New and Improved Unsymmetrical Vertical Curve for Highways

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A new unsymmetrical vertical curve for highways that provides important desirable features is developed. The curve has unequal horizontal projections of the tangents, but its component parabolic arcs are equal. The new curve minimizes the difference between the rates of change of grades of the two arcs and consequently provides a smoother ride and is more aesthetically pleasing. The curve also improves the sight distance, reduces the length requirements, increases rider comfort, and increases the vertical clearance compared with the traditional unsymmetrical vertical curve. These desirable features should make the new unsymmetrical vertical curve an important element in vertical alignment design.

The traditional unsymmetrical vertical curve consists of two unequal parabolic arcs that meet at the tangents' intersection. The geometric characteristics of this curve have been presented in highway and surveying engineering texts (*1–3*). AASHTO points out that on certain occasions, because of critical clearance or other controls, unsymmetrical vertical curves may be required (*4,5*). However, because the need for these curves is infrequent, no information on them has been included by AASHTO.

Approximate relationships between the length of an unsymmetrical crest vertical curve and sight distance have been developed (*6*). These relationships assume that, for minimum sight distance, the line of sight is tangent to the point of common curvature and, consequently, may greatly underestimate the curve length requirements. Exact length requirements of crest and sag vertical curves that satisfy sight distance needs have been developed (*7,8*).

The traditional unsymmetrical vertical curve has some limitations caused by fixing the point of common curvature at the tangents' intersection. First, the difference between the rates of change of grades of the two arcs is generally large. As a result, the minimum available sight distance, which is controlled by the sharper arc, is short and the required curve is long. Second, the curve is not smooth and is less aesthetically pleasing. Third, for fixed ends of the unsymmetrical vertical curve, the required vertical clearance may not be satisfied and the curve is not suitable when it must pass through a fixed intermediate point.

In this paper a new unsymmetrical vertical curve that minimizes the difference between the rates of change of grades of the two arcs is developed. The improvements in sight distance, curve length requirements, rider comfort, and vertical clearance achieved by this curve are examined. Before presenting the new unsymmetrical vertical curve, the traditional unsymmetrical vertical curve is described.

TRADITIONAL UNSYMMETRICAL VERTICAL CURVE

A traditional unsymmetrical crest vertical curve (hereafter called the *traditional curve*) is shown in Figure 1. The curve consists of two parabolic arcs that have a common tangent at the point of common curvature (PCC). The PCC lies at the intersection of the two tangents, known as the point of vertical intersection (PVI). The beginning and end points of the vertical curve (BVC and EVC) have tangents with grades g_1 and g_2 (in percent), respectively. The algebraic difference in grade, A, equals $|g_2 - g_1|$. For crest vertical curves $(g_2 - g_1)$ is negative, and for sag vertical curves it is positive. The absolute value of $(g_2 - g_1)$ is used so that A is positive for both crest and sag vertical curves. The rates of change of grades of the two arcs are given by Hickerson (*1*)

$$r_1 = \frac{AL_2}{100\,LL_1} \qquad (1)$$

$$r_2 = \frac{AL_1}{100\,LL_2} \qquad (2)$$

where

- r_1 = rate of change of grades of the first arc,
- r_2 = rate of change of grades of the second arc,
- A = algebraic difference in grade (in percent),
- L_1 = length of the first arc,
- L_2 = length of the second arc, and
- L = total length of the curve $(L_1 + L_2)$.

Note that r_1 and r_2 will be positive for both crest and sag vertical curves. When $L_1 = L_2 = L/2$, Equations 1 and 2 give $r_1 = r_2 = A/100\,L$, which is the rate of change of grades of a symmetrical vertical curve, r. For an unsymmetrical vertical curve in which $L_1 < L_2$, for example, $r_1 > r$ and $r_2 < r$. This indicates that the first arc has a larger curvature (is sharper) than the symmetrical vertical curve, whereas the second arc has a smaller curvature (is flatter) than the symmetrical vertical curve.

A convenient parameter for describing the unsymmetrical vertical curve is R, which is defined as the ratio of the length of the shorter tangent (or shorter arc in the case of a traditional curve) to the total curve length,

$$R = \frac{L_1}{L} \qquad (3)$$

Expressing Equations 1 and 2 in terms of R gives

$$r_1 = \frac{A(1 - R)}{100\,LR} \qquad (4)$$

$$r_2 = \frac{AR}{100\,L(1 - R)} \qquad (5)$$

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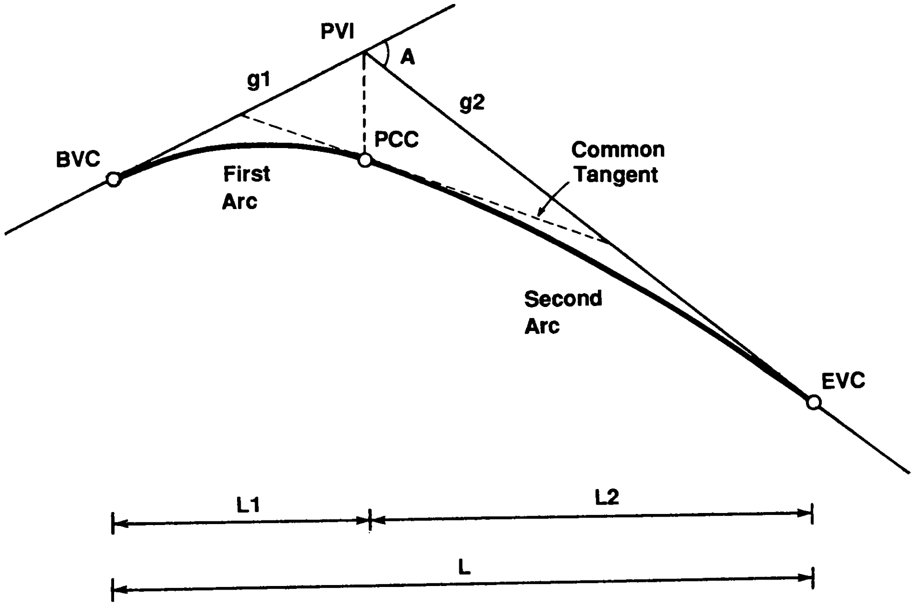

FIGURE 1 Traditional unsymmetrical crest vertical curve ($L_1 < L_2$).

For $R = 0.5$, Equations 4 and 5 give $r_1 = r_2 = A/100\ L$, and the traditional curve reduces to a symmetrical vertical curve.

NEW UNSYMMETRICAL VERTICAL CURVE

The new unsymmetrical vertical curve is derived from a general unsymmetrical vertical curve in which PCC is located at an arbitrary point. A description of the general and new unsymmetrical vertical curves follows.

General Unsymmetrical Vertical Curve

The geometry of a general unsymmetrical vertical curve is shown in Figure 2. PCC is located at a distance d_1 from BVC and a distance d_2 from EVC. Suppose that the horizontal projection of the tangent L_1 is less than L_2. The derivation of the rates of change of grades of the two arcs follows.

From Figure 2, the distances ab, bc, and ac are given by

$$ab = \frac{A}{100}(d_1 - L_1) \tag{6}$$

$$bc = \frac{r_2 d_2^2}{2} \tag{7}$$

$$ac = \frac{r_1 d_1^2}{2} \tag{8}$$

Since $ab = ac - bc$, then

$$\frac{A}{100}(d_1 - L_1) = \frac{r_1 d_1^2}{2} - \frac{r_2 d_2^2}{2} \tag{9}$$

Also, from Figure 2,

$$A_1 + A_2 = A \tag{10}$$

$$r_1 d_1 + r_2 d_2 = \frac{A}{100} \tag{11}$$

Solving Equations 9 and 11 for r_1 and r_2, and noting that $d_2 = L - d_1$, then

$$r_1 = \frac{A(L + d_1 - 2L_1)}{100\ Ld_1}, \qquad L_1 < L_2 \tag{12}$$

$$r_2 = \frac{A(-d_1 + 2L_1)}{100\ L(L - d_1)}, \qquad L_1 < L_2 \tag{13}$$

Equations 12 and 13 are applicable only if $L_1 < L_2$. If $L_2 < L_1$, r_1 and r_2 are given by

$$r_1 = \frac{A(L - 3d_1 + 2L_1)}{100\ Ld_1}, \qquad L_2 < L_1 \tag{14}$$

$$r_2 = \frac{A(3d_1 - 2L_1)}{100\ L(L - d_1)}, \qquad L_2 < L_1 \tag{15}$$

Note that when PCC lies at PVI ($d_1 = L_1$), the preceding equations for $L_1 < L_2$ and $L_2 < L_1$ reduce to Equations 1 and 2 of the traditional curve.

Derivation of New Curve

The variation of r_1 and r_2 of Equations 12 and 13 with d_1 is shown in Figure 3, which corresponds to a general unsymmetrical vertical curve with $g_1 = +2$ percent, $g_2 = -3$ percent, $L_1 = 250$ m, and $L = 800$ m. When $d_1 = 0$, $r_1 = \infty$ and r_2 has a finite value. As d_1

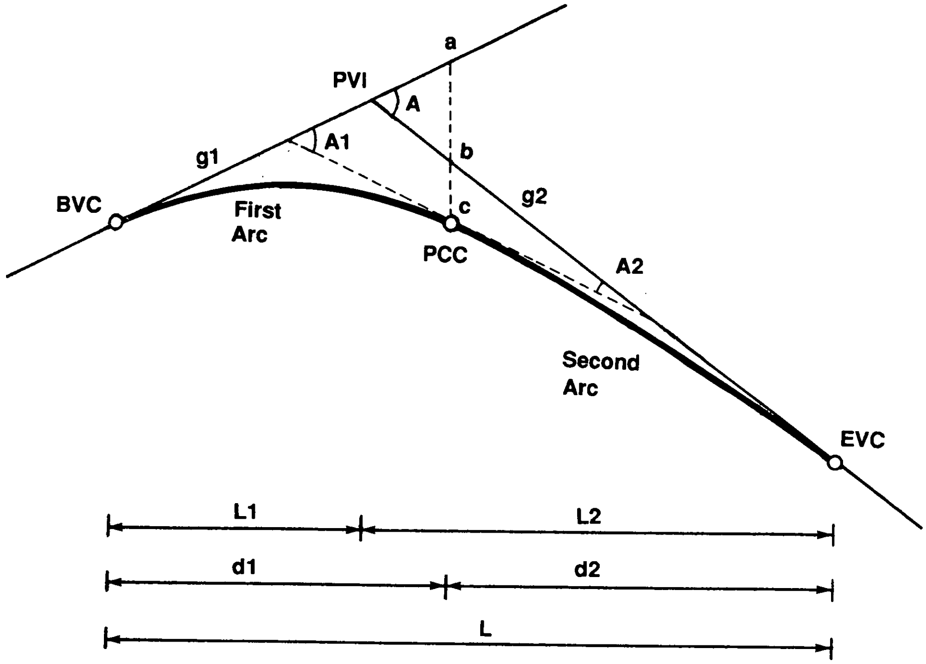

FIGURE 2 General unsymmetrical crest vertical curve ($L_1 < L_2$).

increases, both r_1 and r_2 decrease but r_1 is a convex function of d_1 and r_2 is a concave function of d_1. For $d_1 = 2L_1$, Equations 12 and 13 give $r_1 = A/200L_1$ and $r_2 = 0$, respectively. Therefore, values of d_1 equal to or greater than $2L_1$ are infeasible. The objective is to find d_1 that corresponds to the minimum difference between r_1 and r_2. Let the difference be denoted by F,

$$F = r_1 - r_2 \tag{16}$$

Substituting for r_1 and r_2 from Equations 12 and 13, Equation 16 can be expressed in terms of d_1 as

$$F = \frac{A(L - 2L_1)}{100\, d_1(L - d_1)} \tag{17}$$

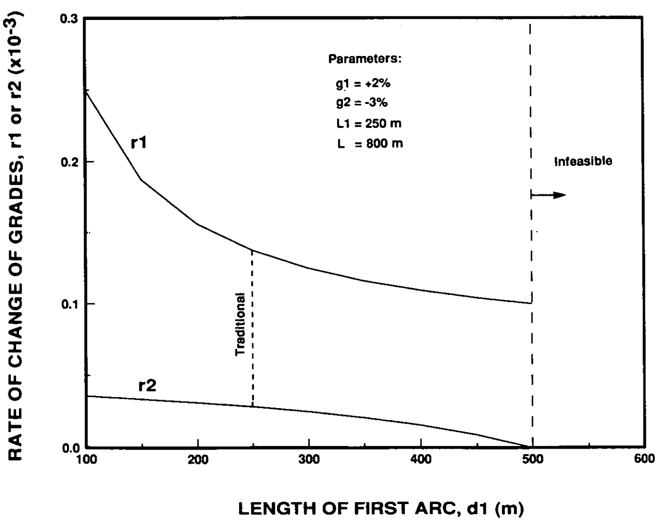

FIGURE 3 Variation of rates of change of grades with length of first arc.

The minimum value of F occurs when the first derivative of F with respect to d_1 equals zero. Differentiating both sides of Equation 17 and equating dF/dd_1 to zero yields

$$A(L - 2L_1)(L - 2d_1^*) = 0 \tag{18}$$

where d_1^* is the length of the first arc corresponding to the minimum value of F. On the basis of Equation 18, then

$$d_1^* = \frac{L}{2} \tag{19}$$

That is, the minimum value of F occurs when the two arcs of the unsymmetrical vertical curve are equal. This curve is referred to throughout as the *equal-arc unsymmetrical* (EAU) curve. The condition of Equation 19 corresponds to a minimum difference between r_1 and r_2 (not a maximum) because the second derivative of F can be shown to be always positive. Figure 4 shows the variation of F with the length of the first arc, d_1, and the minimum point, which occurs at $L/2$.

Substituting for $d_1 = L/2$ into Equations 12 and 13, r_1 and r_2 of the EAU curve are obtained as

$$r_1 = \frac{A(3L - 4L_1)}{100\, L^2}, \qquad L_1 < L_2 \tag{20}$$

$$r_2 = \frac{A(-L + 4L_1)}{100\, L^2}, \qquad L_1 < L_2 \tag{21}$$

Expressing Equations 20 and 21 in terms of R of Equation 3 gives

$$r_1 = \frac{A(3 - 4R)}{100\, L}, \qquad L_1 < L_2 \tag{22}$$

$$r_2 = \frac{A(-1 + 4R)}{100\, L}, \qquad L_1 < L_2 \tag{23}$$

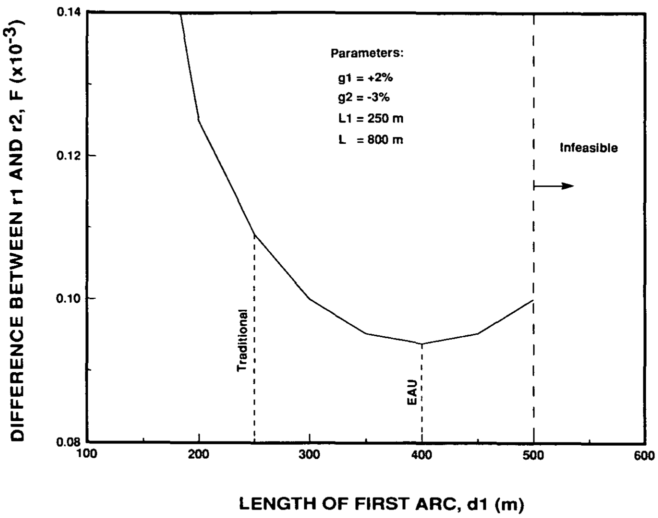

FIGURE 4 Variation of F with length of first arc.

For $L_2 < L_1$, Equations 22 and 23 are applicable, where $R = L_2/L$. Note that for $R = 0.5$, Equations 22 and 23 give $r_1 = r_2 = A/100\,L$ and the EAU curve becomes a symmetrical vertical curve. The traditional and EAU curves are drawn in Figure 5 for $L_1 = 250$ m, $L = 800$ m, $g_1 = +2$ percent, and $g_2 = -3$ percent. A symmetrical vertical curve with the same length as the unsymmetrical vertical curve is also shown.

Clearly, the EAU curve is smoother because the difference between r_1 and r_2 is minimal. The EAU curve is also more aesthetically pleasing because it not only reduces the difference between r_1 and r_2 but also makes the transitions at BVC and EVC less abrupt. This is true because the larger rates of vertical curvature of the two parabolic arcs of the EAU curve are more harmonious with the tangents whose rates of vertical curvature are infinity. The effect of the EAU curve on aesthetic appearance somewhat resembles the effect of a transition (spiral) curve on horizontal alignment.

PRACTICAL CONSIDERATIONS

Besides being smoother and more aesthetically pleasing, the EAU curve improves sight distance, requires a shorter length to satisfy a specific sight distance, increases rider comfort, and increases vertical clearance above that of a traditional crest vertical curve or below that of a traditional sag vertical curve. These benefits are quantified next.

Improving Sight Distance

The EAU curve improves the minimum sight distance compared with that of the traditional curve because both r_1 and r_2 of the EAU curve are smaller. Although exact models for computing the sight distance on the EAU curve are not available, the magnitude of the improvement can be approximately quantified. By using an idea by Guell (*9*), the minimum available sight distance and length requirements can be found on the basis of the sharper arc of the unsymmetrical vertical curve.

For the traditional curve, the minimum required length is

$$L = \frac{KA(1 - R)}{R} \quad \text{(traditional curve)} \qquad (24)$$

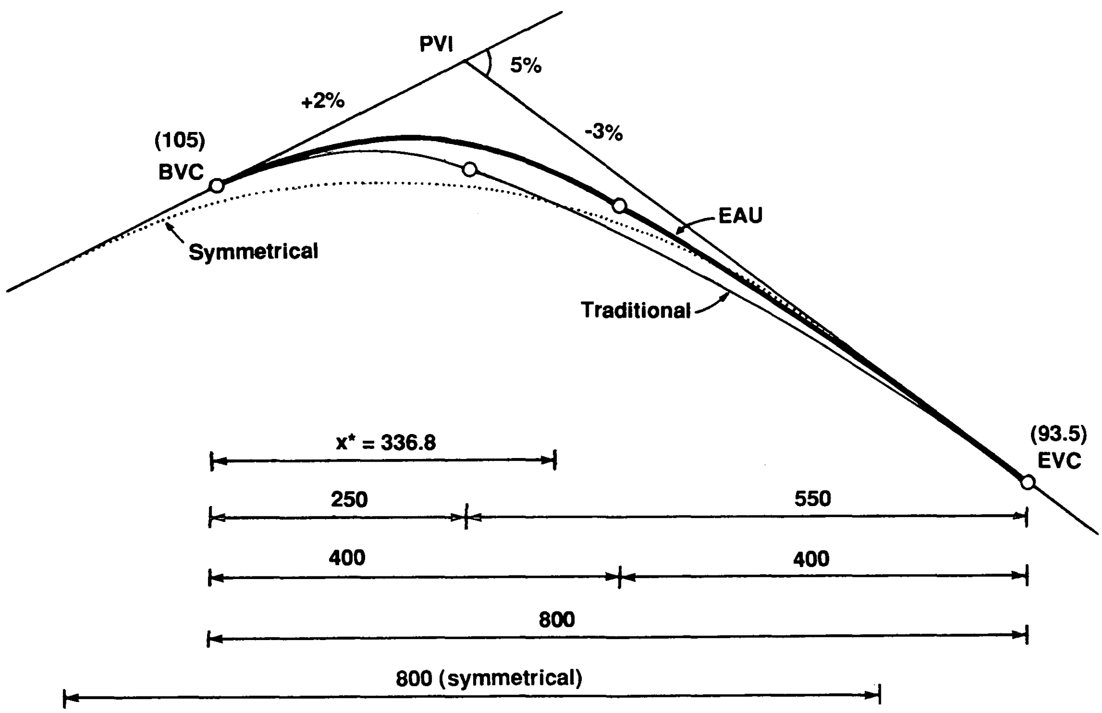

FIGURE 5 Comparison of traditional and EAU crest vertical curves (units in meters).

where K is the required rate of vertical curvature, which is the horizontal distance in meters (or feet) required to effect a 1 percent change in the grade, as given in AASHTO tables (5). For a crest curve, K is given by

$$K = \frac{S_m^2}{100\,(\sqrt{2h_1} + \sqrt{2h_2})^2} \tag{25}$$

where S_m = required minimum sight distance and h_1 and h_2 = the driver's eye and object heights, respectively. From Equations 24 and 25

$$S_m = (\sqrt{2h_1} + \sqrt{2h_2})\left[\frac{100\,LR}{A(1 - R)}\right]^{1/2} \quad \text{(traditional curve)} \tag{26}$$

For the EAU curve, the sight distance is controlled by the first (sharper) arc, which is true only when $L_1 < L_2$. Since $r_1 = 1/100K$, where r_1 is given by Equation (22), then

$$K = \frac{L}{A(3 - 4R)} \tag{27}$$

from which L and S_m are

$$L = KA(3 - 4R) \quad \text{(EAU curve)} \tag{28}$$

$$S_m = (\sqrt{2h_1} + \sqrt{2h_2})\left[\frac{100\,L}{A(3 - 4R)}\right]^{1/2} \quad \text{(EAU curve)} \tag{29}$$

The percentage increase in S_m achieved by the EAU curve, on the basis of Equations 26 and 29, can be obtained as

$$\text{Percent increase in } S_m = 100\left\{1 - \left[\frac{R(3 - 4R)}{(1 - R)}\right]^{1/2}\right\} \tag{30}$$

Table 1 shows the percentage increase in S_m for various values of R. The increase in S_m is greater for smaller R and reaches 26 percent for R = 0.2. For R = 0.5 both the traditional and EAU curves become a symmetrical vertical curve, and therefore no increase exists.

Reducing Curve Length Requirements

The required length of the EAU curve that satisfies a given sight distance (or K) is less than that of the traditional curve. The reduction in curve length, on the basis of Equations 24 and 28, can be obtained as

$$\text{Percent reduction in curve length} = \frac{100(1 - 2R)^2}{(1 - R)} \tag{31}$$

Table 2 shows the percentage reduction in L for various values of R. The reduction in curve length achieved by the EAU curve is significant and reaches 45 percent for R = 0.2.

For example, find the required length of an EAU crest vertical curve with R = 0.3 to satisfy stopping sight distance on a highway with g_1 = +2 percent, g_2 = −4 percent, and an 80-km/hr (50-mph) design speed. For this crest vertical curve, $A = |-4-2| = 6$ percent. From AASHTO (5), K = 36.70 m (120.39 ft) and the minimum required length, on the basis of Equation 28, is 397 m. For comparison, the minimum required length of the traditional unsymmetrical vertical curve is 514 m (the EAU curve length is 23 percent less).

Increasing Rider Comfort

The comfort effect caused by a change in vertical direction is greater on sag than on crest vertical curves because the centrifugal vertical force and the gravitational force are combining rather than opposing forces (5). The EAU curve reduces the centrifugal vertical acceleration on both sag and crest vertical curves and therefore increases comfort, especially on sag vertical curves. The centrifugal vertical acceleration equals the square of the design speed divided by the rate of vertical curvature,

$$C = \frac{V^2}{1{,}300\,K} \tag{32}$$

where

C = centrifugal vertical acceleration (m/sec^2),
V = design speed (km/hr), and
K = rate of vertical curvature (m/percent change of the grade).

By substituting for K of the traditional and EAU curves from Equations 24 and 27, respectively, into Equation 32, the corresponding centrifugal vertical accelerations on the first (sharper) arcs are

$$C = \frac{V^2A(1 - R)}{1{,}300\,LR} \quad \text{(traditional curve)} \tag{33}$$

$$C = \frac{V^2A(3 - 4R)}{1{,}300\,L} \quad \text{(EAU curve)} \tag{34}$$

TABLE 1 Increase in S_m Achieved by EAU Curve

R	Increase in Sm(%)
0.20	26
0.25	18
0.30	12
0.35	7
0.40	3
0.45	1
0.50	0

TABLE 2 Reduction in Curve Length Achieved by EAU Curve

R	Reduction in L(%)
0.20	45
0.25	33
0.30	23
0.35	14
0.40	7
0.45	2
0.50	0

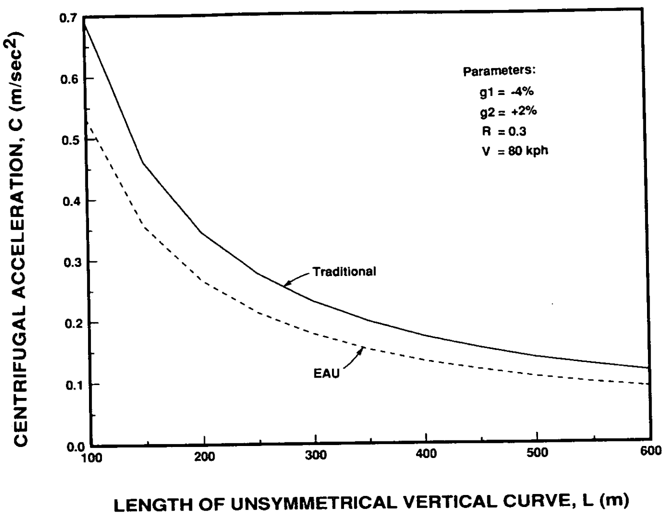

FIGURE 6 Comparison of centrifugal vertical accelerations of traditional and EAU sag vertical curves.

The centrifugal vertical acceleration of the EAU curve is smaller than that of the traditional curve. The percent reduction in C achieved by the EAU curve is given by the right side of Equation 31. Figure 6 shows the variations of C with L for the traditional and EAU curves, for V = 80 km/hr (50 mph), A = 6 percent, and R = 0.3. For this value of R, the centrifugal vertical acceleration on the EAU curve is 23 percent less than that on the traditional curve for any given L.

AASHTO points out that riding is comfortable on sag vertical curves when the centrifugal vertical acceleration does not exceed 0.3 m/sec^2 (1 ft/sec^2). The length of sag vertical curve that satisfies this comfort factor is much less than the headlight sight distance requirement. Therefore, the headlight criterion is used by AASHTO for the design of sag vertical curves (*5*). In Canada, the Roads and Transportation Association of Canada (RTAC) recommends the use of the comfort criterion for computing the length of sag vertical curve when good street lighting normally associated with urban conditions prevails (*10*). Under these conditions, sharper curves can be introduced and comfort is the criterion that limits values. Suostituting for C = 0.3 m/sec^2 in Equations 33 and 34, the minimum required lengths of the traditional and EAU sag vertical curves on the basis of the comfort criterion are

$$L = \frac{V^2A(1 - R)}{390\,R} \qquad \text{(traditional sag curve)} \qquad (35)$$

$$L = \frac{V^2A(3 - 4R)}{390} \qquad \text{(EAU sag curve)} \qquad (36)$$

For R = 0.5, Equations 35 and 36 reduce to $L = V^2A/390$, which is the minimum required length of symmetrical sag vertical curves on the basis of the comfort criterion (*10*). The percent reduction in sag vertical curve length achieved by the EAU curve is given by the right side of Equation 31.

Increasing Vertical Clearance

The EAU curve provides a larger vertical clearance, as shown in Figure 5. The maximum difference in vertical clearance occurs between the first arc of the EAU curve and the second arc of the traditional curve. The derivation of the maximum difference and its location for a crest vertical curve follows.

The elevations of the first arc of the EAU curve and the second arc of the traditional curve at a distance x from BVC are (Figure 7)

$$\text{Elevation (EAU)} = y_{BVC} + \frac{g_1x}{100} - \frac{r_1x^2}{2} \qquad (37)$$

$$\text{Elevation (traditional)} = y_{EVC} - \frac{g_2(L - x)}{100} - \frac{r_2(L - x)^2}{2} \qquad (38)$$

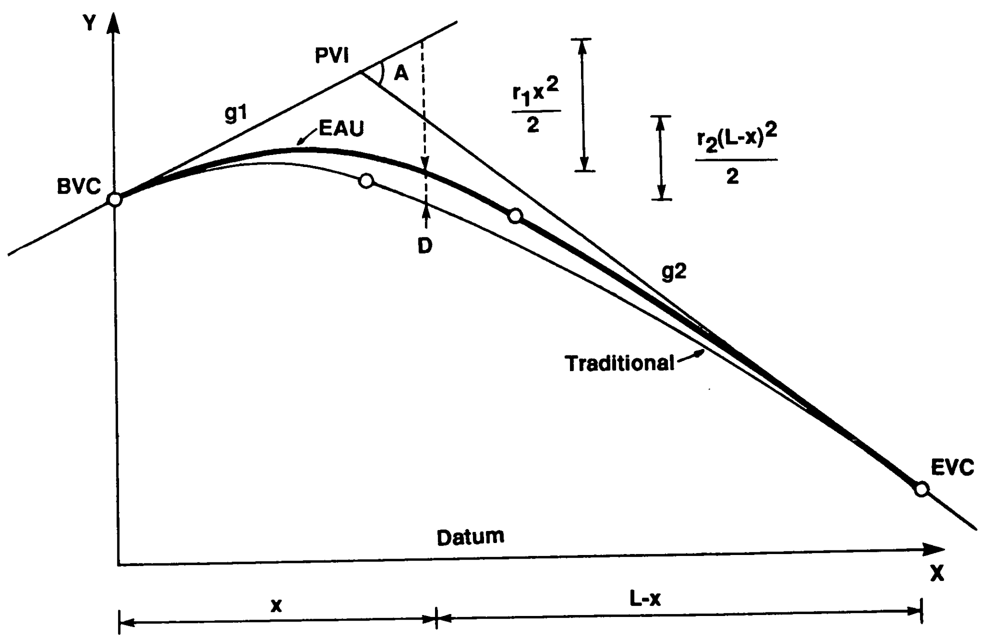

FIGURE 7 Geometry of increased vertical clearance of EAU crest curve.

Substituting for r_1 and r_2 from Equations 22 and 5, respectively, the difference between the elevations of the EAU and traditional curves, D, is

$$D = y_{BVC} - y_{EVC} + \frac{g_2L}{100} - \frac{Ax}{100} - \frac{A(3 - 4R)x^2}{200\ L} + \frac{AR(L - x)^2}{200\ L(1 - R)} \quad (39)$$

Differentiating both sides of Equation 39 with respect to x, equating dD/dx to zero, and solving for x gives

$$x^* = \frac{L}{(3 - 2R)} \quad (40)$$

where x^* is the horizontal distance corresponding to the maximum difference D^*. Substituting for x^* into Equation 39 gives D^* as

$$D^* = y_{BVC} - y_{EVC} + \frac{g_2L}{100} + \frac{AL(1 + 2R)}{200\ (3 - 2R)} \quad \text{(crest curve)} \quad (41)$$

where y_{BVC} and y_{EVC} = elevations of BVC and EVC, respectively. For sag vertical curves, Equation 40 is applicable and Equation 41 becomes

$$D^* = y_{BVC} - y_{EVC} + \frac{g_2L}{100} - \frac{AL(1 + 2R)}{200\ (3 - 2R)} \quad \text{(sag curve)} \quad (42)$$

where D^* will be negative, indicating that the EAU curve lies below the traditional curve.

For example, find the maximum difference in vertical clearance between the EAU and traditional crest vertical curves of Figure 5, where $g_1 = +2$ percent, $g_2 = -3$ percent, $L_1 = 250$ m, $L = 800$ m, $y_{BVC} = 105$ m, and $y_{EVC} = 93.5$ m. For $A = |-3-2| = 5$ percent and $R = L_1/L = 0.3125$, the maximum difference in vertical clearance occurs at $x^* = 336.8$ m and its value is $D^* = 1.18$ m on the basis of Equations 40 and 41, respectively.

CONCLUDING REMARKS

In this paper a new unsymmetrical vertical curve in which the point of common curvature lies at the midpoint of the curve has been described. This EAU vertical curve has two important features: the rates of change of grades of the two arcs are less than those of the traditional unsymmetrical vertical curve and the difference between them is minimal. On the basis of the present study, the following concluding remarks are offered:

1. The EAU curve provides the following practical benefits: (a) improved sight distance, (b) reduced curve length requirements, (c) increased rider comfort, (d) increased vertical clearance, and (e) a more aesthetically pleasing curve.
2. The results show that the required length of the EAU curve that satisfies a specified sight distance is significantly shorter than that of the traditional curve. As a result, when a traditional crest curve lies totally in cut, the shorter EAU crest curve would reduce the cost of excavation. Likewise, when a traditional sag curve lies totally on fill, the shorter EAU sag curve would reduce the cost of fill and compaction. In other situations, the reduction in construction costs would depend on whether the EAU curve fits the terrain better than the traditional curve.
3. The EAU crest curve provides additional clearance over the traditional crest curve. Similarly, the EAU sag curve provides additional clearance below that of the traditional sag curve. This additional clearance would be useful when the vertical clearance restriction is below that of a traditional crest curve or above that of a traditional sag curve. If the vertical clearance restriction is in the opposite direction of the cases noted above, a symmetrical vertical curve should be considered.
4. The general unsymmetrical vertical curve, from which the EAU curve was derived, may also be useful on certain occasions. Because the location of PCC is a variable in this general curve, the curve offers flexibility in satisfying additional design constraints, such as the need for the curve to pass through a fixed point. A specific curve may also be selected to reduce earthwork or other construction costs. In this analysis, it is desirable to consider the curves that provide improvements over the traditional curve ($d_1 > L_1$) and not those curves that increase the difference between r_1 and r_2 ($d_1 < L_1$).
5. The sight distance and length requirements presented in this paper are approximate for $S_m > L_1$ (generally when A is small and when A is large and S_m is small). Exact sight distance models for the EAU crest and sag curves are currently being developed by the author.

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