=1}^{K_{rs}} q^j_{rs} \tag{16d}$$

and the nonnegativity constraints

$$f^k_{rs} \geq 0 \tag{16e}$$

The first sum in the definition of J refers to the travel times proper that people try to minimize. The second one is related to the MEs that people also try to minimize. The third one is close to the opposite of a consumers' surplus.

Existence and Uniqueness of Equilibrium

Theorem 2: Existence

There exists at least one cost-versus-time equilibrium.

Theorem 3: About Uniqueness

If the travel time functions $t_a(x_a)$ are strictly increasing then at equilibrium the arc flows are unique, as are the frontier VOTs. In the fixed-demand case the flow on each ME class of the paths is unique. In the variable-demand case if the demand functions D_{rs} are strictly decreasing, then the trip rates as well as the flows on each ME expense class of paths are unique.

MSA ALGORITHM

Because practitioners are mainly interested in convenient, robust methods, an algorithm that avoids path storage and enumeration is presented. It is a Monte-Carlo method based on random simulation, as will be discussed further.

Two assumptions help to simplify the procedure:

- The price of path k depends only on the prices m_a of the arcs a that are traveled on:

$$P^k_{rs} = \sum_a \delta^{a,k}_{rs} m_a$$

- All O-D pairs with the same origin have the same PDF for VOT to avoid computing the shortest paths for each O-D pair.

Procedure

Step 0: Initialization

- Set iteration counter $n = 0$.
- Choose a sequence α_k of real numbers such that $(0 \leq \alpha_k \leq 1)$, $(\Sigma\alpha_k = \infty)$ and $(\Sigma\alpha^2_k < \infty)$.
- Find an initial feasible flow pattern $[x^{(0)}_a; q^{(0)}_{rs}$ for the variable-demand case only].

In the fixed-demand case the initial feasible flow pattern may be obtained through an all-or-nothing assignment on the basis of times $[t_a(0)]$.

In the variable-demand case the free-flow pattern may be used as an initial flow pattern; set O-D generalized travel time variables $\mathbf{G}_{rs}^{(0)}$ to the most realistic available value (from past assignments or some point of the demand curve that corresponds to a realistic mean generalized time).

Step 1: Arc Travel Time Update

- Set $n = n + 1$.
- Set $t_a^{(n)} = [t_a(x_a^{(n)})]$.

Step 2: Direction Finding

- For each origin r select by random sampling a VOT $v_r^{(n)}$. Compute the shortest paths to all destinations s on the basis of the arc generalized travel times $[t_a^{(n)} + m_a/v_r^{(n)}]$, yielding auxiliary O-D generalized travel times $\mathbf{G}1_{rs}^{(n)}$. For each destination s assign an auxiliary O-D flow $q1_{rs}^{(n)}$ on the shortest path thus determined, in which $q1_{rs}^{(n)}$ is equal either to q_{rs} in the fixed-demand case or to $D_{rs}[\mathbf{G}_{rs}^{(n)}]$ in the variable-demand case.

Assignment of traffic of all O-D pairs yields an auxiliary arc flow pattern $x1_a^{(n)}$.

Step 3: Arc Flow (and O-D Time) Update

- Set $x_a^{(n+1)} = x_a^{(n)} + \alpha_n(x1_a^{(n)} - x_a^{(n)})$.
- In the variable-demand case set $\mathbf{G}_{rs}^{(n+1)} = \mathbf{G}_{rs}^{(n)} + \alpha_n[\mathbf{G}1_{rs}^{(n)} - \mathbf{G}_{rs}^{(n)}]$.

Step 4: Convergence Criterion

- Apply a convergence test, either a maximum number of iterations or a test on the maximum value (among the arcs a of the network) of the change in $\Sigma_{k=1}^{n} \alpha_k \cdot x_a^{(k)}/\Sigma_{k=1}^{n} \alpha_k$ from the previous iteration $n - 1$ to the current one, n. If the test is satisfied, then terminate or go to Step 1.

Comments

The suggested algorithm is a twice-streamlined implementation of the method of successive averages (*17,18*) (with regard to streamlined algorithms).

- Step 2 begins with a random sampling of the VOT; if it was iterated many times the cumulative mean of the auxiliary flow pattern thus obtained would yield a descent direction for J at the current point of Step 1. The streamlined algorithm with one single internal sampling in Step 2 provides the best efficiency (*18*). .
- In the variable-demand case the second part of Step 3 is a further streamlining that allows one to compute the minimum generalized travel time without storing paths.

SHORT EXAMPLE

The following example is aimed at demonstrating that modeling of a continuous-distribution of VOT across travelers may change the results of traffic assignment equilibrium models in the evaluation of a toll highway project.

The Case

Consider an urban context where q vehicles per hour are to make a trip from a single origin r to a single destination s. There are only two available routes, the first one (the basic road) being a free route on the regular urban network and the second one being a toll-charged route designed to allow for quick traveling without congestion. The latter route is named the laser road, from the original idea of GTM (*19*) to build underground, passenger car-only toll motorways in Paris.

Supply Side

Assume that the generalized time on road a for a traveler with VOT is

$$G_a(x_a,v) = t_a(x_a) + \frac{P_a}{v} = T_a \cdot [1 + \gamma_a + \sqrt{\alpha_a^2 \cdot (1 - x_a/N_a)^2 + \beta_a^2} - \alpha_a \cdot (1 - x_a/N_a) - \beta_a] + \frac{P_a}{v}$$

where

P_a = toll fare on road a;
N_a = measure of *practical* capacity (i.e., the traffic flow at which point the service level on the arc decreases sharply); and
α_a, β_a, γ_a = parameters to model the effects of congestion such that (Figure 5):

$$\beta_a = \frac{\gamma_a(\alpha_a - \gamma_a/2)}{\alpha_a - \gamma_a} \text{ and } \alpha_a > \gamma_a \geq 0$$

The values of the parameters are as follows. For the laser arc T_a equals 0.18 hr, N_a equals 1,000 vehicles/hr, α_a equals 4.0, and γ_a equals 0.5. For the basic arc T_a equals 0.30 hr, N_a equals 5,000 vehicles/hr, α_a equals 2.5, and γ_a equals 1.5.

Demand Side

Assume that the VOT is distributed according to a log-normal PDF as depicted in Figure 2. The log-normal PDF is characterized by its median value of \$10/hr, and the standard deviation of its natural logarithm is set equal to 0.6 (*20*). The total trip rate q is fixed to 3,000 vehicles/hr.

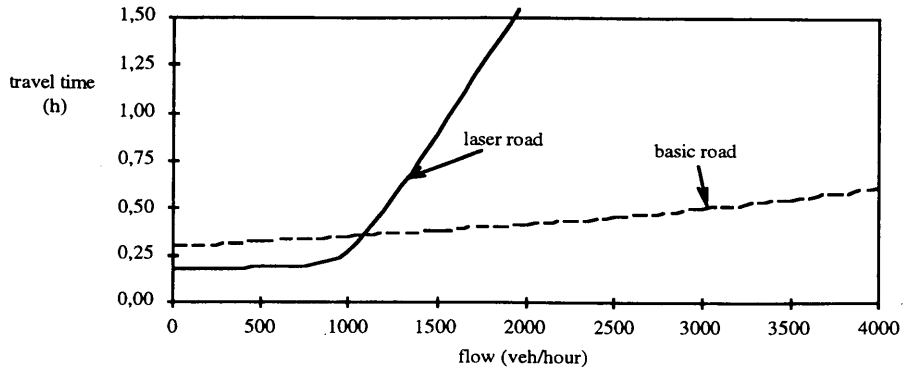

FIGURE 5 Travel time functions for two routes.

Numerical Evidence and Discussion

Calculate the traffic on the toll road and the toll revenues as functions of the toll asked for on the laser road by two different models: a cost-versus-time model and a standard model in which all travelers have the same aggregate VOT (the mean of the VOT distribution in the cost-versus-time model).

As soon as the toll is high enough to significantly differentiate the two routes the drawbacks of the standard, single-VOT model appear; it is unable to calculate either the optimal level of fare or the maximum revenues that are yielded by the more realistic cost-versus-time model.

Furthermore the standard model does not yield robust results; the fare that gives the maximum revenue is very close to another fare at which nobody travels on the toll route (Figures 6 and 7).

CONCLUSION

Cost-versus-time equilibrium assignment, as any assignment over a network, deals with a demand and a supply that are at odds with each other. It has been defined as a double equilibrium:

- An equilibrium between supply and demand, and
- A Pareto equilibrium between suppliers (the paths).

The mathematic and algorithmic tools that enable computation of this equilibrium are especially useful for the evaluation of toll highway projects in urban contexts.

The cost-versus-time model does not invalidate the advantages of the multiple user classes model, which faithfully represents the interaction on the supply side, notably different types of vehicles (e.g., with respect to size or passenger car unit equivalents).

The model described here is primarily demand related. The continuous distribution of the VOT gives robustness to the assignment and also enables one to test sensitivity to parameters like the mean value or the standard deviation of the VOT distribution function. An obvious extension is to introduce several classes of vehicles, each one with a continuously distributed VOT.

A truly disaggregate model requires that nonuniform multicriteria measures of the generalized cost (or of the satisfaction) in the demand be taken into account. The cost-versus-time model is a significant step in that direction for traffic assignment. Although close to the stochastic equilibrium models with respect to the algorithm introduced here, the economic background is quite different, focusing on explaining the deterministic part of the utility function rather than calibrating its random component like current stochastic assignment models do.

From a mathematical point of view researchers are provided with a computationally tractable model that extends the model of Beckmann et al. (*13*). Apart from the method presented here, deterministic algorithms are also available (*21*).

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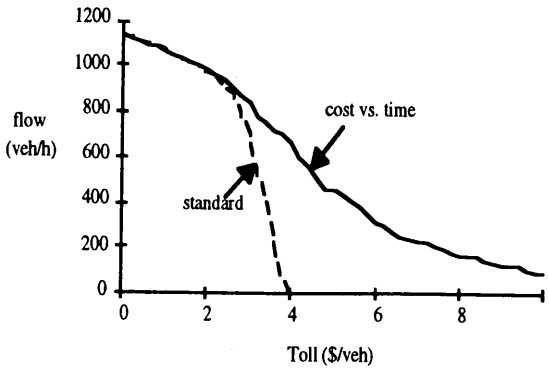

FIGURE 6 Traffic on toll road (laser route).

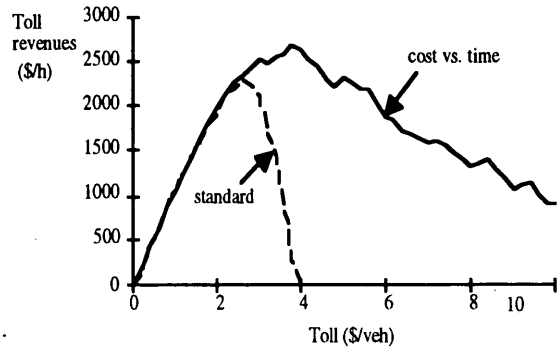

FIGURE 7 Toll revenues.

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