DIPLOMAT: Analysis Program for Bituminous and Concrete Pavements

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Burmister's layered elastic theory is extended to accommodate an interior loading of a multilayered pavement system incorporating an arbitrary sequence of elastic plates and spring beds, in addition to isotropic layers. The formulation is coded into a new computer program, called DIPLOMAT, thereby allowing for the first time direct comparisons between the distinct analytical approaches conventionally used for so-called rigid and flexible pavements. The new program is as user friendly and computationally efficient as the widely used layered elastic analysis program BISAR. In addition to permitting multiple-wheel loads, DIPLOMAT may be used to investigate the effects of a variety of interface and boundary conditions, including that of a rigid base. When considering a plate on grade, DIPLOMAT reproduces the closed-form solutions by Westergaard (dense liquid foundation) and by Losberg (elastic solid foundation). On the other hand, for a pavement section consisting of up to five isotropic layers, the new program reproduces the BISAR solutions for bonded or unbonded layers. A number of applications of the program are presented, including an assessment of the structural contribution of compressible (granular or bituminous) bases under a concrete pavement slab, and determination of the interface spring stiffness that accounts for constructed layer compressibility. Some implications of program results to pavement design are also discussed.

In determining the structural response of highway and airport pavements, one of two fundamentally different hypotheses has been traditionally used to idealize the properties of the subgrade. For portland cement concrete (PCC) pavement systems, the simplest of these theories is used: the supporting soil medium is considered a bed of closely spaced, independent, linear springs. Each spring deforms in response to the stress applied directly to it, and neighboring springs remain unaffected. The spring stiffness, k, is called the modulus of subgrade reaction and is assumed to be spatially independent. This idealization is commonly called a "dense liquid" and is almost universally ascribed to Winkler (*1*).

For asphalt concrete (AC) pavements, on the other hand, a second support characterization theory is conventionally used, in which the soil is regarded as a linearly elastic, isotropic, homogeneous solid, of semi-infinite extent. The terms "elastic solid," "elastic continuum," or "Boussinesq's half-space" are often applied to this idealization (*2*). It is regarded as a more realistic representation of actual subgrade behavior than the dense liquid model, inasmuch as it takes into account the effect of shear interaction between adjacent subgrade support elements.

In a parallel fashion, two distinct theoretical models have been traditionally used in idealizing the constructed layers in the pavement system. For PCC pavements, medium-thick plate theory is conventionally used (*3*). This approach is again the simpler of the two and proceeds from the assumption that the constructed layer, typically a PCC slab, resists the applied loads by bending alone, experiencing no compression through its thickness in the process. A more realistic representation of in situ behavior of constructed pavement layers may be obtained by assuming that they behave as linearly elastic, homogeneous, and isotropic materials not subject to the restrictive assumptions of plate theory. In view of the relatively higher compressibility of asphalt concrete, the layered elastic approach has been adopted in current analysis procedures for AC pavements (*4,5*).

These conventional choices may lead to the impression that the elastic solid foundation is inextricably associated with layered elastic analysis. It has been recently demonstrated not only that is this not the case but also that a formulation based on the isotropic layer–dense liquid subgrade combination might in fact have numerical advantages over conventional applications of layered elastic analysis (*6*). A literature survey conducted recently (*7*) revealed that similar ideas had also been promulgated in the former Soviet Union (*8*).

This paper describes the development of a new structural analysis program, code-named DIPLOMAT, which can be used for both PCC and AC pavement systems. Much like conventional layered elastic analysis programs, DIPLOMAT can accommodate multilayered pavement sections, loaded by multiple-wheel loads. In addition, however, it allows the user the option to specify that the last layer in the pavement system be a bed of springs and that one or more of the constructed layers be treated as plates. Such a structural model is interesting from theoretical as well as practical perspectives. For the first time, it allows analyses of both PCC and AC pavement systems based on the same assumptions and can facilitate comparisons between the behavior and performance of these two heretofore distinct pavement types. In this respect, the formulation in DIPLOMAT constitutes an extension and generalization of Burmister's layered elastic theory.

FORMULATION OF BOUNDARY VALUE PROBLEM

The formulation of the boundary value problem (BVP) posed by a multilayered pavement system involves four major components: the equilibrium equations, the strain-displacement relationships, the constitutive law, and the boundary and initial conditions. Complete details of this formulation are provided in a work by Khazanovich (*9*). Only the highlights are presented herein, with emphasis placed on the boundary (or interface) conditions used.

The equilibrium equations, constitutive laws, and strain-displacement relationships for a uniform, isotropic (Burmister) layer have been presented elsewhere (*10*). For an axisymmetric problem, it is convenient to use the cylindrical coordinate system

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(r, Φ, z). In this coordinate system, some components of the displacement vector and stress tensor are identically equal to zero due to axisymmetry. The remaining stress and displacement components can be rewritten in terms of the biharmonic stress function, Φ (*11*). The fact that function Φ is biharmonic, that is,

$$\nabla^4 \Phi = 0 \tag{1}$$

provides satisfaction of the equilibrium condition in this case. Following Burmister (*4,5*), the biharmonic function Φ for isotropic layer i is expressed in the following form:

$$\Phi_i = \int_0^\infty [A_i(m)e^{mz} - B_i(m)e^{-mz} + C_i(m)ze^{mz} - D_i(m)ze^{mz}]J_0(m\,r)\,dm \tag{2}$$

where

J_n = Bessel function of nth order,
z = local vertical coordinate, which in contrast to Burmister's convention is measured here down from top surface of each layer i, and
$A_i(m)$, $B_i(m)$, $C_i(m)$, and $D_i(m)$ = unknown functions that do not depend on coordinates.

These constants are evaluated by satisfying the boundary conditions.

The governing equilibrium equations, strain-displacement relationships, and constitutive law for a uniform Kirchhoff plate are given elsewhere (*12*). Accordingly, the plate deflection profile is given by

$$\nabla^4 w = \frac{q^*}{D} \tag{3}$$

where q^* is the net (total) pressure applied to top and bottom surfaces of plate, and D is plate flexural stiffness, defined as

$$D = \frac{E\,h^3}{12(1 - \mu^2)} \tag{4}$$

Here h, E, and μ denote the plate thickness, modulus of elasticity, and Poisson's ratio, respectively. Note that for a general multilayered system, the applied pressure, q^*, is unknown and is determined from the boundary conditions for the adjacent layers, above and below the plate. The deflection profile for a plate, w^i, can be presented in a form similar to that used by Losberg (*13*):

$$w^i = \int_0^\infty \frac{W_i(m)}{E_i} J_0(m\,r)\,m\,dm \tag{5}$$

where function $W_i(m)/E_i$ is the zero-order Hankel transform of the deflection profile and is the only function to be determined for a plate.

The governing equilibrium equations, strain-displacement relationships, and constitutive law for a bed of springs can be derived from the corresponding equations for the isotropic layer, if the horizontal displacements u and v are set to zero, along with the two transverse shear strains and the layer's Poisson's ratio, μ (*14*). Thus, the following equation can be written for the spring bed:

$$\sigma_z = -k(w_t - w_b) \tag{6}$$

where

σ_z = vertical stress in springs,
w_t and w_b = vertical displacements of top and bottom ends of springs, and
k = spring stiffness.

It should be noted that the presence of spring beds does not introduce additional unknown functions. The response of these springs can be expressed in terms of the responses of the adjacent layers, above and below the springs.

To complete the formulation of the BVP for a multilayered system, it is necessary to specify boundary conditions between the layers. The presence of different layer options makes formation of the BVP in this case more difficult than for the original Burmister (*4*) problem. Here, the boundary conditions for each layer depend on the kind of model adopted for its adjacent layers (above and below it). To reduce the number of possible combinations, the following restrictions have been accepted: (a) a plate cannot have a common border with another plate, and (b) a spring bed cannot have a common border with another spring bed.

These restrictions impose no reduction in the generality of the proposed approach. Two plates resting on one another can be replaced by an equivalent plate with parameters as defined by Ioannides et al. (*15*). At the same time, n spring beds in series can also be replaced by one equivalent spring bed with effective spring stiffness, k_{eff}

$$k_{eff} = \frac{1}{\sum_{i=1}^{n} \frac{1}{k_i}} \tag{7}$$

where k_i is stiffness of each individual spring bed. Stresses in every spring bed are equal to the corresponding stress in the equivalent bed.

In what follows, the boundary conditions for the BVP will be formulated for different combinations of isotropic layers, plates, and spring beds. It is assumed that all layers are numbered sequentially from top to bottom as 1 to n, with the nth layer being an elastic solid half-space, or a finite isotropic layer or spring bed resting on a rigid base. The case of the pavement surface layer being a spring bed is trivial and is not considered below.

Boundary Conditions at Interface Between Two Isotropic Layers

Following Burmister (*4,5*), two kinds of boundary conditions at the interface between two isotropic layers are usually considered: a rough interface or a smooth interface. Both kinds of conditions assume continuity of vertical displacements and of vertical stresses across the interface, whose normal is in the z direction. Thus, the following continuity conditions are always satisfied:

$$\sigma_z^{i,b} = \sigma_z^{i+1,t} \tag{8}$$

$$w^{i,b} = w^{i+1,t} \tag{9}$$

where $\sigma_z^{i,b}$ *and* $\sigma_z^{i+1,t}$ are vertical stresses at the bottom surface of the upper layer and at the top surface of the lower layer, respectively; $w^{i,b}$ *and* $w^{i+1,t}$ are vertical displacements at the bottom surface of the upper layer and at the top surface of the lower layer, respectively.

The rough interface condition also assumes continuity of horizontal displacements and of shear stresses across the interface. This assumption can be written in the following form:

$$\tau_{rz}^{i,b} = \tau_{rz}^{i+1,t} \tag{10}$$

$$u^{i,b} = u^{i+1,t} \tag{11}$$

where $\tau_{rz}^{i,b}$ *and* $\tau_{rz}^{i+1,t}$ are shear stresses at the bottom surface of the upper layer and at the top surface of the lower layer, respectively; $u^{i,b}$ and $u^{i+1,t}$ are radial displacements at the bottom surface of the upper layer and at the top surface of the lower layer, respectively.

The smooth interface condition does not require continuity of horizontal displacements or of shear stresses across the interface, but allows free slip of one layer with respect to the other in the horizontal direction. Therefore, this kind of interface leads to the following conditions:

$$\tau_{rz}^{i,b} = 0 \tag{12}$$

$$\tau_{rz}^{i+1,t} = 0 \tag{13}$$

One can note that either Equation 12 or Equation 13 can be replaced by Equation 10.

Boundary Conditions at Interface Between Isotropic Layer and Rigid Base

The interface between an isotropic layer and a rigid base can also be rough or smooth. Both types of interface conditions require zero vertical displacement at the bottom of the isotropic layer, n, above the rigid base:

$$w^{n,b} = 0 \tag{14}$$

The rough interface condition requires that radial displacements also vanish at the bottom surface of the isotropic layer:

$$u^{n,b} = 0 \tag{15}$$

The smooth interface does not resist horizontal displacements along the bottom of the isotropic layer and leads to the condition of vanishing shear forces at this interface:

$$\tau_{rz}^{n,b} = 0 \tag{16}$$

Boundary Conditions for Elastic Solid Half-Space

In this case, displacements should vanish as depth tends to infinity, that is,

$$w = u = 0 \qquad \text{as } z \rightarrow \infty \tag{17}$$

It may be verified that satisfaction of these conditions also leads to vanishing stresses at infinite depth.

Boundary Conditions at Interface Between Isotropic Layer and Plate or Spring Bed

If an isotropic layer, i, has a common boundary with a plate or a spring bed, this interface is always smooth and free of shear stresses

$$\tau_{rz}^{i,s} = 0 \tag{18}$$

where superscript s denotes t or b (top or bottom surface, respectively), depending on which surface of the layer is considered.

If the adjacent layer is a plate, then its vertical deflections are equal to the corresponding layer vertical deflections at the interface:

$$w^{i,s} = w^P \tag{19}$$

where w^P is the adjacent plate deflection.

If the adjacent layer is a spring bed, then normal stresses in the isotropic layer at the interface are equal to the vertical stresses in the spring bed, σ_z^S

$$\sigma_z^{i,s} = \sigma_z^S \tag{20}$$

Boundary Conditions for Plate

As mentioned earlier, to specify boundary conditions for a plate means to specify equations for the applied net (total) pressure, q^*, in Equation 3. If plate i is not the surface layer of the multilayered pavement system, then this pressure is the difference between the vertical stress at the top surface of the layer right below the plate and the vertical stress at the bottom surface of the layer right above the plate, as follows:

$$q^* = -\sigma_z^{i-1,b} + \sigma_z^{i+1,t} \tag{21}$$

If either adjacent layer is a spring bed, then the normal stress in the springs can be written in terms of the plate deflection and of the deflection of the other end of the spring bed, in a manner similar to Equation 6.

Boundary Conditions for Uppermost Surface of Multilayered System

It is assumed in this study that the applied loading is normal to the uppermost surface of the multilayered system. Therefore, if the first layer is an isotropic layer, then two boundary conditions should be satisfied at this surface: equality of normal stress to the applied pressure, p, and presence of no shear stresses. These conditions can be presented as follows:

$$\sigma_z^{1,t} = p \tag{22}$$

$$\tau_{rz}^{1,t} = 0 \tag{23}$$

If the first layer is a plate, then the following equation should be satisfied:

$$\nabla^4 w^1 = p + \sigma_z^{2,t} \tag{24}$$

where $\sigma_z^{2,t}$ is the vertical stress at the top surface of the second layer.

Equations 8 through 24 allow the formation of a complete system of equations for a BVP for any combination of isotropic layers, plates, and spring beds. In solving these equations, the Hankel transforms of the boundary conditions are first obtained, and then these transforms are rewritten in terms of the unknown functions A_i, B_i, C_i, D_i, and W_i. If the multilayered system consists only of N_L

isotropic layers, then the stress and deflection distributions are described by four N_L unknown functions. N_L layers have $N_L - 1$ shared interfaces, which give rise to four $(N_L - 1)$ equations. The boundary conditions for the uppermost surface of the multilayered system gives two more equations. The final two equations necessary are obtained from the boundary conditions at the bottom surface of the last member of the multilayered system. If this member rests on a rigid base, the pertinent conditions are zero vertical displacements, and either zero horizontal displacements or zero shear stresses, depending on the nature of the interface (rough or smooth). If the last member of the multilayered system is a semi-infinite half-space, the pertinent conditions are vanishing displacements, u and w. Thus, the complete system consists of four N_L equations with four N_L unknown functions.

It should be noted that every time an additional plate is inserted into a multilayered system consisting of N_L isotropic layers, the total number of equations increases by one. It should also be noted that introduction of a spring bed at any existing interface does not change the total number but only the form of the equations. Therefore, the total number of equations in the system is equal to $(4N_L) + N_P$, which is also the number of unknown functions. Solution of the linear system of equations with N_T unknown functions leads to the determination of the responses for the multilayered pavement system.

VERIFICATION OF PROGRAM DIPLOMAT

The formulation presented has been coded in FORTRAN into program DIPLOMAT (*9*). The program is capable of analyzing up to five layers over a rigid base or up to four layers over an elastic solid half-space. To accommodate multiwheel loading, superposition is used, and Cartesian coordinates (x, y) are adopted, instead of polar coordinates (r, ϕ). To verify the program, several series of runs were performed. Some of the results obtained will be presented below where they will be compared with solutions obtained from other programs when appropriate.

Comparison With BISAR

If all layers in the multilayered system are isotropic, such a system may be analyzed using any of the conventional computer programs for layered elastic systems. In this study, results obtained using programs DIPLOMAT and BISAR (*16*) were compared. The following multilayered system, representing a typical AC pavement system, was analyzed:

Layer 1—127-mm-thick (5-in.) AC layer with modulus of elasticity E_1 = 5 517 MPa (800,000 psi), Poisson's ratio $\mu_1 = 0.25$;
Layer 2—152-mm-thick (6-in.) base layer with modulus of elasticity E_2 = 207 MPa (30,000 psi), Poisson's ratio $\mu_2 = 0.3$;
Layer 3—508-mm-thick (20-in.) subbase layer with modulus of elasticity E_3 = 103 MPa (15,000 psi), Poisson's ratio $\mu_3 = 0.45$; and
Layer 4—Subgrade with modulus of elasticity E_s = 34.5 MPa (5,000 psi), Poisson's ratio $\mu_s = 0.45$.

All layers were assumed to be unbonded. The radius of the applied load was set at 150 mm (5.9 in.) (falling weight deflectometer load), and the applied pressure at 689 kPa (100 psi). Calculations were performed for the following locations:

Point A—AC layer, top surface, under the center of applied load.
Point B—AC layer, top surface, 305 mm (12 in.) from the center of applied load.
Point C—AC layer, top surface, 610 mm (24 in.) from the center of applied load.
Point D—AC layer, top surface, 914 mm (36 in.) from the center of applied load.
Point E—AC layer, bottom surface, under the center of applied load.
Point F—Base layer, top surface, under the center of applied load.
Point G—Base layer, bottom surface, under the center of applied load.
Point H—Subbase layer, top surface, under the center of applied load.
Point I—Subbase layer, bottom surface, under the center of applied load.
Point J—Subgrade layer, top surface, under the center of applied load.

The results of these calculations are presented in Table 1. It can be observed that BISAR and DIPLOMAT produce identical results for this system. Equally satisfactory coincidence between BISAR and DIPLOMAT has been obtained for sections with bonded layers as well.

Plate Over Isotropic Elastic Solid Half Space

A series of runs was performed involving a plate resting on an isotropic elastic solid half-space, to compare the maximum plate bending stresses obtained by DIPLOMAT with the closed-form solution presented by Losberg (*13*). The modulus of elasticity and Poisson's ratio for the plate were set equal at 27.6 GPa (4 Mpsi) and 0.15, respectively. The modulus of elasticity, E_s, and Poisson's ratio, μ_s, for the elastic solid half-space were set equal to 276 MPa (40,000 psi) and 0.45, respectively. The plate thickness varied from 102 mm (4 in.) to 406 mm (16 in.). The total applied load was 178 kN (40,000 lb) and the applied pressure was 2 759 kPa (400 psi). Table 2 presents the maximum plate bending stresses obtained by using DIPLOMAT and Losberg's closed-form solution

$$\sigma = \frac{-6\,P(1+\mu)}{h^2}\left[0.1833\,\mathrm{Log}_{10}\left(\frac{a}{\ell_e}\right) - 0.049 - 0.012\left(\frac{a}{\ell_e}\right)^2 + 3.537\;10^{-3}\left(\frac{a}{\ell_e}\right)^3 - 5.012\;10^{-4}\left(\frac{a}{\ell_e}\right)^4\right] \quad (25)$$

where ℓ_e is the radius of the relative stiffness of the plate-on-elastic solid system, defined as follows:

$$\ell_e = \sqrt[3]{\frac{2\,D(1-\mu_s^2)}{E_s}} \quad (26)$$

Again, excellent agreement is observed between results obtained using DIPLOMAT and Losberg's closed-form solution. Near-perfect agreement has also been obtained between DIPLOMAT and Westergaard's closed-form solution.

Isotropic Layer Over Spring Bed

This problem was analyzed by Glazyrin (*8*) and by van Cauwelaert (*6*). In this study, a series of runs for an isotropic layer with modu-

TABLE 1 Stresses and Displacements in Four-Layered AC Pavement System

(a) Using *BISAR*

Point	w mm	u mm	σ_y MPa	σ_x MPa	σ_z MPa
A	1.097	0.000	-2.522	-2.522	-0.689
B	0.899	-0.050	-0.978	2.990	0.000
C	0.660	-0.042	-0.346	0.137	0.000
D	0.485	-0.030	-0.004	0.160	0.000
E	1.087	0.000	2.411	2.411	-0.115
F	1.087	0.000	-0.118	-0.118	-0.115
G	1.021	0.000	0.076	0.076	-0.085
H	1.021	0.000	-0.105	-0.105	-0.085
I	0.815	0.000	0.051	0.051	-0.026
J	0.815	0.000	-0.025	-0.025	-0.026

(b) Using *DIPLOMAT*

Point	w mm	u mm	σ_y MPa	σ_x MPa	σ_z MPa
A	1.0965	0.0000	-2.5238	-2.5238	-0.6890
B	0.8987	-0.0500	-0.9784	-2.9882	0.0000
C	0.6612	-0.0422	-0.3462	0.1371	0.0000
D	0.4844	-0.0302	-0.1417	0.1600	0.0000
E	1.0876	0.0000	2.4115	2.4115	-0.1151
F	1.0876	0.0000	-0.1178	-0.1178	-0.1151
G	1.0206	0.0000	0.0760	0.0760	-0.0854
H	1.0206	0.0000	-0.1050	-0.1050	-0.0854
I	0.8164	0.0000	0.0511	0.0511	-0.0265
J	0.8164	0.0000	-0.0252	-0.0252	-0.0265

Notes: E_1 = 5517 MPa (800,000 psi); μ_1 = 0.25; h_1 = 127 mm (5 in.); E_2 = 207 MPa (30,000 psi); μ_2 = 0.30; h_2 = 152 mm (6 in.); E_3 = 103 MPa (15,000 psi); μ_3 = 0.45; h_3 = 508 mm (20 in.); E_4 = 34.5 MPa (5,000 psi); μ_4 = 0.45.
Load = 44.5 kN (10,000 lbs); pressure = 689 kPa (100 psi); Unbonded interfaces. Tension is positive.
(w, u): Displacements in z and x directions, respectively.
(σ_y, σ_x, σ_z): Stresses in y, x, and z, directions, respectively.

TABLE 2 Maximum Bending Stresses in Plate on Elastic Solid Foundation Under Interior Loading

h	Bending Stress, MPa	
mm	Losberg (13)	*DIPLOMAT*
102	7.655	7.655
203	4.901	4.902
254	3.392	3.394
305	1.919	1.918
356	1.526	1.525
406	1.246	1.246

Notes: E_1 = 276 GPa (4 Mpsi); μ_1 = 0. 15; E_s = 207 MPa (30,000 psi); μ_s = 0.45;
Load = 178 kN (40,000 lbs); pressure = 689 kN (100 psi).

lus of elasticity equal to 27.6 GPa (4 Mpsi) and Poisson's ratio equal to 0.15, resting over a spring bed with stiffness, k, of 54.3 kPa/mm (200 psi/in.) was performed. The isotropic layer thickness varied from 102 mm (4 in.) to 408 mm (16 in.). The total applied load was 44.5 kN (10,000 lb) and the applied pressure was equal to 689 kPa (100 psi). The solution obtained using DIPLOMAT was compared with van Cauwelaert's solution evaluated using the commercial software MATHEMATICA (*17*). The results of calculations are presented in Table 3. Excellent agreement is observed in the results obtained using these two different numerical solutions.

APPLICATIONS OF PROGRAM DIPLOMAT

Using DIPLOMAT To Obtain Interlayer Spring Stiffnesses

In a previous paper (*18*), the authors presented a finite-element formulation that accommodates through-the-thickness compressibility and separation of the constructed layers in a multilayered PCC pavement system. Accordingly, a bed of springs is inserted between consecutive plates, as proposed by Totsky (*19*). The interface spring stiffness, k_I, is an important parameter to be determined. DIPLOMAT is a good tool for this purpose. Determination of k_I using DIPLOMAT involves the following steps:

1. Consider a multilayered system consisting of isotropic layers resting on a dense liquid foundation. The elastic parameters of the isotropic layers and their thicknesses are equal to the corresponding elastic parameters and thicknesses of the constructed layers in the field. The stiffness of the dense liquid foundation is equal to the field modulus of subgrade reaction. Using computer program DIPLOMAT, find the maximum bending stress in each of the layers in this system under interior loading.
2. Consider another multilayered system consisting of alternating plate and spring beds. The elastic parameters of the plates and their thicknesses are equal to the corresponding elastic parameters and thicknesses of the constructed layers in the field. Using a trial-and-error approach and computer program DIPLOMAT, find the values of the spring interlayer stiffnesses, which lead to maximum bending stresses in the plates close to the corresponding maximum bending stresses in the isotropic layers, obtained in Step 1.

To establish the validity of this suggestion, predictions for the maximum bending stresses at the bottom of the constructed layers in multilayered pavement systems obtained using different models have been compared. The following models were used in analyzing two unbonded constructed layers resting on a dense liquid foundation: (a) isotropic layers, (b) plates separated by a Totsky spring interlayer, and (c) plates resting on one another. All three of these models can be accommodated in DIPLOMAT. For Model b, the interlayer spring stiffness, k_I, may be calculated using the aforementioned DIPLOMAT-based iterative procedure, or using one of two equation-based approaches. The first of these, involves the following equation originally presented by Totsky (*19*):

$$k_I = K\left(\frac{E_j\,E_{j+1}}{h_j\,E_{j+1} + h_{j+1}\,E_j}\right) \tag{27}$$

where subscripts j and $j + 1$ denote the plates just above and just below the springs, respectively. The constant K is set at 2.461, per Totsky's own recommendation. An alternative simple mathematical expression, which results in a good first estimate of k_I, has been derived by the authors (*18*):

TABLE 3 Maximum Tensile Bending Stresses in Isotropic Layer on Dense Liquid Foundation Under Interior Loading

h	Bending Stress, MPa	
mm	van Cauwelaert (*6*)	*DIPLOMAT*
102	4.588	4.587
152	2.383	2.383
203	1.469	1.469
254	0.994	0.994
305	0.713	0.713
356	0.535	0.535
406	0.415	0.415

Notes: E_1 = 276 GPa (4 Mpsi); $\mu_1 = 0.15$;
k = 54.3 kPa/mm (200 psi/in.)
Load = 44.5 kN (10,000 lbs); pressure = 689 kPa (100 psi).
Tension is positive.

$$k_l = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}} \tag{28}$$

where

$$k_1 = \frac{2E_1(1 - \mu_1)}{h_1(1 - \mu_1 - 2\mu_1^2)} \tag{29}$$

and

$$k_2 = \frac{2E_2(1 - \mu_2)}{h_2(1 - \mu_2 - 2\mu_2^2)} \tag{30}$$

with subscripts 1 and 2 denoting the upper and lower plates, respectively. Model c can be considered a special case of Model b, in which k_l becomes extremely large. In this study, the plate theory closed-form approach presented by Ioannides et al. (*15*) was used for Model c.

Eight pavement sections were considered. Three of these represent a concrete slab resting on an unbonded stabilized base of variable thickness. Four others model an unbonded concrete overlay over an existing concrete slab of variable thickness. The final section represents a thick unbonded AC overlay on an existing concrete slab.

Table 4 indicates that the maximum bending stresses obtained using the plate theory closed-form approach can differ from those obtained by employing the more realistic model of two unbonded isotropic layers resting on a dense liquid foundation, especially for relatively thin unbonded concrete overlays. At the same time, for all cases considered, the iterative DIPLOMAT-based approach results in estimates of the stiffness parameter for the spring interlayer in the Totsky model that achieve closer maximum bending stresses in the corresponding plates and isotropic layers than either of the two equation-based approaches.

Spring interlayer stiffnesses from the two equation-based approaches produce similar results for all cases considered. For the slab-on-stabilized–base cases, these solutions are close to the corresponding isotropic-layers-on-dense-liquid solutions. For the unbonded concrete overlay cases, the equation-based approaches produce solutions that overestimate the upper layer stresses, yet are in better agreement with the isotropic-layers-on-dense-liquid solutions than the predictions of the plate theory closed-form approach. Therefore, they may serve in obtaining a first approximation of the interlayer stiffness. Moreover, in the actual pavement, the PCC overlay is often separated from the existing concrete slab by a bond-breaker layer. This layer is neglected in this analysis but can increase overlay stresses (E. J. Barenberg, personal communication, 1992). Therefore, using Equation 28 may be more appropriate.

Consider, for example, a typical airport pavement section consisting of a 203-mm (8-in.) unbonded PCC overlay over an existing 406-mm (16-in.) PCC slab, resting on a dense liquid foundation [k = 27.1 kPa/mm (100 psi/in.)]. The radius of the applied interior load is 150 mm (5.9 in.), and the applied pressure is 689 MPa (100 psi). Both constructed layers have Young's modulus values of 27.6 GPa (4 Mpsi) and Poisson's ratios of 0.15. Assume that the overlay and the existing slab are separated by a bituminous interlayer with a thickness (h_b) of 51 mm (2 in.), a modulus of elasticity (E_b) of 5517 MPa (800 ksi), and a Poisson's ratio (μ_b) of 0.35. For this case, the three isotropic-layers-on-dense-liquid model predicts that the maximum tensile stresses in the overlay and existing slab are 510 kPa (74 psi) and 347 kPa (50.4 psi), respectively. These values are close to those predicted by the Totsky model with k_l defined by Equation 28, which ignores the bituminous interlayer. If the bituminous interlayer modulus of elasticity were only 2 759 MPa (400,000 psi), the isotropic-layers-on-dense-liquid maximum tensile stresses in the overlay and existing slab would be 568 kPa (82.4 psi) and 341 kPa (49.5 psi), respectively. In this case, it is preferable to use the following equation for the spring interlayer stiffness:

$$k_l = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_b}} \tag{31}$$

with k_1 and k_2 as given by Equations 29 and 30 and

$$k_b = \frac{2E_b(1 - \mu_b)}{h_b(1 - \mu_b - 2\mu_b^2)} \tag{32}$$

Equation 31 leads to a k_l value equal to 54.3 MPa/mm (200,000 psi/in.) in this case. For this interlayer stiffness, the Totsky model predicts maximum tensile stresses in the overlay and the existing slab equal to 568 kPa (82.4 psi) and 341 kPa (49.5 psi), respectively, that is, the same values as predicted by the isotropic-layers-on-dense-liquid solution.

Estimation of Base Layer Contribution to Stress Reduction

In a previous paper (*15*), the authors presented a method for assessing the structural contribution of base layers in PCC pavement systems in a manner that accounted for the through-the-thickness compressibility of both constructed layers. This entailed adjusting the value of the maximum bending stress in the upper layer obtained on the basis of plate theory, by calculating a correction increment. An equation for the latter was derived using layered elastic analysis results, which used the elastic solid idealization. Application of the same correction increment to dense liquid cases was cautiously recommended at that time, pending the development of more appropriate numerical analysis tools. DIPLOMAT provides an opportunity to assess the validity of the correction increment, while

TABLE 4 Comparison Between Totsky Model and Isotropic Layers on Dense Liquid Models for Three-Layered System (*a*) Model Parameters, (*b*) Maximum Bending Stresses in Layers 1 and 2

(a)

	E_1 GPa	μ_1	h_1 mm	E_2 GPa	μ_2	h_2 mm	Interlayer Spring Stiffness, k_I, from Eq. (27) MPa/mm	Eq. (28) MPa/mm	*DIPLOMAT* MPa/mm
CASE 1	276	0.15	203	13.8	0.45	152	20.3	55.3	135.6
CASE 2	276	0.15	203	13.8	0.45	203	15.5	43.4	27.1
CASE 3	276	0.15	203	13.8	0.45	254	12.5	35.8	135.6
CASE 4	276	0.15	203	276	0.15	203	162.8	143.2	271.3
CASE 5	276	0.15	203	276	0.15	254	144.6	127.2	257.7
CASE 6	276	0.15	203	276	0.15	305	130.2	114.5	244.1
CASE 7	276	0.15	203	276	0.15	406	108.5	95.5	230.6
CASE 8	55.2	0.30	152	276	0.15	203	38.2	41.5	263.1

(b)

MODEL	Isotropic Layers		Plate Theory		Totsky Model with k_I from Eq. (27)		Eq. (28)		*DIPLOMAT*	
LAYER	1	2	1	2	1	2	1	2	1	2
CASE 1	1.571	0.089	1.536	0.090	1.543	0.082	1.543	0.086	1.543	0.088
CASE 2	1.530	0.103	1.495	0.116	1.509	0.099	1.502	0.106	1.502	0.103
CASE 3	1.481	0.112	1.426	0.138	1.461	0.110	1.447	0.120	1.454	0.111
CASE 4	0.889	0.772	0.841	0.841	0.944	0.744	0.951	0.737	0.923	0.765
CASE 5	0.696	0.653	0.593	0.744	0.751	0.637	0.765	0.631	0.717	0.662
CASE 6	0.563	0.537	0.414	0.621	0.625	0.528	0.636	0.523	0.573	0.551
CASE 7	0.415	0.362	0.214	0.429	0.490	0.360	0.504	0.356	0.417	0.378
CASE 8	0.171	1.447	0.138	1.516	0.358	1.406	0.347	1.412	0.172	1.495

Notes: No curling; interior loading; radius of applied load = 150 mm (5.9 in.); pressure = 689 kPa (100 psi); k = 27.1 kPa/mm (100 psi/in.).

retaining the conventional dense liquid subgrade idealization. For each of the two conditions for the interface between the two constructed layers (unbonded or bonded), eight runs were conducted, selected to correspond to a wide variety of practical cases. The results are shown in Table 5. It is observed that for the unbonded interface condition, retaining the elastic solid-based correction increment equation yields results that are in very reasonable agreement with DIPLOMAT. On the other hand, DIPLOMAT indicates that the corrected plate theory solution for bonded layers may result in bending stress overestimation by about 25 percent.

DESIGN IMPLICATIONS

With the increasing popularity of multilayered concrete pavement systems in recent years, DIPLOMAT can contribute toward bridging an apparent gap in the pavement engineering tool chest. The new program provides the ability to analyze a concrete pavement system as a truly multilayered one. Individual layers in the system may be assumed to be incompressible through their thickness (e.g., PCC slab or stabilized base) or compressible (e.g., AC overlay or granular base). This new capability will be particularly useful in the area of maintenance and rehabilitation of concrete pavements. Although at the first design stage it is feasible—and even desirable—to treat base and subbase layers as nonstructural layers, placed only for construction expediency and drainage purposes, their structural function cannot be ignored in forensic studies aimed at realistic characterization of in situ pavement properties and the concomitant design of overlays. DIPLOMAT provides the opportunity to establish more conclusively the structural contribution of base and subbase layers, without the need to resort to questionable empirical concepts, such as "bumping-the-*k*-value" or establishing correlations between *k* and the soil Young's modulus, E_s (*15*). In addition, the algorithm developed may be easily incorporated in a unified multilayered pavement moduli backcalculation scheme, whose absence is a severe inhibitor to current rehabilitation efforts. The development of a unified backcalculation procedure is particularly called for following the FAA's adoption of a unified design procedure based on layered elastic analysis (*20*).

Another potential benefit from the development of DIPLOMAT is that for the first time it can provide a two-dimensional comprehensive approach that retains, as an option, all assumptions conventionally made in the analysis of both concrete and bituminous pavements. It is a well-documented axiom that the parameter used to characterize the dense-liquid foundation, the modulus of subgrade reaction *k*, cannot be reliably correlated to the elastic modulus E_s, used with the elastic solid subgrade characterization (*21*). If the last isotropic layer in DIPLOMAT is extended so that its thickness tends to infinity (or is simply made large enough), the model adopted would correspond to the conventional layered elastic analysis. If on the other hand the thickness of the last isotropic layer tends

TABLE 5 Verification of Corrected Plate Theory Solution Using DIPLOMAT

E_1 GPa	h_1 mm	E_2 GPa	h_2 mm	k kPa/mm	S.R.	LS.R.	Max. Bend. Stress in Upper Layer Using *DIPLOMAT* MPa	Corrected Pl.Theory MPa
UNBONDED LAYERS								
27.6	254	2.76	320	17.1	1.2	0.1	0.950	0.993
20.7	237	2.07	406	13.6	1.5	0.1	0.964	1.001
34.5	203	2.41	493	18.3	2.0	0.1	1.191	1.207
41.3	152	4.82	394	14.1	3.0	0.1	1.591	1.556
13.8	176	0.69	279	45.6	1.2	0.2	1.590	1.610
13.8	178	1.89	274	58.9	1.5	0.2	1.315	1.336
16.5	127	2.07	254	34.5	2.0	0.2	2.121	2.126
15.2	152	4.13	297	81.6	3.0	0.2	1.152	1.040
BONDED LAYERS								
15.2	381	1.36	152	32.3	1.2	0.1	0.362	0.445
27.6	406	2.41	287	90.9	1.5	0.1	0.268	0.359
27.6	254	2.76	254	32.0	2.0	0.1	0.574	0.714
20.7	257	2.07	406	46.1	3.0	0.1	0.446	0.590
13.6	229	0.69	127	99.8	1.2	0.2	0.865	0.959
41.3	152	4.12	102	115.3	1.5	0.2	1.513	1.718
34.5	126	3.44	127	77.3	2.0	0.2	1.659	1.954
10.3	127	1.38	190	48.8	3.0	0.2	0.997	1.310

Notes: $\mu_1=\mu_2=0.15$; Load=44.5 kN (10,000 lbs) @ 861 kPa (125 psi) (radius, a= 128 mm (5.05 in.).

S.R. = Stiffness Ratio

$$= \left(\frac{h_{eU}}{h_1}\right)^3 \text{ for unbonded}$$

$$= \left(\frac{h_{eB}}{h_{1F}}\right)^3 \text{ for bonded}$$

$$\beta = \left(x - \frac{h_1}{2}\right) = \left(\frac{h_1 + h_2}{2}\right) - \alpha$$

$$x = \frac{E_1 h_1 \frac{h_1}{2} + E_2 h_2 \left(h_1 + \frac{h_2}{2}\right)}{E_1 h_1 + E_2 h_2}$$

$$LS.R. = \text{Load Size Ratio} = \left(\frac{a}{\ell_m}\right)$$

$$h_{eU} = \sqrt[3]{\left(h_1^3 + \frac{E_2}{E_1} h_2^3\right)}$$

$$h_{2F} = \sqrt[3]{\left(h_2^3 + 12\,\alpha^2\,h_2\right)}$$

$$\alpha = \left(h_1 + \frac{h_2}{2} - x\right)$$

$$h_{eB} = \sqrt[3]{\left(h_{1F}^3 + \frac{E_2}{E_1} h_{2F}^3\right)}$$

$$\ell_m = \sqrt[4]{\frac{E\,h_m^3}{12\,(1 - \mu^2)\,k}}$$

$$h_{1F} = \sqrt[3]{\left(h_1^3 + 12\,\beta^2\,h_1\right)}$$

$$h_m = h_{eU} \text{ or } h_{eB}$$

to zero and the constructed layers are assumed to behave as plates resting on a bed of springs, this would correspond to the conventional Westergaard problem.

A number of follow-up possibilities and enhancements to DIPLOMAT are possible. Prominent among these is the capability to account for the dynamic effects on pavements of moving wheel loads. Findings and conclusions from recent work conducted by a number of investigators (*22,23*) could easily be incorporated into DIPLOMAT, in view of the retention in the latter of both layered elastic and plate theory assumptions.

To address the issue of edge loading within the context of a comprehensive pavement analysis and design procedure, an interesting formulation proposed by researchers in the former Soviet Union could be adapted and enhanced for use in DIPLOMAT. Called the Method of Compensative Loads, this approach can lead to an analytical (closed-form) solution for the edge-loading problem, using any chosen subgrade idealization (*7*). Edge loading is solved by superposition of the corresponding interior loading solution plus a solution for a set of comprehensive loads that restore the boundary conditions along the location of the edge. Such a solution would be much easier to implement in a design algorithm than current finite-element techniques.

CONCLUSIONS

In this study, Burmister's layered elastic theory has been extended to accommodate a multilayered pavement system incorporating an arbitrary sequence of plates and spring beds, in addition to isotropic layers. The formulation has been coded into a new computer program, called DIPLOMAT, thereby allowing for the first time direct comparisons between the distinct analytical approaches conventionally employed for so-called rigid and flexible pavements. The new program is as user friendly and computationally efficient as the widely used layered elastic analysis program BISAR. In addition to permitting multiple-wheel loads, DIPLOMAT may be used to investigate the effects of a variety of interface and boundary conditions, including that of a rigid base. When considering a plate on grade, DIPLOMAT reproduces the closed-form solutions by Westergaard (dense liquid foundation) and by Losberg (elastic solid foundation). On the other hand, for a pavement section consisting of up to five isotropic layers, the new program reproduces the BISAR solutions for bonded or unbonded layers. A number of applications of the program are presented, including an assessment of the structural contribution of compressible (granular or bituminous) bases under a concrete pavement slab, and the determination of the interface spring stiffness that accounts for constructed layer compressibility. It is illustrated with several examples that using a DIPLOMAT-based iterative procedure it is possible to find a set of values for the spring interlayer stiffnesses in a given pavement system, which produce an adequate match of maximum bending stresses obtained using the plate and the isotropic layer models for the constructed layers in the pavement system. Some implications of program results to pavement design are also discussed.

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