Bus Transit Service Coverage for Maximum Profit and Social Welfare

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A framework for finding the optimal bus transit service coverage in an urban corridor is presented. The service variables considered are a combination of route length, route spacing, headway, and fare. The criterion for optimality is either operator profit or social welfare maximization. The social welfare, a sum of user and operator surplus, is optimized with both unconstrained subsidy and breakeven constraints. The equations for the optimal design variables that maximize operator profit and social welfare are derived analytically for a rectangular transit corridor with elastic demand, uniformly distributed passenger trip density, and many-to-one travel patterns. The equations provide considerable insight into the optimality conditions and interrelations among variables. These equations are also incorporated within an efficient algorithm that computes optimal values for the decision variables for a more realistic model with vehicle capacity constraints. The numerical results show that at the optimum the operator profit and welfare functions are rather shallow, thus facilitating the tailoring of design variables to the actual street network and particular operating schedule without substantial decreases in profit or welfare. The social welfare function is relatively flat near the optimum for a relatively large range of subsidies. This result implies that for a given set of input data the breakeven constraint may be an economically preferable objective because it eliminates subsidy, whereas it reduces social welfare only marginally. The sensitivities of the design variables to some important exogenous factors are also presented. The presented methodology is also applicable to the problem of optimal service coverage of feeder bus systems serving rapid rail line stations.

The basic elements that must be determined in planning bus transit service in an area are route lengths, route spacing (or density), headways, and fares. Determining how far outward to extend transit routes from the central business district (CBD) is particularly important. The general trade-off is between the cost of service to the operator and the cost of travel to users. Operators prefer short routes to minimize costs. Passengers, especially those from the outer suburbs, prefer longer routes to minimize their access impedance. When the demand for transit service is elastic [i.e., passengers are sensitive to the level of service (LOS) characteristics and the fare], shorter routes and thus higher access impedance will decrease the attractiveness of the service and cause potential travelers to switch to other modes. Because the route length has a significant impact on both operator costs and passenger impedance, its value should be carefully selected.

The purpose of this paper is to develop a method for optimizing the lengths of bus transit routes that extend radially outward from the CBD or those of a feeder bus system serving rapid rail line stations. However this problem may not be considered independently of route location and service scheduling. Therefore the problem considered here is that of finding an optimal combination of route length, route spacing, headway, and fare that maximizes operator profit and social welfare for a rectangular-shaped urban corridor with uniformly distributed passenger trip densities.

Demand is considered to be elastic. Service characteristics affect ridership, which in turn has an impact on revenue. Ridership also affects service characteristics, and thus operator cost. The method proposed in this paper recognizes these interactions between demand and supply (operator cost) and calculates equilibrium LOS characteristics and fare that optimize transit service coverage under several design objectives, which are (a) maximization of operator profit, (b) maximization of social welfare with unconstrained subsidy, and (c) maximization of social welfare with a breakeven constraint.

BACKGROUND

Several previous studies sought to optimize various elements of transit service and network design by using calculus and, to a lesser extent, mathematical programming methods (*1–23*). An extensive review of optimization models can be found in Chang and Schonfeld (*20*). A summary of pertinent analytical models classified according to the design variables optimized is presented in Table 1. In most studies travel demand was inelastic and uniformly distributed over the service area. The usual travel pattern was many to one, whereas the most common objective function was the minimization of the sum of operator cost and user time cost. The assumptions of inelastic demand precluded the models from analyzing the impacts of pricing policies and subsidies.

Kocur and Hendrickson (*12*) developed an analytical model with elastic demand and derived closed-form solutions for optimal route spacing, headway, and fare but not route length for different design objectives. Morlok and Viton (*21*) and Viton (*22*) developed a similar model to evaluate the profitability of bus transit service.

A literature review revealed only two published papers (*15,16*) that dealt with the optimization of a radial transit route length in an urban transportation corridor, which is the focus of this paper. Wirasinghe and Seneviratne (*15*) developed closed-form solutions for the optimal rail transit line length for sectorial and rectangular corridors with inelastic demand and uniformly distributed passenger trip density. The objective function to be minimized included the total rail fleet cost, rail and feeder bus operating cost, and passenger time cost. Spasovic and Schonfeld (*16*) presented a model for optimal service coverage for rectangular and sectorial urban corridors with uniform and linearly decreasing density func-

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TABLE 1 Pertinent Analytical Models for Transit Network Design

Decision Variables	Objective Function	Transit Mode	Street Network Geometry	Passenger Demand	Authors
Route Length, Spacing, Headway, Stop Spacing	Min. operator and user cost	bus	rectangular and sectorial grid	Uniform and Linear Decreasing, inelastic, many-to-one	Spasovic and Schonfeld (1993)
Route Length	Min. operator and user cost	rail	rectangular grid	General, inelastic, many-to-one	Wirasinghe and Seneviratne (1986)
Route Spacing, Zone Length, Headway	Min. operator and user cost	bus	rectangular grid	Uniform, inelastic, many-to-one	Chang and Schonfeld (1992)
Route Spacing, Lengths and Headway	Min. operator and user cost	bus and rail	rectangular grid	Uniform, inelastic, many-to-one	Byrne (1976)
Route Spacing	Min. operator and user cost	bus	rectangular grid	Uniform, inelastic, many-to-many	Holroyd (1967)
Route Spacing and Headway	Min. operator and user cost	bus	rectangular grid	Uniform, inelastic, many-to-one	Byrne and Vuchic (1972)
Route Density and Frequency	Min. operator and user cost	bus	rectangular grid	General linear, inelastic, many-to-one	Hurdle (1973)
Route Spacing, Headway and Fare	Max. operator profit, Max. user benefit, etc.	bus	rectangular grid	Uniform elastic, many-to-one	Kocur and Hendrickson (1982)
Route Spacing, Headway and Stop Spacing	Min. operator and user cost	feeder bus to rail	rectangular grid	General, inelastic, many-to-one	Kuah and Perl (1988)
Route Spacing, Headway and Fare	Max. profit, max. welfare, min. cost	bus	rectangular grid	Irregular, elastic, many-to-many, time dependent	Chang and Schonfeld (1989)
Route Spacing, Headway	Max. profit	bus	sectorial grid	Uniform, elastic, many-to-one	Morlok and Viton (1984)

tions that were inelastic. The model jointly optimized route length, headway, route, and stop spacing, and it also considered stations along the line and the associated access cost.

This paper extends the methodology of Spasovic and Schonfeld (*16*) to the case of a rectangular corridor with elastic demand. The assumption of elastic demand enables the model to analyze the impacts of pricing policies and subsidies on the system's design characteristics and service coverage.

EQUILIBRIUM FRAMEWORK

The framework for planning optimal bus transit service coverage in which the resources and costs of providing the service are related to its operating characteristics and the induced ridership is presented in Figure 1. In this process the values of the service characteristics such as route length, route spacing (or route density), headway (or its inverse, the frequency), and fare must be carefully selected to satisfy prespecified design objectives.

Because the demand is elastic the service characteristics chosen will have an impact on ridership, and thus system revenue. On the other hand ridership will have an impact on the service characteristics, and thus operator cost.

The LOS characteristics and fare could be optimized by using several objectives. For example the maximization of operator profit—the difference between the fare box revenue and operating cost—could be one objective. However most transit systems do not recover their operating cost from the fare box and need to be subsidized from additional external revenue sources.

As mentioned earlier, there is a conflict between the operator's and users' objectives. Users prefer to have short access to the route and short waiting time, whereas the operator would prefer to have a very long headway and shorter, sparsely located routes with few stops to minimize costs. To alleviate the perceived con-

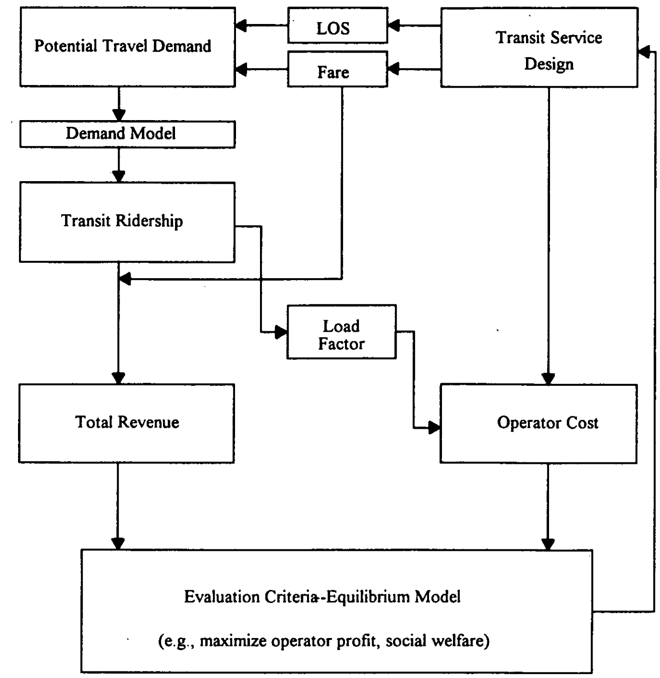

FIGURE 1 Equilibrium framework for optimal transit service coverage.

flict between user and operator objectives, the sum of operator and user time costs (i.e., access, waiting, and in-vehicle riding times multiplied by the value of user time) is often used as a suitable design criterion. In the case of elastic demand, with no requirements for minimum service provision, it is possible to find a set of LOS and fare that minimizes operator and user costs by effectively eliminating ridership. In this case the objective of minimizing the total system cost should be replaced with the maximization of social welfare (defined as the sum of consumer surplus and operator surplus, or profit) subject to a budget constraint.

In this paper the bus service coverage problem of Figure 1 is formulated as an optimization problem wherein the route length, route spacing, headway, and fare must be chosen to maximize either profit or social welfare. The optimization process yields optimal values for service characteristics taking into consideration the interaction of demand and operator cost.

STUDY APPROACH

The problem under consideration is to provide optimal transit service coverage with a simplified bus transit system in an urban corridor as illustrated in Figure 2. The corridor of length E and width Y is divided into two zones. Zone 1 is the area between the end of the corridor and the route terminus, and Zone 2 is the area between the CBD and the route terminus.

The basic approach of this paper is to formulate design objectives as functions of the decision variables. The optimal values of the decision variables are found by taking partial derivatives of the objective function with respect to all decision variables, setting them equal to zero, and solving them simultaneously. This approach, as will be seen later, resulted in a simple model that offered considerable insight into the optimality conditions and interrelations among variables. The equations obtained are incorporated within an efficient algorithm that optimizes service coverage for a more realistic model that includes a vehicle capacity constraint.

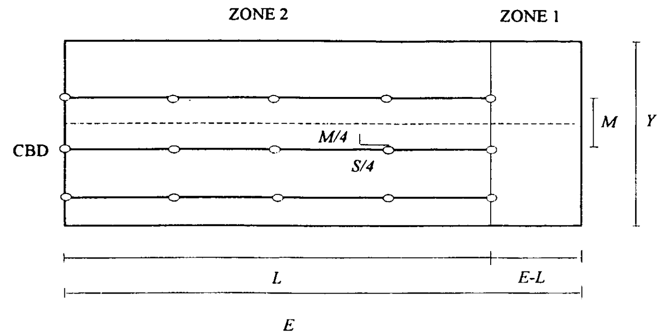

FIGURE 2 Urban corridor and transit network under study.

BUS SYSTEM CHARACTERISTICS AND DEMAND/SUPPLY FUNCTIONS

This section describes briefly the assumptions of the bus system's operating characteristics and presents the derivation of the system's passenger demand and cost functions.

Assumptions About Bus System Characteristics

1. An urban rectangular corridor is served by a bus transit system consisting of n parallel routes of uniform length L separated laterally by route spacing M.
2. The routes extend from the CBD outward.
3. The total transit demand is uniformly distributed along the entire corridor and over time and is sensitive to the quality of transit service and fare.
4. The commuter travel pattern consists of many-to-one or one-to-many trips focused on the CBD.
5. A dense rectangular grid street network allows passengers orthogonal access movements (i.e., access paths are parallel and perpendicular to the route).
6. Transit vehicles operate in local service (i.e., all vehicles serve all stations).
7. The average access speed is constant. Walking is the only access mode.
8. Average waiting time equals half the headway. The headway is uniform along the route and among all parallel routes.
9. Operator costs are limited to those for vehicles (i.e., the infrastructure is free).
10. There is no limit on vehicle fleet size.

Demand Functions

The urban corridor demand is assumed to be a linear function sensitive to price and various travel time components (waiting, access, and in-vehicle times). A conceptual form of the demand density function is as follows:

$$q = P[1 - e_w * \text{wait time} - e_a * \text{access time} - e_{iv} * \text{in-vehicle time} - e_p * \text{fare}] \quad (1)$$

where

q = unit transit demand density (passengers/mi^2-hr),
P = potential travel demand density (passengers/mi^2-hr),
e_w = sensitivity factor for waiting time,
e_a = sensitivity factor for access time,
e_{iv} = sensitivity factor for in-vehicle time, and
e_p = sensitivity factor for fare.

The demand function is similar to the one suggested by Kocur and Hendrickson (*12*) and is almost identical to that of Chang and Schonfeld (*20*).

For the particular application presented in this paper total demand consists of the sum of Zone 1 and Zone 2 demands. It is obvious that access, waiting, and in-vehicle times will affect demand in both zones. Because the trip origins are uniformly distributed over the corridor an average passenger accessing the route walks perpendicularly one-quarter of the spacing between the two routes—an access distance of $M/4$. The access distance parallel to the route depends on whether the trip originated within Zone 1 or Zone 2. Passengers originating in Zone 1 must board vehicles at the terminus, thus having a total average access distance of $(E - L)/2 + M/4$. A passenger from Zone 2 walks along the route one-quarter of the local stop spacing S to reach a stop. The total access time for an average passenger in Zone 1 equals the average access distance divided by the access speed g [i.e., $(E - L)/2g + M/4g$]. For a passenger in Zone 2 the access time is $(M + S)/4g$.

The in-vehicle time is the actual riding time between the stop of origin and the CBD. The average in-vehicle time is obtained as the average distance traveled divided by the average transit speed V and is different for each zone. Passengers originating in Zone 1 travel the whole length of route L, whereas those from Zone 2 travel approximately an average distance of $L/2$. According to Assumption 8 passengers wait $H/2$.

The hourly transit demand in Zone 1 (in passengers per hour) is then given as

$$Q_1 = PY(E - L)\left[1 - e_w\frac{H}{2} - e_a\left(\frac{M}{4g} + \frac{E - L}{2g}\right) - e_{iv}\frac{L}{V} - e_p f\right] \quad (2a)$$

where:

Q_1 = transit demand in Zone 1 (passengers/hr),
P = potential transit trip density (passengers/km^2-hr),
Y = corridor width (km),
E = corridor length (km),
L = length of transit route (km),
H = route headway (hr/vehicle),
M = route spacing (km/route),
g = access speed (km/hr), and
V = average transit speed (km/hr).

The hourly transit demand in Zone 2 (in passengers per hour) is as follows:

$$Q_2 = PYL\left[1 - e_w\frac{H}{2} - e_a\frac{M + S}{4g} - e_{iv}\frac{L}{2V} - e_p f\right] \quad (2b)$$

where S is average stop spacing (km/stop).

The total hourly corridor demand, Q, is the sum of Q_1 and Q_2.

Operator Cost

The operator cost includes maintenance and overhead as well as the more direct cost of operation (driver wages, fuel, spare parts, etc.) and is represented by the all-inclusive hourly operating cost per vehicle, c. The total hourly operator cost is obtained by multiplying the active fleet size by the hourly operating cost per vehicle. Fleet size is the number of on-line vehicles required to provide service and is obtained by dividing the total round-trip time (running time and layover time) by the headway. The total round-trip time is the round-trip route length divided by the average speed. The total hourly operator cost is then

$$C = \frac{2cYL}{HMV} \quad (3)$$

where

C = operator cost (\$/hr),
c = vehicle operating cost (\$/vehicle-hr),
Y = corridor width (km),
L = length of transit route (km/route),
H = route headway (hr/vehicle),
M = route spacing (km/route), and
V = average transit speed (km/hr).

TRANSIT SERVICE DESIGN OBJECTIVES

The two objectives considered in this paper are maximization of operator profit and maximization of social welfare. The analysis consists of optimizing service coverage under each objective, comparing the results, and deriving insights about the optimal coverage.

Maximizing Operator Profit

Operator profit (Π) is defined as a difference between the fare box revenue R and operator cost C

$$\Pi = R - C \quad (4)$$

Revenue R is defined as the fare multiplied by ridership

$$R = PYE\left(1 - e_w\frac{H}{2} - e_a\frac{M}{4g} - e_p f\right)f + PY(E - L)\left(- e_a\frac{E - L}{2g} - e_{iv}\frac{L}{V}\right)f + PYL\left(- e_a\frac{S}{4g} - e_{iv}\frac{L}{2V}\right)f \quad (5)$$

The hourly operator profit (Π) is the difference between the total operator revenue (Equation 5) and operator cost (Equation 3)

$$\Pi = PYE\left(1 - e_w \frac{H}{2} - e_a \frac{M}{4g} - e_p f\right)f + PY(E - L)\left(- e_a \frac{E - L}{2g} - e_{iv} \frac{L}{V}\right)f + PYL\left(- e_a \frac{S}{4g} - e_{iv} \frac{L}{2V}\right)f - \frac{2cYL}{HMV} \quad (6)$$

The operator profit function can be maximized by setting its partial derivatives with respect to the route length L, headway H, route spacing M, and fare f, to zero. When the resulting equations are solved independently, the following expressions for route length L, headway H, spacing M, and fare f are obtained:

$$L^* = E - \frac{2cg}{PHMf(e_aV - e_{iv}g)} - \frac{e_aSV}{4(e_aV - e_{iv}g)} \quad (7a)$$

$$H^* = \left(\frac{4cL}{e_wMPEVf}\right)^{1/2} \quad (7b)$$

$$M^* = \left(\frac{8cLg}{e_aHPEVf}\right)^{1/2} \quad (7c)$$

$$f^* = \frac{2 - e_wH}{4e_p} - \frac{e_a[ME + 2(E - L)^2 + SL]}{8e_pEg} - \frac{e_{iv}(2LE - L^2)}{4e_pEV} \quad (7d)$$

Solving Equations 7b and 7c simultaneously yields the following expressions for H and M:

$$H^* = \left(\frac{2cLe_a}{PEVfe_w^2g}\right)^{1/3} \quad (8a)$$

$$M^* = \left(\frac{16cLe_wg^2}{PEVfe_a^2}\right)^{1/3} \quad (8b)$$

When the route length, route spacing, headway, and fare are optimized independently of each other, their relation to the other decision variables can be read directly from Equations 7a to 7d. These equations provide the optimal value of one of the decision variables as a function of the other three. For exmaple Equation 7a can be used to find the optimal route length when the headway, route spacing, and fare are given. Equations 8a and 8b may be useful by themselves in cases in which the route length L or fare f cannot be modified.

Equations 7a to 7d also provide useful insights into the relationship between the decision variables and the various parameters. For example according to Equation 7a the optimal route length varies directly with the corridor length E, passenger density P, headway H, route spacing M, fare f, sensitivity factor for access time e_a, and transit speed V. It varies inversely with the vehicle operating cost c, access speed g, stop spacing S, and the sensitivity factor for in-vehicle time e_{iv}.

It should be noted that the simultaneous solution of Equations 8a and 8b produces an interesting result. Optimally the ratio of route spacing and headway is constant and has the following value:

$$\frac{M^*}{H^*} = 2 \frac{e_wg}{e_a} \quad (9)$$

Unfortunately all four equations, Equations 7a to 7d, cannot be solved simultaneously by algebraic methods.

Maximizing Social Welfare

Social welfare (W) is defined as the sum of consumer surplus (T) and producer surplus or profit (Π)

$$W = T + \Pi \quad (10)$$

Consumer surplus (T) is the total social benefit minus the total cost that users actually pay. The total social benefits (also known as the users' willingness to pay) for each of the zones can be obtained by inverting the demand functions (Equations 2a and 2b) to find the fare as a function of demand and by integrating the inverted functions from zero to Q_1 and Q_2, respectively. Then the total consumer surplus (T) can be stated as

$$T = \frac{PY(E - L)}{2e_p}\left[1 - e_w \frac{H}{2} - e_a\left(\frac{M}{4g} - \frac{E - L}{2g}\right) - e_{iv} \frac{L}{V} - e_p f\right]^2 + \frac{PYL}{2e_p}\left(1 - e_w \frac{H}{2} - e_a \frac{M + S}{4g} - e_{iv} \frac{L}{2V} - e_p f\right)^2 \quad (11)$$

Therefore, the social welfare objective can be formulated as follows:

$$W = \frac{PY(E - L)}{2e_p}\left[1 - e_w \frac{H}{2} - e_a\left(\frac{M}{4g} + \frac{E - L}{2g}\right) - e_{iv} \frac{L}{V} - e_p f\right]^2 + \frac{PYL}{2e_p}\left(1 - e_w \frac{H}{2} - e_a \frac{M + S}{4g} - e_{iv} \frac{L}{2V} - e_p f\right)^2 + PY(E - L)\left[1 - e_w \frac{H}{2} - e_a\left(\frac{M}{4g} + \frac{E - L}{2g}\right) - e_{iv} \frac{L}{V} - e_p f\right]f + PYL\left(1 - e_w \frac{H}{2} - e_a \frac{M + S}{4g} - e_{iv} \frac{L}{2V} - e_p f\right)f - \frac{2cYL}{HMV} \quad (12)$$

In solving for the maximization of social welfare a deficit constraint is considered. This constraint states that the operator cost must be equal to the sum of the total revenue R and a prespecified acceptable level of subsidy K, namely

$$C = R + K \quad (13)$$

Therefore, the deficit constraint is as follows:

$$\frac{2cYL}{HMV} - PY(E - L)\left[1 - e_w \frac{H}{2} - e_a\left(\frac{M}{4g} + \frac{E - L}{2g}\right) - e_{iv}\frac{L}{V} - e_p f\right]f - PYL\left(1 - e_w \frac{H}{2} - e_a \frac{M + S}{4g} - e_{iv}\frac{L}{2V} - e_p f\right)f - K = 0 \tag{14}$$

The breakeven constraint is introduced by eliminating subsidies (i.e., $K = 0$) from Equation 14.

Unconstrained Subsidy Results

If the subsidy is unconstrained the first-order conditions at optimum are

$$\frac{\partial W}{\partial L} = 0 \tag{15}$$

$$\frac{\partial W}{\partial M} = 0 \tag{16}$$

$$\frac{\partial W}{\partial H} = 0 \tag{17}$$

$$\frac{\partial W}{\partial f} = 0 \tag{18}$$

The optimized fare can be immediately obtained for Equation 18 and is $f^* = 0$.

This result is not surprising because the marginal operator cost is zero according to the assumptions made so far. The marginal cost, and thus the fare, would become positive if a vehicle capacity constraint is introduced, as will be shown later.

By substituting a zero fare back into Equations 15 to 17, the expressions for the optimal route length, spacing, and headway are obtained, and they are given in Appendix A.

Results with Breakeven Constraint

To solve the problem by using the breakeven constraint, the constraint was introduced into the objective function (Equation 12) with a multiplier, λ. The purpose of λ is to introduce a penalty for violating the constraint. In economic terms it is the "shadow price" of the subsidy (i.e., it indicates the change in welfare that will result from a \$1 subsidy).

The expressions for the optimal route length, spacing, headway, and fare assuming a breakeven constraint (i.e., no subsidy) are also shown in Appendix A. A detailed derivation of these expressions can be found in Spasovic et al. (*23*).

Capacity Constrained Headway

The models for maximizing either operator profit or social welfare presented so far have not taken into account a vehicle capacity constraint. This constraint ensures that the total capacity provided on the routes satisfies the demand at some reasonable LOS by restricting the maximum allowable headway. The constraint is written as

$$PYE\left(1 - e_w \frac{H}{2} - e_a \frac{M}{4g} - e_p f\right) + PY(E - L)\left(- e_a \frac{E - L}{2g} - e_{iv}\frac{L}{V}\right) + PYL\left(- e_a \frac{S}{4g} - e_{iv}\frac{L}{2V}\right) \leq k \frac{Y}{MH} l \tag{19}$$

where k is the capacity of transit vehicle (in spaces), and l is the allowable peak load factor at the CBD. The expression for maximum allowable headway, derived from Equation 19, is used within an optimization algorithm that is described next.

OPTIMIZATION ALGORITHM

Although the models presented so far provided valuable insights into the relations among decision variables and exogenous parameters, they are too complex for simultaneously optimizing all of the decision variables algebraically. To solve the model an algorithm that sequentially used Equations 7a to 7d (or Equations a1 to a3 or Equations b1 to b5 in the Appendix, depending on the objective to be optimized) was developed to advance from an initial feasible solution toward the optimal solution. The algorithm starts with a trivial feasible solution and in each step improves the value of the objective function by computing an optimal value of one decision variable while keeping the others at their feasible levels. In computing the optimal values of decision variables, the algorithm computes sequentially the route length, route spacing, headway, and finally fare. In each step the value of a newly computed variable is recorded and used in the next step for computing the optimal values of the other decision variables. The algorithm keeps improving the objective function until it converges to an optimal solution. It terminates when the values of the objective functions from two successive iterations are sufficiently close and no significant further improvement can be expected. The objectives turned out to be relatively flat (shallow, four-dimensional, U-shaped) functions. Thus small deviations from the optimal decision variables result in even smaller relative changes in the values of the objectives.

It is quite possible that buses may overload if no capacity constraint is introduced. Instead of formulating a model as a constrained optimization problem with a nonlinear objective function and a linear constraint and solving it by using a penalty method, the following modification of the algorithm is made to incorporate the vehicle capacity constraint:

1. Examine whether the newly obtained optimal headway satisfies the capacity constraint, by computing the optimal busload and checking whether the busload exceeds capacity.
2. If the busload is smaller than the available capacity there is no need for capacity-constrained results.
3. Otherwise set the optimal headway equal to the maximum

allowable headway (obtained by solving Equation 19), which is as follows:

$$H^* = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \tag{20}$$

where

$$B = -(E - L)^2 \frac{e_a}{2g} + E\left(1 - e_a \frac{M}{4g} - e_p f\right) - e_{iv} \frac{(E - L)L}{V} - L\left(e_a \frac{S}{4g} + e_{iv} \frac{L}{2V}\right),$$

$$A = -e_w \frac{E}{2}, \text{ and}$$

$$C = -\frac{k}{PM}.$$

Then calculate the set of decision variables that satisfies the capacity constraint. This is considered to be the optimal solution.

NUMERICAL EXAMPLE

A numerical example is developed to demonstrate how the model optimizes transit service coverage.

Table 2 gives results from the maximization of operator profit and social welfare (with both the unconstrained subsidy and breakeven constraint) objectives. The results include optimal route length, route spacing, headway, fare, operator profit, social welfare, and consumer surplus for a rectangular corridor of 8.045 × 4.824 km (5 × 3 mi) with a potential demand density of 77.35 passengers/km^2-hr (200 passengers/m^2-hr). The hourly operating cost of the bus is assumed to be \$40/vehicle, the average transit speed is assumed to be 16.09 km/hr, and the average access speed is assumed to be 4.02 km/hr. The transit vehicle capacity is 50 seats/vehicle, the allowable peak load factor is 1, the stop spacing is 0.402 km/stop, and the sensitivity factors for waiting time, access time, in-vehicle time, and fare are 0.7, 0.7, 0.35, and 0.5, respectively.

Under the profit maximization objective the optimal route length is 5.3 km (3.296 mi), route spacing is 1.614 km (1.004 mi), headway is 0.201 hr, and the fare is \$0.88. The induced hourly ridership is 743 passengers, yielding a \$264 profit.

Under the welfare maximization objective with unconstrained subsidy the optimal service design has a route length of 4.560 km (2.834 mi), route spacing of 1.120 km (0.699 mi), headway of 0.140 hr, and a \$0.36 fare. The induced hourly ridership is 1,536 passengers, the social welfare is \$719, and the subsidy is \$150. The introduction of the breakeven constraint results in an optimal service design with a route length of 4.57 km (2.838 mi), route spacing of 1.19 km (0.739 mi), headway of 0.148 hr, and a \$0.45 fare. These service variables induce an hourly ridership of 1,372 passengers, yielding a social welfare of \$710.

A comparison of the welfare maximization results with the unconstrained subsidy and breakeven constraints reveals that when the breakeven constraint is removed the welfare increases slightly (by \$8.50 or approximately 1.2 percent), whereas the deficit increases much more (from \$0 to \$150). The welfare function appears to be relatively flat near the optimum. This indicates that minor deviations from the optimum will not decrease welfare significantly. This result is similar to the one found by Chang and Schonfeld (*20*). This implies that for a given set of input data the welfare objective with a breakeven constraint seems quite reasonable and far more desirable from an economical standpoint than the welfare objective with unconstrained subsidy.

The shadow price in Table 2 implies that relaxing the breakeven constraint and thus increasing the operator deficit from \$0 to \$1/hr (i.e., to a \$1/hr subsidy) will result in a \$0.128/hr increase in welfare.

The equilibrium demand is strongly influenced by the level of service and the fares optimized under different objectives. The total hourly demand level is 24.8 percent of the potential demand under profit maximization. It is 51.2 and 45.76 percent of the potential demand under welfare maximization for the unconstrained subsidy and breakeven conditions, respectively.

A comparison of the optimal route length for different objectives indicates that profit maximization yields longer routes of 5.3

TABLE 2 Optimal Objectives and Design Variables

	Objective Functions		
	Profit	Social Welfare	
		Unconstrained	Break-Even
Route Length (km)	5.3 (3.296 mi)	4.56 (2.834 mi)	4.57 (2.838 mi)
Route Spacing (km)	1.61 (1.004 mi)	1.12 (0.699 mi)	1.19 (0.739 mi)
Headway (hr)	0.201	0.140	0.148
Fare (\$)	0.88	0.36	0.45
Ridership (pass/hr)	744	1536	1372
Operator Cost (\$/hr)	392	696	623
Revenue (\$/hr)	656	546	623
Profit (\$/hr)	264	-150	0
Consumer Surplus (\$/hr)	2367	869	710
Welfare (\$/hr)	501	719	710
Bus Load (pass/bus)	50	50	49.267
Fleet Size (buses)	3.28	4.05	3.83
Shadow Price of Subsidy Increase (\$/hr)	N/A	N/A	0.128

km (3.29 mi) than welfare maximization [4.56 km (2.834 mi)]. Also the optimal design for profit maximization has longer headway and route spacing than that of welfare maximization [i.e., 0.201 hr and 1.61 km (1.004 mi) versus 0.14 hr and 1.120 km (0.699 mi)]. This can be explained by the presence of the vehicle capacity constraint and the customers' higher value for waiting and access times than for in-vehicle time (as indicated by the values of e_w, e_a, and e_{iv}) that replace the transit system with one with denser routes and more frequent service so that welfare can be maximized.

A sensitivity analysis was performed to show how changes in the more important exogenous parameters given in the numerical example affect the values of the decision variables and objective functions. The changes in design variables, namely route length, spacing, headway, and fare with respect to the corridor length, passenger density, transit and access speed, operator cost, sensitivity factors, and fare, are shown in Table 3. The two values used for each parameter are between 10 and 20 percent above and below those that were used to generate the basic results of Table 2.

Table 3(*A*) illustrates the effects that changes in parameters have on the optimal design variables under profit maximization. For example if corridor length is increased by 10 percent (from 8.045 to 8.8495 km) the route length is increased by 10.4 percent. This implies that the optimal route length L is elastic (i.e., the absolute

TABLE 3 Sensitivity Analysis

A) FOR PROFIT MAXIMIZATION

Parameters		Length (km)	Spacing (km)	Headway (hr/veh)	Fare ($/pass)	Profit ($/hr)	Demand (pass/hr)
Corridor Length	7.24	4.761	1.636	0.203	0.87	286.33	725
(km)	8.85	5.845	1.610	0.200	0.89	234.96	749
Density	69.53	5.303	1.71	0.213	0.87	227.66	664
(pas/km^2-hour)	84.98	5.303	1.535	0.191	0.89	301.39	824
Transit Speed	14.48	4.962	1.684	0.209	0.86	211.68	684
(km/hr)	17.7	5.573	1.562	0.194	0.90	313.02	795
Access Speed	3.62	5.601	1.556	0.215	0.89	239.27	722
(km/hr)	4.425	4.997	1.672	0.189	0.87	287.96	764
Stop Spacing	0.362	5.314	1.614	0.201	0.89	265.96	746
(km)	0.442	5.292	1.617	0.201	0.88	262.53	742
Operator Cost	36	5.565	1.570	0.195	0.89	305.61	787
($/hr)	44	5.041	1.664	0.207	0.87	227.06	702
(e_w, e_a)	0.6	4.788	1.574	0.196	0.88	318.01	784
	0.8	5.673	1.655	0.206	0.87	217.66	708
(e_{iv})	0.25	4.788	1.578	0.196	0.92	298.08	779
	0.45	5.673	1.655	0.206	0.84	232.34	708
(e_p)	0.4	5.828	1.527	0.190	1.11	439.37	832
	0.6	4.777	1.712	0.213	0.72	161.46	662
Optimal Results*		5.300	1.61	0.201	0.88	264.24	744

B) FOR WELFARE MAXIMIZATION WITH BREAK-EVEN CONSTRAINT

Parameters		Length (km)	Spacing (km)	Headway (hr/veh)	Fare ($/pass)	Welfare ($/hr)	Demand (pass/hr)
Corridor Length	7.24	4.22	1.199	0.149	0.42	733.53	1353
(km)	8.85	4.854	1.195	0.149	0.48	678.40	1358
Density	69.53	4.563	1.263	0.157	0.45	623.86	1218
(pas/km^2-hour)	84.98	4.568	1.128	0.140	0.45	797.96	1528
Transit Speed	14.48	4.149	1.250	0.155	0.45	618.77	1243
(km/hr)	17.7	4.927	1.144	0.142	0.44	798.07	1483
Access Speed	3.62	4.798	1.152	0.159	0.47	663.54	1315
(km/hr)	4.425	4.327	1.224	0.138	0.43	754.92	1424
Stop	0.362	4.574	1.187	0.148	0.46	714.47	1376
Spacing	0.442	4.557	1.190	0.148	0.45	706.60	1369
Operator Cost	36	4.894	1.150	0.143	0.44	785.33	1466
($/hr)	44	4.256	1.231	0.153	0.46	644.46	1283
(e_w, e_a)	0.6	4.167	1.145	0.142	0.41	805.65	1479
	0.8	4.849	1.236	0.154	0.48	627.74	1272
(e_{iv})	0.25	4.761	1.157	0.144	0.47	768.93	1451
	0.45	4.37	1.224	0.152	0.43	657.04	1295
(e_p)	0.4	5.239	1.115	0.139	0.52	1087.39	1564
	0.6	3.966	1.274	0.158	0.39	488.42	1197
Optimal Results*		4.57	1.19	0.148	0.45	710.52	1372

*** For the values of the exogenous parameters given in the numerical example**

value of the elasticity exceeds 1.0) with respect to the corridor length E. The reason for this is that, as the length of the corridor E is increased, the length of the area between the terminus and the end of the corridor $(E - L)$ is increased very slowly, thus increasing L faster than E. This result is consistent with those obtained by Spasovic and Schonfeld (*16*) for fixed-demand systems. Also if the passenger density is increased by 10 percent the headway will be reduced by 5 percent. This result confirms that headway varies inversely with the cube root (approximately) of the passenger density. Table 3(A) also shows that the route length would decrease by 10 percent if the sensitivity factor for fare is increased by 20 percent (from 0.5 to 0.6).

Table 3(B) shows the effect that changes in parameters have on the optimal design variables under welfare maximization with a breakeven constraint.

The effect of the route length on profit and on welfare is shown in Figure 3. For a given route length the system design variables have been reoptimized, yielding the optimal profit or welfare. The profit and welfare functions are relatively flat near the optimum. A practical application of this result is that, for a given set of data, the optimal design variables can be tailored to the actual street network without substantially reducing the optimal profit or welfare.

The effect of subsidy on welfare and consumer surplus is shown in Figure 4. For a given subsidy level, the system design variables have been reoptimized, yielding the optimal welfare. The consumer surplus increases with subsidy. For no subsidy the breakeven constraint holds and the social welfare equals consumer surplus. The net effect of the profit and consumer surplus interactions is that the welfare function is relatively flat near the optimum for a relatively large range of subsidies. A practical implication of this result is that for a given set of data the breakeven constraint may be economically and politically preferable because it eliminates subsidy and marginally reduces social welfare. Furthermore Figure 4 shows that a negative subsidy (profit) can be obtained by marginally decreasing welfare.

CONCLUSIONS

The paper presented a model of optimal bus transit service coverage that was optimized to maximize profit and social welfare with unconstrained subsidy and a breakeven constraint. The model provides simple guidelines for optimizing the extent of transit routes and other major operating characteristics such as route

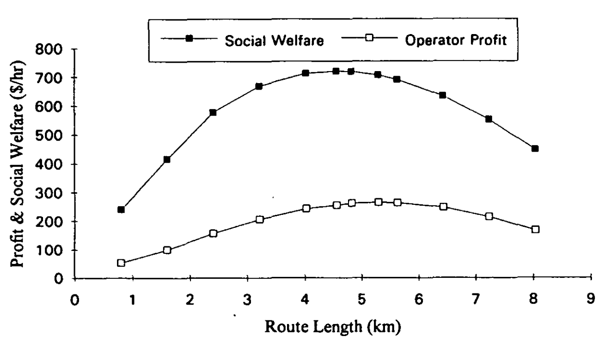

FIGURE 3 Impact of route length on design objectives.

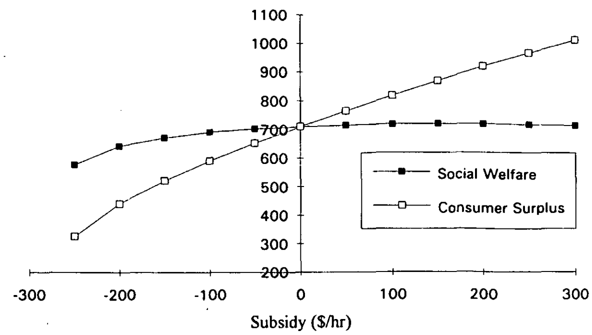

FIGURE 4 Social welfare and consumer surplus for various subsidy levels.

spacing, headway, and fare. Equations 7a to 7d can be used to optimize route length, route spacing, headway, and stop spacing separately. They provide insights into the interrelationships among the optimized variables. For example the cube root in Equations 8b and 8c indicates that optimal solutions for headway and route spacing are relatively insensitive to changes in system parameters.

The optimality of a constant ratio between route spacing and headway, which has been found in previous studies for various bus network and demand conditions (*12.20*), is also found to be maintained in the present study, which optimized the route length as well. The route spacing and headway that optimize profit, welfare with unconstrained subsidy, and welfare with a breakeven constraint closely maintain a ratio of 5.00, irrespective of the values of the other parameters such as potential demand density, sensitivity factors, or speed.

The profit and social welfare functions are relatively flat near the optimum. For practical applications this implies that a near-optimal profit or welfare can be attained while fitting the transit network to the particular street network or modifying its operating schedule.

The results of maximization of social welfare for different subsidy levels indicate that the welfare function is relatively flat near the optimum. A practical application of this result is that for a given set of data the subsidy can be reduced (or eliminated) by providing passengers a service with marginally worse quality. Therefore the welfare objective under a breakeven constraint seems reasonable and more desirable from an economical standpoint than the welfare objective with unconstrained subsidy. Furthermore for a given set of input data in the numerical example a negative subsidy or profit can be obtained for a marginal decrease in social welfare.

FUTURE RESEARCH POSSIBILITIES

Several simplifying assumptions could be relaxed in future models. The linear demand function may be replaced by a nonlinear function that more precisely reflects traveler behavior. More realistic and irregular distributions of temporal and spacial demand (e.g., nonuniform lateral distributions) could be used. The model could be improved to handle non-CBD trips (e.g., many-to-many travel pattern) and access modes other than walking. A modified model could handle sectorial service areas with possible overlaps in service coverage among the routes, and the assumption that average stop spacing is constant could be relaxed. The impacts of passengers boarding and alighting on bus dwell time, cruising speeds, and the cost of operations may also be included.

ACKNOWLEDGMENT

The authors appreciate the partial support provided by the National Center for Transportation and Industrial Productivity at the New Jersey Institute of Technology funded by the University Transportation Centers Program.

APPENDIX A
Optimal Bus Transit Service Design Variables

ROUTE LENGTH

(*a*) With unconstrained subsidy

$$L^* = R - \tfrac{1}{2}(S + T)^{1/2}/(C^2 - A^2)\sqrt{HMVP} \quad \text{(a-1)}$$

(*b*) With breakeven constraint

$$\begin{aligned} L^* = R - \tfrac{1}{2}\{&S(1 + \lambda) + T \\ &+ fHMVPe_p 2(A + B)[B(A + 3C) \\ &+ D(3A + C)](1 + \lambda) \\ &+ fEHMVPe_p 2A(A - 3C)(A + C) \\ &+ f^2 HMVPe_p^2 4(A + C)^2(1 + \lambda)^2\}^{1/2} \\ &\div (C^2 - A^2)\sqrt{HMVP} \end{aligned} \quad \text{(b1)}$$

where, for Equations a1 and b1

$$A = \frac{e_a}{2g} - \frac{e_{iv}}{V},$$

$$B = 1 - e_w\frac{H}{2} - e_a\frac{M}{4g} - e_a\frac{E}{2g},$$

$$C = \frac{e_{iv}}{2V},$$

$$D = 1 - e_w\frac{H}{2} - e_a\frac{M + S}{4g},$$

$$R = \frac{-[2(AB + CD) - A^2E]}{3(A^2 - C^2)},$$

$S = 12ce_p(A^2 - C^2)$, and

$$\begin{aligned} T = &HMVP[(AB + CD)^2 + 3(AD + BC)^2] \\ &+ EHMVPA(A^3E + 2A^2B - 4ACD - 6BC^2). \end{aligned}$$

HEADWAY

(*a*) With unconstrained subsidy

$$H^* = \frac{\sqrt{J}}{A + B(1 - C - D + F - 2G) + L(1 - G - I)} \quad \text{(a2)}$$

(*b*) With breakeven constraint

$$H^* = \frac{\sqrt{J(1 + \lambda)}}{\begin{gathered} A + B(1 - C - D + F - 2G) + L(1 - G - I) \\ + epf[-B + (E - L) + E\lambda] \end{gathered}} \quad \text{(b2)}$$

ROUTE SPACING

(*a*) With unconstrained subsidy

$$M^* = \frac{\sqrt{J'}}{A' + B(1 - C' - D + F - 2G) + L(1 - G - I')} \quad \text{(a3)}$$

(*b*) With breakeven constraint

$$M^* = \frac{\sqrt{J'(1 + \lambda)}}{\begin{gathered} A' + B(1 - C' - D + F - 2G) + L(1 - G - I') \\ + e_p f[-B + (E - L) + E\lambda] \end{gathered}} \quad \text{(b3)}$$

where, for Equations a2, b2, a3, and b3

$$\begin{aligned} A &= -Ee_w/2, \\ A' &= -Ee_a/2, \\ B &= E - L, \\ C &= e_a M/4g, \\ C' &= e_w H/2, \\ D &= e_a E/2g, \\ F &= e_a L/2g, \\ G &= e_{iv} L/2V, \\ I &= e_a(M + S)/4g, \\ I' &= e_a(2Hg + S)/4g, \\ J &= (4ce_p L)/(MVPe_w), \text{ and} \\ J' &= (8cge_p L)/HVPe_a. \end{aligned}$$

FARE

(*a*) With breakeven constraint

$$f^* = \frac{\begin{gathered} 2g\lambda Le_{iv}(-2E + L) - 2e_a\lambda V(E - L)^2 \\ + e_a\lambda V(LS - EM) + 2Eg\lambda V(2 - He_w) \end{gathered}}{4Ege_p V(1 + 2\lambda)} \quad \text{(b4)}$$

(*b*) Shadow price for breakeven constraint:

$$\lambda^* = \frac{-X + \sqrt{X^2 - 4(A - B)(X + ICDJ + IFGJ - Z)}}{2(X + ICDJ + IFGJ - Z)} \quad \text{(b5)}$$

where, for Equation b5

$X = 4A - 4B + ICDJ + IFGJ,$
$Z = I^2 e_p(C + F)$, and
$A = 2cLY/HMV,$
$C = (E - L)PY,$
$D = 1 - e_a(E - L)/2g - e_a M/4g - Le_{iv}/V - He_w/2,$
$F = LPY,$
$G = 1 - e_a(M + S)/4g - Le_{iv}/2V - He_w/2,$
$J = -4ELge_{iv} + 2L^2 ge_{iv} - 2E^2 Ve_a + 4EVg + 4ELVe_a - 2e_a L^2 V - e_a EMV - e_a LSV - 2EgHVe_w$, and
$I = 4EgPV.$

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Publication of this paper sponsored by Committee on Public Transportation Planning and Development.