Computationally Efficient Method for Inclusion of Nonprismatic Member Properties in a Practical Bridge Analysis Procedure

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For purposes of bridge analysis, bridges typically are classified as either statically determinate or statically indeterminate. Continuous-span bridge structures, which generally are statically indeterminate, offer advantages over statically determinate, simply supported bridge systems, such as lighter weight, lower cost, greater stiffness, and smaller deflections. Therefore, most multispan bridge structures are designed to be continuous. In the past, one of the simplifying assumptions generally made during the bridge analysis process was that the bridge superstructure members could be analyzed as prismatic members. This was primarily because of the computationally intensive nature of the calculations necessary to include consideration of nonprismatic member properties. However, the widespread proliferation of the digital computer has eliminated the need for this particular analysis simplification. A comprehensive outline of a computer-based bridge analysis process that incorporates nonprismatic member behavior is prescribed. Application of this bridge analysis procedure is shown to be remarkably reliable and accurate compared with theoretically exact analysis results and is considerably more accurate than utilizing a simple prismatic analysis. The analysis procedure presented is economical in terms of computational time and computer memory requirements and is a practical alternative to currently used analysis methods that consider only prismatic member properties.

Bridges can be classified in many ways: for example, by type of girders or by type of material. With respect to analysis, bridges typically are classified as either statically determinate, for which all reactions and internal forces can be obtained directly from static equilibrium equations, or indeterminate, which requires a more sophisticated analysis. Continuous-span bridge structures, which are statically indeterminate (assuming the absence of interior moment releases), offer advantages over statically determinate, simply supported bridge systems. These advantages include lighter weight, lower cost, greater stiffness, and smaller deflections, as well as greater overload capability caused by stress redistribution.

In the past, analysis of indeterminate, continuous-span bridge structures has been performed by applying analysis methods that make use of several simplifications. These simplifications include the use of the AASHTO wheel load distribution factor, the AASHTO load impact factor, and the assumption of prismatic member section properties, among others. The use of any one of these analysis simplifications can cause an error in the analysis results, and using all of these can yield a significant discrepancy between the results of the simplified analysis and the theoretically exact values.

Steps recently have been taken to provide means by which to eliminate some of these simplifications from the analysis of indeterminate bridge structures. For example, NCHRP recently funded the development of a computer program to generate more accurate wheel load distribution factors. This program, LDFAC, calculates wheel load distribution factors using a finite element-based structural analysis, which extends the range of applicability of the wheel load distribution factors by a wide margin and allows more reliable and accurate analysis results to be obtained for cases such as bridges with skewed supports, continuous spans, and bridges for which geometric parameters such as span length or girder spacing fall outside the range of simplified formulas (*1*). Additionally, in a study sponsored by the National Science Foundation, the dynamic influence of moving vehicular traffic was investigated (*2*). In this study it was shown that the AASHTO load impact factor yields results that can be significantly in error and it presented an alternative method for including the effects of impact on bridge structures.

This paper focuses on the minimization of error introduced into girder bridge analysis results as a result of making the assumption that all of the structural members behave as prismatic members. The nonprismatic behavior of bridge structural members was ignored in the past primarily because of the computationally intensive nature of the mathematical calculations. However, the widespread proliferation of the digital computer makes unnecessary a continuation of this analysis simplification. This paper presents the development of an analysis methodology that incorporates rapid and accurate dead and live load internal force evaluation for nonprismatic girder bridges. This analysis procedure is remarkably successful in significantly reducing the error between the results of the bridge analysis procedure and the theoretically exact solution.

TYPICAL BRIDGE ANALYSIS METHODS

Classical methods, approximation methods, and numerical methods are the three types of analysis methods generally applied to civil engineering systems, including bridge structures. Classical analysis methods are based on the exact solution of the governing differential equations of the system. However, the limitations of these methods, which are applicable only to systems that possess relatively simple geometry, loading, and boundary conditions, restrict the usefulness to a narrow range of problems.

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More complex problems are typically solved using an approximation method or a numerical method. Approximation methods include energy methods, such as the principle of minimum potential energy; variational principles, such as the Galerkin method and the Ritz method; and perturbation methods. However, the application of approximate methods are limited to uncomplicated boundary conditions and simple variation of thickness. Also, it is important to remember that the use of an approximation method will yield just that—an approximate analysis solution.

The numerical methods of analysis are based on the principles of finite elements and finite differences. Numerical analysis allows for a more accurate analysis than can be achieved using the approximation methods and is applicable to a far wider range of problems than the other types of analysis methods. Numerical methods have been applied successfully to problems such as those that include tapered plates, circular plates, elements of varying thickness, and nonprismatic members.

All three analysis procedures are currently being used in bridge analysis. A classical approach based on the flexibility method is used in most older programs with a constant flexural stiffness *EI* to generate influence lines. A numeric successive approximation method known as Newmark's method is used in several computer programs because of the ease and simplicity of implementation. A number of direct stiffness matrix analysis programs are available that use a moving load to generate influence diagrams. Several of these programs use the "transfer matrix" approach to reduce the demands on computer memory and computation time. The analysis approach introduced in this paper has proven to be more accurate, based on a one-dimensional analysis, and is significantly more efficient in terms of computer storage requirements and execution time.

DEVELOPMENT OF GIRDER BRIDGE ANALYSIS METHODOLOGY

The bridge analysis approach developed in the computer program, termed GBRIDGE, consists of three segments: structural analysis based on the direct stiffness method, influence line generation, and determination of maximum moments and shears.

The direct stiffness method of structural analysis uses the principles of joint equilibrium and compatibility to solve for joint displacements. These actions and displacements are related through the matrix equilibrium equation

$$[A] = [S][D]$$

where

$[A]$ = action matrix of applied forces and moments,
$[S]$ = global stiffness matrix assembled from the member stiffness matrixes, and
$[D]$ = unknown displacement matrix.

In a physical sense, the global stiffness matrix contains coefficients that represent the actions taking place at a node caused by a unit displacement of a member end. This matrix, along with the appropriate boundary conditions, is then used to calculate the actual displacements caused by the actual dead load and live load forces and moments by the solution of simultaneous equations. After the joint displacements have been found, the forces, stresses, and displacements at the internal analysis points and at material breaks can be calculated through application of superposition for the dead load and superimposed dead load cases and through use of influence lines and superposition for the live load case.

A unique feature of the GBRIDGE analysis procedure is the inclusion of nonprismatic member behavior. Indeterminate bridge member section properties vary as a function of the construction process. In the dead load condition, prismatic bridge girders, such as AASHTO prestressed beams, can be analyzed on the basis of prismatic section properties before the hardening of the roadway deck. However, in the case of the composite bridge system, which has both positive and negative moment areas, the concrete roadway deck can contribute to composite action only in the positive moment area because concrete is effective only under compressive stress. In the negative moment area, the reinforcing steel can be considered in section property evaluation. Therefore, even in general continuous bridge systems with prismatic members, the bridge systems are composed of nonprismatic members if composite construction is used.

Incorporating nonprismatic member properties into the bridge analysis procedure presents particular problems when employing the direct stiffness analysis approach. The direct stiffness procedure assumes a continuous shape (displacement) function or interpolation polynomial in formulating the element stiffness matrix. Therefore, this method can lead to exact answers only when the displacement of the member's neutral axis is continuous. The difficulty associated with using the direct stiffness method becomes apparent when it is recognized that virtually all girder bridges are composed of segmental, nonprismatic supporting girders.

In the case of segmentally nonprismatic beams, any approximating shape function that represents the entire girder length must be discontinuous. This is explained through examination of the moment-curvature equation

$$\frac{d^2y}{dx^2} = \frac{M_x}{EI_x}$$

in which

y = displacement of the neutral axis,
x = location at any point on the member,
M_x = moment at location x,
I_x = moment of inertia at location x, and
E = modulus of elasticity at location x.

In this equation, y represents the displacements caused by bending of the beam member's neutral axis as a function of the member's length. However, on either side of a material change, the internal resisting moment M_x is the same but the member's neutral axis location and moment of inertia are different, as shown in Figure 1. Therefore, any analysis formulation must account for this discontinuity.

This discontinuity problem can be overcome provided that the girder is modeled by a series of prismatic beam elements. Each prismatic segment can utilize a continuous shape function because, for each segment, the neutral axis location remains constant. This type of formulation requires the use of a large number of prismatic beam elements to obtain accurate analysis results. This segmental formulation requires a large amount of the available computer random access memory and requires considerably longer execution time to solve the greater number of simultaneous equations that result. Even when using the "transfer matrix" approach, considerable central processing unit time is required. This difficulty is overcome in the GBRIDGE analysis procedure by integration of classical beam theory employing numeric integration and the traditional displacement-based direct stiffness analysis.

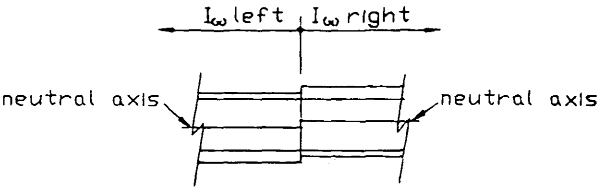


FIGURE 1 Nonprismatic girder material change.

Nonprismatic Stiffness Matrix

The development of the nonprismatic element stiffness matrix can be divided into two parts: the flexural contribution and the axial contribution. The flexural contributions to the girder stiffness matrix assume that the girder is bent in a principal plane and the effects of shear deformations can be neglected. In addition, it is assumed that the angle change between two adjacent cross sections is small after bending has occurred. The nonprismatic element formulation process employs the classical analysis approach of superposition in which the indeterminate structure is reduced to a statically stable and determinate structure by removing the redundant end moments M_L and M_R. These redundant end moments then are reapplied and the resulting member end rotations are related to the fact that the actual rotations at fixed ends are 0. Manipulating the solution of the resulting simultaneous equations will yield the nonprismatic element stiffness matrix and the equivalent nodal forces. The flexural stiffness components are

$$S_e = \frac{EI_L}{L^3(AC - B^2)} \begin{vmatrix} A & BL & -A & (A-B)L \\ BL & CL^2 & -BL & (B-C)L^2 \\ -A & -BL & A & -(A-B)L \\ (A-B)L & (B-C)L^2 & -(A-B)L & (A-2B+C)L^2 \end{vmatrix}$$

where

$$A = I_L \int_0^L \left(\frac{1}{LI_x} \right) dx$$

$$B = I_L \int_0^L \left(\frac{x}{L^2 I_x} \right) dx$$

$$C = I_L \int_0^L \left(\frac{x^2}{L^3 I_x} \right) dx$$

and I_L is the moment of inertia at the left end of the member, L is the length of the member, and x is a variable location along the member length.

The nonprismatic stiffness coefficients caused by flexure have been derived as closed-form integrals in terms of natural or global coordinates. The formal integration of these coefficients is tedious and susceptible to error and, because each new girder would require individual evaluation, formal integration is neither practical nor efficient for computer implementation. Instead, numerical integration is performed using Gaussian quadrature. The accuracy of this approach is shown via application to two illustrative problems shown in Figures 2 and 3; the results for each are given in Tables 1 and 2, respectively. In the segmental nonprismatic beam problem, all of the results obtained by the various methods are identical; however, the segmental beam approach (using three beam segments) required twice the amount of computer memory space and execution time compared with the single nonprismatic element method. In the tapered nonprismatic beam problem, not only did the segmental approach (15 segments) require eight times more memory and execution time, but it was also considerably less accurate.

The axial contribution to the element stiffness matrix is based on the standard displacement-based direct stiffness approach by employing the assumption of centroid segment alignment. The concrete roadway system is neglected in considering axial effect, that is, only the supporting girders are considered to carry axial loads. Therefore, the axial stiffness components are

$$S_e = \frac{A_R E}{L} \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$$

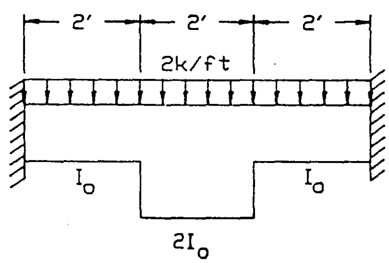


FIGURE 2 Segmental nonprismatic beam example.

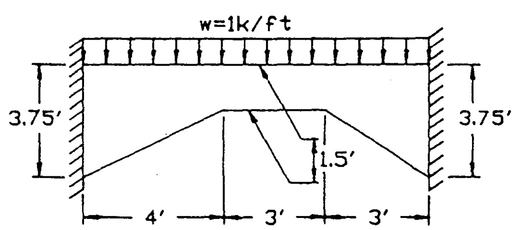


FIGURE 3 Tapered nonprismatic beam example.

TABLE 1 Solution Comparison for Segmental Beam

	Member End Moments (ft-k)		
Member End	Theoretically Exact	GBRIDGE Quadrature Solution	Traditional FEM
Left End	5.47	5.47	5.47
Right End	5.47	5.47	5.47

TABLE 2 Solution Comparison for Tapered Beam

	Member End Moments (ft-k)		
Member End	Theoretically Exact	GBRIDGE Quadrature Solution	Traditional FEM
Left End	11.37	11.39	11.74
Right End	10.28	10.34	10.71

where A is the equivalent cross-sectional area, expressed as

$$A_x = A_L\,[1 + t_w(h_R - h_L)x]$$

The complete element stiffness matrix is formulated by adding the two matrixes together. If the local element axes are not parallel to the global structure axes, the stiffness matrix coefficients must be adjusted to correspond to the global axes through the use of direction cosines. Then, the global stiffness matrix for the bridge structure is formulated by summing the element stiffness matrixes for each structural member.

Structural Analysis

Once the global stiffness matrix has been obtained and appropriate boundary conditions applied, the analysis is performed for each loading condition. These loading conditions, per AASHTO specifications, are dead load, superimposed dead load, and live load plus impact. For the conditions of dead load and superimposed dead load, only the member end actions need to be computed. Once these member end actions are determined, the internal shears and moments at any point along the member can be evaluated directly from superposition, given the assumption of a uniformly distributed loading. The actual shears and moments for each analysis point for the dead load and superimposed dead load conditions can be calculated using the following:

$$M_{ap} = M_x = -M_n + (M_{n+1} + M_n)\,\frac{x}{L} + \frac{wx}{2}(L - x)$$

$$V_{ap} = V_x = \left(\frac{M_n + M_{n+1}}{L}\right) + w\left(\frac{L}{2} - x\right)$$

which are developed from the illustrations indicated in Figure 4. M_n is the member end moment at the left end of the member, M_{n+1} is the member end moment at the right end of the member, x is the location of the analysis point of interest, and L is the member length.

The analysis of the live load condition can be accomplished by using influence lines. An influence line shows the value of any action (shear, moment, deflection) as a result of a unit point load moving across the structure. (Note that the influence line unit load must represent the function sought at each analysis point, that is, a unit load for shear and a unit moment for moment.) In GBRIDGE, actual shear influence lines are used; however, the moment influence line used is actually an analogous end-moment distribution line generated for each girder analysis point. The use of this end-moment distribution line as the moment influence line is an important feature of GBRIDGE.

To develop the GBRIDGE live load influence lines, it is necessary only to obtain the end moments over the supports and apply distribution equations. The determination of member end moments can be accomplished by indirectly considering the effects of the fixed end moments for any specific unit loading. Final member end moment equations then can be developed for an arbitrary application of a 1,000 ft-k joint moment to each unrestrained rotational degree of freedom. The resulting end moments divided by 1,000 are the coefficients that, when multiplied by the fixed end moments, result in the true member end force. The fixed end moments are computed numerically for a unit load placed successively at each analysis point. Utilization of this analysis technique significantly reduces the computation time required in the evaluation of final member end moments for the multitudes of loads that must be considered since only a relatively few analyses are performed on the basis of applied joint moments. Also, the required amount of computer memory needed to accomplish the analysis is minimized

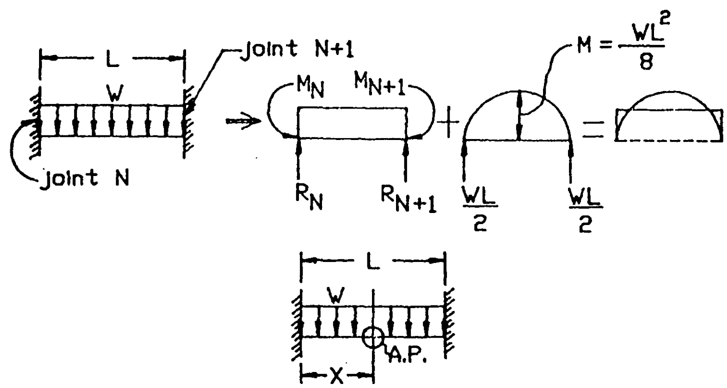


FIGURE 4 Analysis point forces for uniform loads.

because only the member end moments are stored, not the individually calculated ordinate values for each influence line. Rather, the influence lines for each analysis point can be rapidly computed as needed. This approach is much less expensive in terms of computation time and memory requirements.

On the basis of this approach, the true member end forces can be obtained for any loading condition without actually analyzing that loading condition. The ordinates for the moment and shear influence lines can be easily evaluated using the following equations:

$$x \leq kL \quad M_x = -M_n + (M_{n+1} + M_n)\,\frac{x}{L} + (1 - k)x$$

$$x > kL \quad M_x = -M_n + (M_{n+1} + M_n)\,\frac{x}{L} + (L - x)k$$

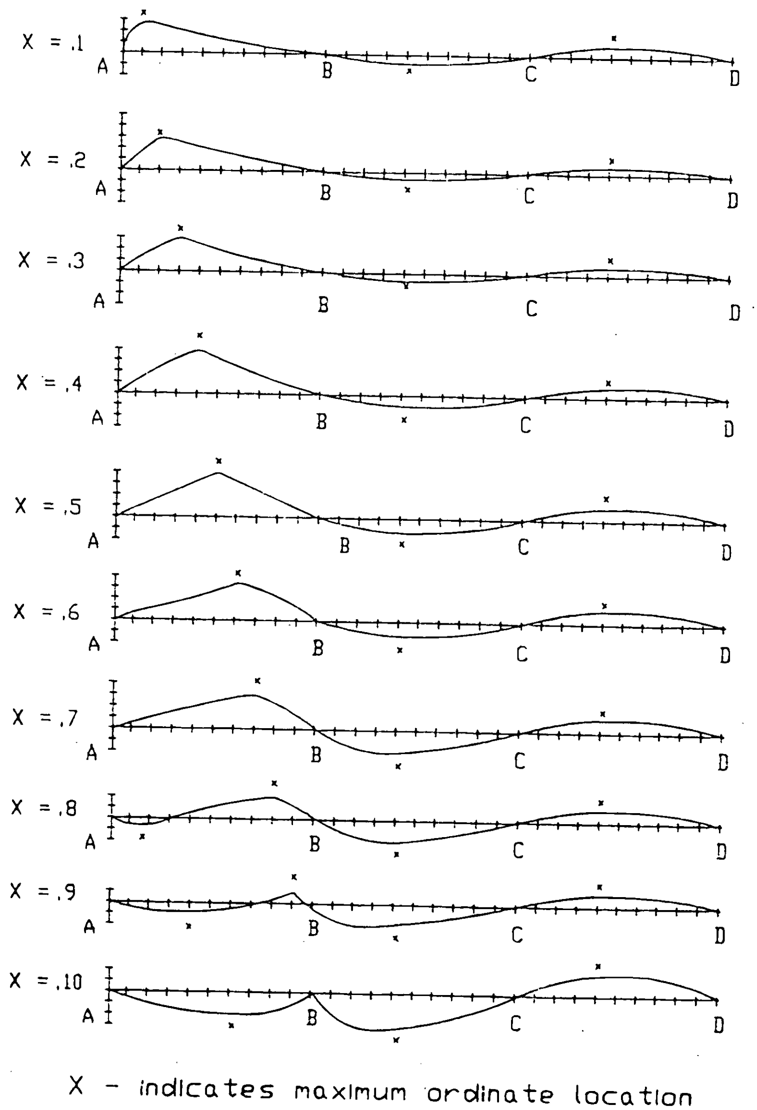
$$x > L \quad V_{ap} = V_x = \frac{M_n + M_{n+1}}{L} - k$$

$$x \leq kL \quad V_{ap} = V_x = \left(\frac{M_n + M_n + 1}{L}\right) + (1 - k)$$

where

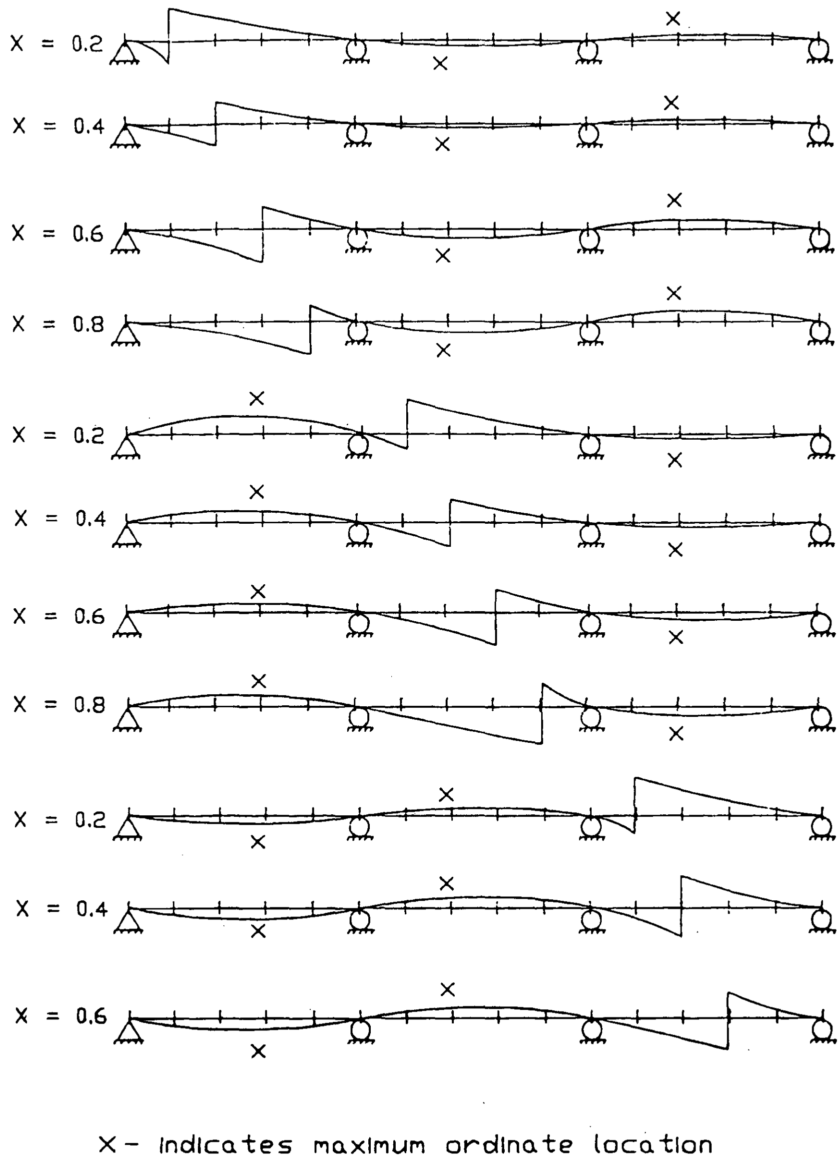
M_{ap} and V_{ap} = moment and shear ordinates at the analysis point of interest,
x = location along the member length of the analysis point of interest, and
kL = location of the applied load.

Figure 5a and b shows examples of typical moment and shear influence lines for a three-span continuous bridge system that has been



(a)

FIGURE 5 ***(a)*** **Moment influence lines,** ***(b)*** **shear influence lines** ***(continued on next page).***



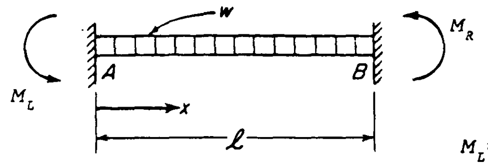
(b)

FIGURE 5 ***(continued).***

developed using the approach outlined above. The nonprismatic fixed end moment equations used in the dead load, superimposed dead load, and live load cases are shown in Figure 6.

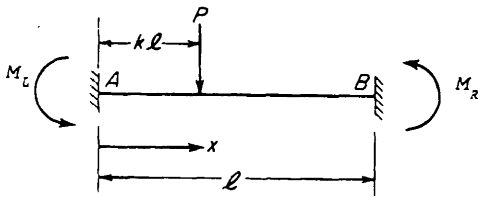
In addition to the efficient method developed for generating influence lines, GBRIDGE also incorporates a technique for rapid internal force evaluation. When the analysis point location and the position of the applied loading are in the same span, the internal force evaluation process is dependent on the analysis point position. If the analysis point is located farther than $0.2L$ away from the member ends, the maximum ordinate will be located at the location of the analysis point. However, if the analysis point is located within $0.2L$ of the member ends, then both positive and negative ordinate values will occur along the span and both the maximum positive and negative values must be determined.

A significant detail can be observed through study of Figure 5a and b. The location of the maximum ordinate for spans other than the loaded span will not vary, no matter where the load is applied on the loaded span. The magnitude of the ordinate value will vary,



$$M_L=\left[\frac{-wl^2}{2(AC-B^2)}\right][BD-C^2]$$

$$M_R=\left[\frac{-wl^2}{2(AC-B^2)}\right][AC-B^2+BD-C^2-AD+BC]$$



$$M_L=\left[-\frac{Pkl}{(AC-B^2)}\right]\left[AC-B^2+kC(F-E)-k^2B(G-F)\right]$$

$$M_R=\left[-\frac{Pkl}{(AC-B^2)}\right]\left[k(C-B)(F-E+k^2(A-B)(G-F)\right]$$

$$A=I_L\int_0^L\left(\frac{1}{LI_x}\right)dx \qquad D=I_L\int_l\frac{x^3dx}{L^4I_x}$$

$$B=I_L\int_0^L\left(\frac{x}{L^2I_x}\right)dx \qquad E=I_L\int_l\frac{dx}{(kl)I_x}$$

$$C=I_L\int_0^L\left(\frac{x^2}{L^3I_x}\right)dx \qquad F=I_L\int_l\frac{x\,dx}{(kl)^2I_x}$$

$$G=I_L\int_l\frac{x^2dx}{(kl)^3I_x}$$

FIGURE 6 Fixed end moment equations.

but not the location. This is because, in elastic analysis, the distribution of loading is dependent only on the member properties. GBRIDGE makes efficient use of this phenomenon by storing the location of the maximum ordinates in the unloaded spans after the initial internal force evaluation. As a result, the maximum shears and moments in the unloaded spans can be calculated directly for each successive evaluation of the loaded span.

The procedure for calculating maximum moments and maximum shears are identical to this point. In addition, both positive and negative shear effects must be examined for absolution maximum shear load. Also, fatigue and shear stud spacing both are dependent on shear range (i.e., the maximum difference between positive and negative shear forces), which varies only slightly throughout the bridge system, as indicated in Figure 7.

SUMMARY

This paper has presented a comprehensive outline of an analytical bridge evaluation process that incorporates nonprismatic member behavior. This behavior is considered in the analysis through the development of nonprismatic element stiffness matrixes. Applica-

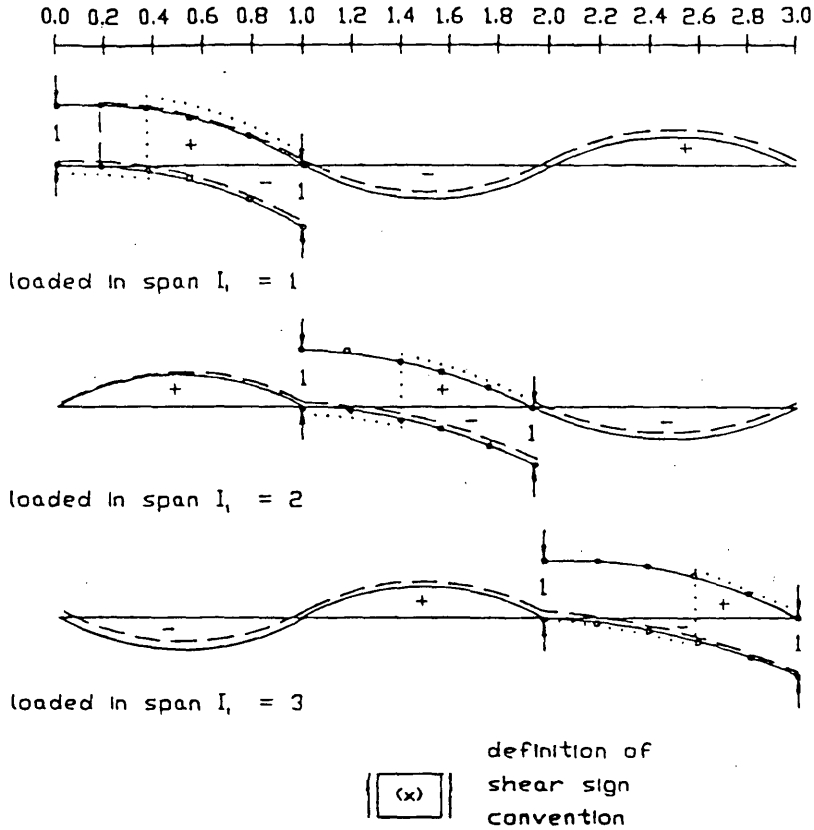


FIGURE 7 Shear range.

tion of this bridge analysis procedure is remarkably reliable and accurate, as shown in Figure 8, in which moment diagrams for a composite, two-span girder bridge are presented. These moment diagrams represent a comparison of the theoretically exact analysis results versus the results of the GBRIDGE analysis procedure plus a comparison of nonprismatic versus prismatic member analysis. The maximum percentage difference between the theoretically exact values and the GBRIDGE numeric solution is less than 1 percent. In addition, the maximum difference between the results obtained from a prismatic analysis and those obtained from a nonprismatic analysis is approximately 15 percent. This fact alone dramatically underscores the need to incorporate nonprismatic member properties into the general bridge analysis process.

CONCLUSIONS

The GBRIDGE analysis procedure is a practical alternative to currently used bridge analysis methods that consider only prismatic member properties. The GBRIDGE computer program is fully implemented and operational. The authors believe that the performance of the GBRIDGE program, in terms of accuracy of results and computation time, will impress the bridge designer when compared with other available software. To that end, the authors/developers of GBRIDGE will be pleased to share a scaled-down version of GBRIDGE with any not-for-profit organization, such as a state department of transportation, which may be interested in testing GBRIDGE in practice.

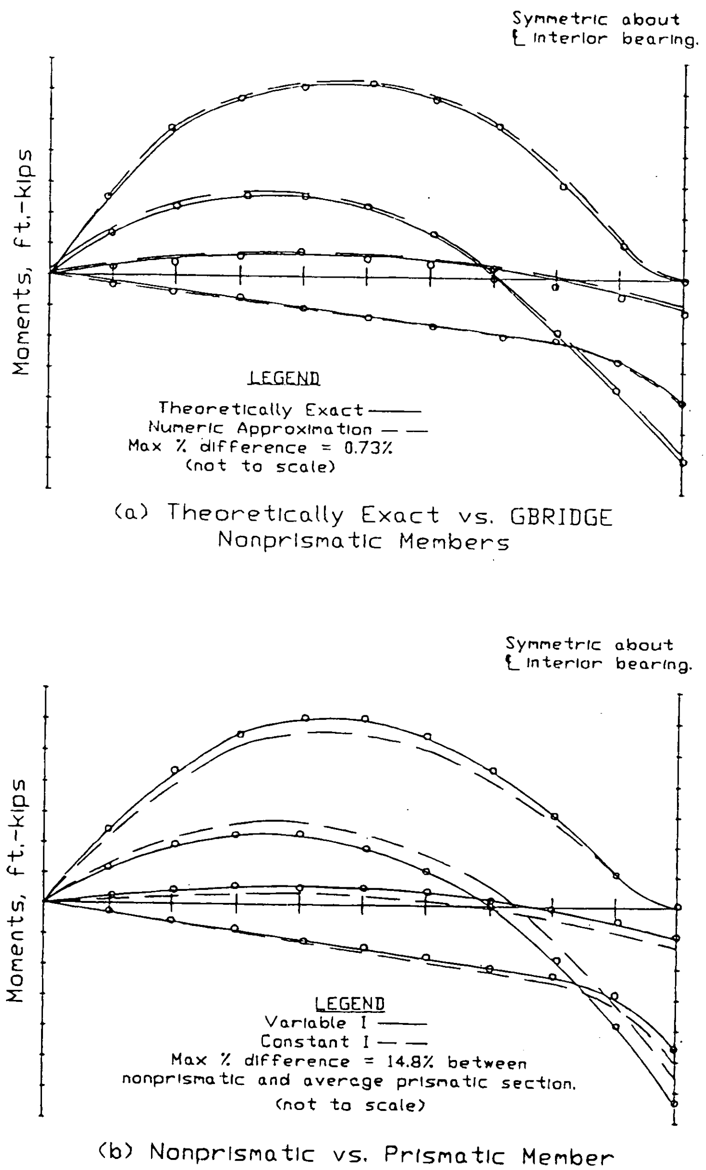


FIGURE 8 Span moment variation.

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