Recursive Structure for Exact Line Probabilities and Expected Waiting Times in Multipath Transit Assignment

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Exact analytical expressions for incorporation into transit network assignment frameworks are presented. These expressions apply to the case of random uniform passenger arrivals and fixed constant line headways. Previously, difficulty in specification has led to assumptions such as Poisson line arrivals; the reality, however, conforms more to fixed schedules than to Poisson line arrivals. The exact expressions that are derived define expected waiting times and line ridership probabilities. Recursive schemes are developed for computational implementation of these expressions by which to facilitate their application in practical transit assignment. The expressions were developed for multipath assignment schemes and can be used to enhance existing transit assignment algorithms in commonly used planning packages; applications can be either in the line enumeration phase or in the line ridership probability calculations. Numerical examples are provided to illustrate the application of the recursive schemes, and the predicted line probabilities are compared with simulated passenger and line arrivals.

The motivation for this paper is to improve on certain assumptions and methods that are used in existing transit assignment models and applied in practice for transit planning and operations. The focus is on developing exact expressions for finding line selection probabilities and expected waiting times without making traditional assumptions such as Poisson line arrival probability distributions. Recursive procedures are developed to facilitate the practical implementation of these expressions. The successful design and operation of an efficient transit system rely heavily on the successful implementation of the assignment models utilized in systems planning packages. The transit assignment algorithm developed by Spiess and Florian (*1*) is one of the most popular models and, hence, this algorithm is used as a benchmark to apply the exact expressions and calculation schemes developed. Although the proposed models are possibly most applicable to the Spiess-Florian algorithm, they can be used in other assignment schemes as well.

The paper presents a brief discussion on the history of transit assignment through current multipath assignment models used in practice. It also highlights the relative merits of existing transit assignment models and the implications of passenger arrival distributions and waiting strategies. The proposed models are applied to candidate transit networks, including a simplified real-world network.

OVERVIEW OF TRANSIT ASSIGNMENT

An important component of transit planning is transit path assignment, the prediction of how transit passengers choose paths. Path assignment involves determining the level of service on various paths serving an origin and destination (O-D) and assigning passenger demand to these paths. There are various ways by which traveler route choice may be formulated. One possible assumption would be that all used paths will have the same minimum expected travel time and any unused paths will have travel times that are at least as great as this minimum (a variation of Wardrop's first principle). This principle implies that transit passengers choose a path from a set of paths with the minimum expected travel time.

Transit path assignment, however, rarely considers other important aspects of passenger behavior. Some of the conventional transit assignment methods assume negative exponential headways for transit services. Although it does not necessarily reflect actual behavior, this assumption has been attractive to researchers in the past because of its simplicity. Most multipath assignment models have used approximate expressions for the expected waiting time at a stop and the resulting ridership probabilities. Multipath assignment models assign passengers to a set of paths on the basis of some optimal strategy that typically seeks to minimize the expected total travel time to the destination.

Transit assignment is different from conventional traffic assignment because of the waiting aspect and line transfer requirements. Associated techniques can be broadly classified on the basis of the nature of the assignment as deterministic and probabilistic. Deterministic transit assignment models find a single shortest path between an O-D pair by considering waiting time at a node and possible transfers to other lines. The earliest model developed is Dial's transit pathfinder algorithm (*2*), which is an extension of Moore's shortest-path algorithm accommodating the peculiarities of transit minimum paths. Le Clercq (*3*) suggested a different shortest-path algorithm known as the once-through algorithm for transit assignment as an improvement over the pathfinder algorithm.

Probabilistic models consider the possibility of choosing from a set of lines. Perhaps the earliest work that considers transit assignment on the basis of perceived travel time is by Chriqui and Robillard (*4*). The idea of choosing the first line among a set of lines has been adopted by recent transit researchers (*1,5–7*); the nature of the algorithms differs in the definition of the choice set of transit lines. Spiess and Florian (*1*) developed a mathematical model for enumerating an optimal strategy set of transit lines that aims to minimize the expected total travel time from an origin to a destination. Horowitz (*5*) modified Dial's multipath assignment model to accommodate various level-of-service parameters (such as travel time, waiting time, and capacity), which are used to find the set of reasonable paths between an origin and destination; a logit model was employed to estimate the probability of choosing a particular transit line.

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Passenger Arrivals and Waiting Strategies

In most of the conventional transit assignment methods, the expected waiting time for each route is assumed to be one-half of the route headway (assuming that a passenger arrives randomly in a perfectly reliable system). The expected waiting time for any transfer is again equal to one-half the headway on the connecting route (regardless of transfer timing). Even recently proposed multipath assignment models still assume random passenger arrivals and approximate waiting times as some fraction of the headway. The assumption of random uniform passenger arrivals at a node might hold true in the case of frequent transit service but may not necessarily be warranted for less-frequent bus service (for example, headways greater than 10 min). For such cases, several researchers have suggested that alternate distributions of passenger arrivals be used (*8–10*).

Route Choice

The choice of a route depends on an optimal strategy (or a waiting strategy). A strategy is a set of rules that, when applied, allows a traveler to reach the desired destination. Early transit assignment models specify a single shortest path, whereas probabilistic assignment models specify a set of paths on the basis of an optimal strategy. Actual route choice is more complex because it is a function of a passenger's perception of different level-of-service parameters, such as waiting time, number of transfers, and line capacity.

The validity of Wardrop's principles in transit path choice is questionable because riders often may not select one initial complete path but may show adaptive behavior. In a comprehensive study of the transit route choice problem, Hall (*11*) suggested that passengers are able to improve their travel time over the Wardrop optimum by adaptively selecting routes. The research focus was to determine the importance of real-time information to passengers in making route choice decisions. Passengers can improve their path choice after arriving at a node by using additional information available at that node. The advantages of knowing how much time a passenger has waited at a node in determining route choice when there are overlapping bus routes also was investigated by Marguier and Ceder (*12*).

MULTIPATH ASSIGNMENT MODELS

The most common multipath transit assignment models used in practice are those developed by Spiess and Florian (*1*) and Horowitz (*5*). Spiess and Florian enumerate a set of paths on the basis of an optimal strategy that minimizes the total expected travel time; this model has been implemented in EMME/2. Horowitz applied Dial's stochastic multipath assignment to enumerate the set of reasonable paths from a node to a destination on the basis of the disutility of a transit trip; this transit assignment model is implemented in QRS II.

The Spiess-Florian transit assignment model is an algorithm for solving the transit assignment problem with a fixed set of transit lines. The traveler chooses the strategy that allows a desired destination to be reached at minimum expected cost. A strategy is a set of rules that, when applied, allows the traveler to reach that destination. For the special case in which the waiting time at a stop depends only on the combined line frequency, the problem may be formulated as a linear program of a size that increases linearly with network size. The problem of transit assignment is solved by a label-setting algorithm in polynomial time. In the algorithm's first pass (from the destination node to all origins), the optimal strategy is enumerated and the expected travel times from each node to the destination are computed. In the second pass (from all origins to the destination), the demand is assigned to the network according to the optimal strategy.

The multipath transit assignment model developed by Horowitz (*5*) is a modification of the general stochastic multipath assignment approach of Dial (*13*). Although Dial's model is considered to be efficient, there are instances where travel behavior is inaccurately represented. However, in the case of transit assignment, it can be shown that the potential inaccuracies of Dial's model are of little consequence relative to the anomalies of Dial's algorithm that arise in its performance in automobile networks. Any multipath assignment is based on hypothesized behavior of users; therefore, it is important to understand the passenger's perception of the (dis)utility of the individual routes. The disutility of a transit trip may be represented as a weighted function of the components of travel time, including access and egress walking time, waiting time, and transfer time.

DEVELOPMENT OF EXACT EXPRESSIONS

The major focus of this paper is the development of exact expressions for the line probabilities and expected waiting time for passengers randomly arriving (via a uniform distribution) at a node and choosing a transit line from a candidate set of lines. This candidate set could, for instance, be based on an optimal strategy that minimizes the expected total travel time (such as that proposed by Spiess and Florian). Of particular interest is the case of uniform random arrivals of passengers and constant interarrival times of vehicles, which have been considered difficult in the past and have led to assumptions of Poisson line arrivals. The Spiess-Florian algorithm, for instance, makes such an assumption for finding the line choice probabilities (based on line frequencies).

The expressions developed for the expected waiting time under fixed uniform line arrivals and random uniform passenger arrivals are applicable to assignment algorithms that use calculated expected waiting times during the candidate transit line enumeration phase. The second set of expressions developed for the line selection probabilities for a random passenger is of use in any assignment algorithm, which may be identifying candidate lines using different variables other than travel times and expected waiting times.

Link Probability Expressions

Link selection probability calculations are required in an assignment algorithm that develops the candidate line set by adding links to the set rather than paths. A link here refers to a transit line between two nodes in the network. The link probabilities can be derived from line frequencies, such as in the original Spiess-Florian algorithm:

$$P_a(A_i^+) = \frac{f_a}{\sum_{\forall a' \in A_i^+} fa'} \qquad a \in A_i^+ \tag{1}$$

The probability $P_a(A_i^+)$ of selecting link a at node i from a set of links A_i^+ is the ratio of the frequency of transit service f_a for link a

to the sum of the frequencies of the transit services for all the links. This expression holds true only for negative exponential vehicle headways. This expression is only an approximation in the case of constant headways. Because transit services tend to follow a fixed schedule (with some variance around the schedule), the Poisson arrival assumption does cause significant errors in some cases, and this is examined in a later section. The developments in this paper are based on the belief that a constant interarrival time is a much more logical assumption.

Consider a set of m transit lines at a node that the traveler may pick from to reach the desired destination. Let $H_1, H_2, \ldots, H_m$ be the headways for Lines $1, 2, \ldots, m$, respectively. Let H_k be the minimum of all these headways (that is, the kth line has the smallest headway). Ties are not restricted and either of the tying lines can be selected.

The assumption of constant headways and uniform passenger arrivals results in a waiting time probability density function for Line i given by $1/H_i$, implying that the randomly arriving passenger will find the arrival of line i to be after any waiting time of up to H_i with equal probability. The joint probability density function of all the m lines is simply the reciprocal of the product of the headways of all the transit lines.

The expression "a line arriving" is equivalent to a vehicle from a particular line arriving. The probability P_i of line i being selected from the set of m lines is the probability of that transit line arriving first among the set of lines. The bus headways $H_1, H_2, \ldots, H_m$ may be in any order. This probability is given by the following m-space integral:

$$P_i = \frac{1}{\prod_{j=1}^{m} H_j}\left[\int_0^{H_k} dh_i \int_{h_i}^{H_1} dh_1 \int_{h_i}^{H_2} dh_2 \ldots\ldots\ldots \int_{h_i}^{H_m} dh_m\right] \tag{2}$$

The motivation behind this expression is simple: if Line i is to be the first line selected, then it must arrive within a time interval equal to the minimum headway among the set of lines under consideration, and all the remaining lines must arrive at a later time. This exact probability expression is a polynomial integration of degree $(m-1)$ and hence results in a polynomial of degree m. To find a general expression, consider the case for two and three lines. Without loss of generality, assume that the minimum headway among the set of lines is H_1. Expanding that expression yields the probability of selecting line i from the two or three line choice set (the number of lines are indicated parenthetically):

$$P_1(2) = \frac{1}{\prod_{j=1}^{2} H_j}\left[\frac{(-1)H_1^2}{2} + H_1 H_2\right] \tag{3}$$

$$P_1(3) = \frac{1}{\prod_{j=1}^{3} H_j}\left[\frac{H_1^3}{3} + \frac{(-1)H_1^2}{2}(H_2+H_3) + H_1H_2H_3\right] \tag{4}$$

Defining $S_j^m(i)$ as the sum of the products of headways of all possible combinations from the set of m lines, not including line i, leads to a general expression for the ridership probability.

$$S_1^2(1) = H_2$$

$$S_1^3(1) = H_2 + H_3$$

$$S_2^3(1) = H_2 H_3$$

From induction, the general result is as follows:

$$P_i = \frac{1}{\prod_{j=1}^{m} H_j}\left[\frac{(-1)^{m-1} H_k^m}{m} + \frac{(-1)^{m-2}H_k^{m-1}}{m-1} S_1^m(i) + \ldots + H_k S_{m-1}^m(i)\right] \tag{5}$$

The probability expression derived has lost the simplicity of the proportionality expression originally advocated by Spiess and Florian; however, the expression manifests the theoretical predictions without the use of approximations. This is a more precise approach than that of Jansson and Ridderstolpe (*14*), where the probability of selecting a line is found by converting the already selected set to an "equivalent" route. Once a new route is added with share p_u each of the shares in the "equivalent" set is scaled by the factor $(1 - p_u)$.

Expected Waiting Time Formulation

Just as the probability of picking a link is a function of the line frequencies, waiting time is also a function of the headway distribution of the transit services. Again, an example is the expression for the expected waiting time at a node as specified by Spiess and Florian:

$$E[\text{wait}] = \frac{\alpha}{\sum_{a \in A_i^+} f_a} \qquad \alpha > 0 \tag{6}$$

Spiess and Florian state that

> The case $\alpha = 1$ corresponds to an exponential distribution of interarrival times of the vehicles with mean $1/f_a$ and a uniform passenger arrival rate at the nodes. The case $\alpha = ½$ is an approximation of a constant interarrival time $1/f_a$ for the vehicles on link a. This measure of waiting time is the most widely used approach in practice, in spite of the fact that it is based on a rough approximation. (*1*, p.91)

Although this is a widely used expression, it significantly underestimates the expected waiting time at a node for the special case of constant interarrival times of the transit lines. The actual expected waiting time is the expected value of min $[t_1, t_2, \ldots, t_m]$ where t_i is the time a randomly arriving passenger has to wait until the arrival of a line i service. This is the expected waiting time at a node because the passenger waits for the first transit line that arrives from among the optimal strategy set. The theoretical expression for expected waiting time in the case of constant line headways and uniform random passenger arrivals is

$$E[\text{wait}] = \frac{1}{\prod_{i=1}^{m} H_i}\int_0^{H_k}\left[t\sum_{j=1}^{m}\prod_{\substack{i=1,2,..m\\ i\neq j}}(h_i - t)\right]dt \tag{7}$$

The probability density function (PDF) for Min $[t_i]$ (that is, the PDF for the line i to be picked first from the set of transit lines) is given by:

$$\text{PDF for } \min[t_i] = \frac{1}{\prod_{i=1}^{m} H_i} \left[\prod_{\substack{i=1,\ldots,m \\ i \neq j}} (h_i - t) \right] \tag{8}$$

The summation term over all the lines is to take into account the case for each line being the minimum one to be selected from the set of lines. The limits of integration are 0 and H_k because the maximum waiting time is equal to the minimum headway among the set of transit lines. This seemingly cumbersome equation takes a similar expression as that of the link probabilities, with the polynomial integration being of the order m resulting in a polynomial expression of degree $(m + 1)$. To find a general expression, consider, as before, the expected waiting times for the two and three-line cases (the number of lines are indicated parenthetically):

$$E[\text{wait}](2) = \frac{1}{\sum_{i=1}^{3} H_i} \left[\frac{(-1)2H_k^3}{3} + \frac{H_k^2}{2} (H_1 + H_2) \right] \tag{9}$$

$$E[\text{wait}](3) = \frac{1}{\prod_{i=1}^{2} H_i} \left[\frac{3H_k^4}{4} + \frac{(-1)2H_k^3}{3} (H_1 + H_2 + H_3) + \frac{H_k^2}{2} (H_1H_2 + H_1H_3 + H_2H_3) \right] \tag{10}$$

Denote as R_j^m the sum of the products of the headways of all possible combinations of j lines taken from the set of m lines. Note that the joint probability density function of headways (or waiting time) for the m lines can be denoted as $1/R_m^m$. The above cases result in the following:

$$R_1^2 = H_1 + H_2$$

$$R_2^2 = H_1H_2$$

$$R_1^3 = H_1 + H_2 + H_3$$

$$R_2^3 = H_1H_2 + H_2H_3 + H_1H_3$$

$$R_3^3 = H_1H_2H_3$$

From induction, it is possible to find the general form of the R_j^m expressions as

$$E[\text{wait}] = \frac{1}{\prod_{i=1}^{m} H_i} \left[\frac{(-1)^{m-1} m H_k^{m+1}}{m+1} + \frac{(-1)^{m-2}(m-1)H_k^m}{m} R_1^m + \ldots \frac{H_k^2}{2} R_{m-1}^m \right] \tag{11}$$

RECURSIVE SOLUTION FOR MODIFIED ALGORITHM

Theoretical expressions for the expected waiting time at a node and for the corresponding link probabilities have been derived from basic probability fundamentals. However, these expressions increase the computational complexity when implementing the modified model in practice. Computational time is of significant concern in assignment for large transit networks; thus, the expressions must be specified efficiently.

The method proposed here is suitable for algorithms based on shortest-path finding (such as Spiess-Florian), where the process of enumeration of the optimal strategy set of transit lines at a node is started by selecting the link with the least expected travel time. Another link is then added to the optimal set if it improves the expected total travel time. If it does not improve the expected travel time, the link is discarded and never again considered in the enumeration. Thus, links are added sequentially to define the optimal strategy.

A major portion of the computational time is spent in the calculation of the expressions for R and S at each stage of the algorithm. For example, when the fourth line is added, R_1^4, R_2^4, R_3^4, and R_4^4 must be calculated. It is possible, however, to compute these values at each stage of the algorithm in a more elegant way without a significant increase in computational burden. The following steps will illustrate the manner in which the algorithm can incorporate the expected waiting time and link probability expressions while maintaining control of the computational expense.

Consider an optimal strategy set consisting of m lines and assume that each line is added in the order 1, 2, . . . , m (i.e., Line 1 is selected first, then Line 2, and so on, until Line m is selected last). The algorithm has as many stages as there are transit lines in the optimal strategy. Stage 1 is executed when the first line is added to the optimal strategy set at a node. Stage 2 is executed when the second line is added to the optimal strategy set and so on until Stage m, which is executed when the mth line is added to the optimal strategy set. The R_j^m values at stage m can be obtained from the R_j^{m-1} values of the prior stage; the S-values for each stage are directly obtained from the R-values of that stage. The expected waiting times and ridership probabilities are calculated by using the following recursive structure:

- **Stage 1: Number of lines = 1; Line 1 enters.**
  $R_1^1 = H_1$
  $E[\text{wait}] = H_1/2$
  $P_1 = 1.00$
- **Stage 2: Number of lines = 2; Line 2 enters.**
  $R_1^2 = R_1^1 + H_2$
  $R_2^2 = R_1^1 H_2$
  $S_1^2(1) = R_1^2 - H_1$
  Find min $[H_1, H_2]$
  Find E [wait]
  Find P_i, $i = 1,2$
- **Stage 3: Number of lines = 3; Line 3 enters.**
  $R_1^3 = R_1^2 + H_3$
  $R_2^3 = R_2^2 + R_1^2 H_3$
  $R_3^3 = R_2^2 H_3$
  $S_1^3(i) = R_1^3 - H_i$ **for** $i = 1,2$
  $S_2^3(i) = R_2^3 - H_i [S_1^3(i)]$ **for** $i=1,2$
  Find min $[H_1, H_2, H_3]$
  Find E [wait]
  Find P_i, $i=1,3$
- **Stage** m**: Number of lines =** m**; Line** m **enters.**
  $R_1^m = R_1^{m-1} + H_m$
  $R_j^m = R_j^{m-1} + R_{j-1}^{m-1} H_m$ **for** $j=2, (m-1)$
  $R_m^m = R_{m-1}^{m-1} H_m$
  Do i=1, $(m-1)$
  $S_1^m(i) = R_1^m - H_i$
  $S_j^m(i) = R_j^m - H_i[S_{j-1}^m(i)]$ **for** $j=2, (m-1)$

Continue
Find min $[H_1, H_2, \ldots, H_m]$
Find E [wait]
Find P_i, $i=1,m$

The recursive steps developed facilitate the computation of the values of the R's and the S's at each stage on the basis of the values from the prior stage.

ILLUSTRATED EXAMPLE

A four-line case has been chosen to illustrate the recursive algorithm; let the headways of these four lines be 30, 20, 12, and 4 min. Assume that these lines form the optimal strategy set. Note that whether a link becomes a part of the optimal strategy is a function of link cost (which typically in the case of the transit assignment is the link travel time), whereas the expected total waiting time and link probabilities are functions of headways only. Therefore, it is not necessary to consider the link travel times for each transit line.

- **Stage 1: Line 1 enters the optimal strategy set.**
  $R_1^1 = H_1 = 30$
  E [wait] = 15 min
  $P_1 = 1.00$
- **Stage 2: Line 2 enters the optimal strategy set.**
  $R_1^2 = R_1^1 + H_2 = 50$
  $R_2^2 = R_1^1 H_2 = 600$
  $S_1^2(1) = R_1^2 - H_1 = 20$
  Min $[H_1, H_2] = 20$ min
  E [wait] = 7.78 min
  $P_1 = 0.3333$
  $P_2 = 0.6667$
- **Stage 3: Line 3 enters the optimal strategy set.**
  $R_1^3 = R_1^2 + H_3 = 62$
  $R_2^3 = R_2^2 + R_1^2 H_3 = 1{,}200$
  $R_3^3 = R_2^2 H_3 = 7{,}200$
  $S_1^3(1) = R_1^3 - H_1 = 32$
  $S_1^3(2) = R_1^3 - H_2 = 42$
  $S_2^3(1) = R_2^3 - H_1 S_1^3(1) = 240$
  $S_2^3(2) = R_2^3 - H_2 S_1^3(2) = 360$
  Min $[H_1, H_2, H_3] = 12$ min
  E [wait] = 4.24 min
  $P_1 = 0.16$
  $P_2 = 0.26$
  $P_3 = 0.58$
- **Stage 4: Line 4 enters the optimal strategy set.**
  $R_1^4 = R_1^3 + H_4 = 66$
  $R_2^4 = R_2^3 + R_1^3 + H_4 = 1{,}448$
  $R_3^4 = R_3^3 + R_2^3 H_4 = 12{,}000$
  $R_4^4 = R_3^3 h_4 = 28{,}800$
  $S_1^4(1) = R_1^4 - H_1 = 36$
  $S_1^4(2) = R_1^4 - H_2 = 46$
  $S_1^4(3) = R_1^4 - H_3 = 54$
  $S_2^4(1) = R_2^4 - H_1 S_1^4(1) = 368$
  $S_2^4(2) = R_2^4 - H_2 S_1^4(2) = 528$
  $S_2^4(3) = R_2^4 - H_3 S_1^4(3) = 800$
  $S_3^4(1) = R_3^4 - H_1 S_2^4(1) = 960$
  $S_3^4(2) = R_3^4 - H_2 S_2^4(2) = 1{,}440$
  $S_3^4(3) = R_3^4 - H_3 S_2^4(3) = 2{,}400$
  Min $[H_1, H_2, H_3, H_4] = 4$ min
  E [wait] = 1.60 min
  $P_1 = 0.055$
  $P_2 = 0.085$
  $P_3 = 0.150$
  $P_4 = 0.710$

Note that the link probabilities of $(m - 1)$ lines are calculated using the expression that, for the mth line, is computed as follows:

$$1 - \sum_{i=1}^{m-1} P_i \tag{12}$$

DISCUSSION OF RESULTS

The comparison of the Spiess-Florian method with the exact method indicates a significant difference in the values of the expected waiting time and the probabilities predicted for uniform passenger arrivals at a node (Table 1). It is important to note that the actual probabilities depend on the initial line starting times (the relative headway gaps at the start of the time horizon). The link probabilities obtained from the exact method give the probability of taking a line, which is unconditional on the initial arrival times. If two routes have identical 10 min headways, with the first arrival on Line 1 on the hour and arrivals on Line 2 lagged by 2 min, then the first route receives 80 percent of the randomly arriving passengers and the other only 20 percent. This is not the equal split that would be expected using either the exact method or the Spiess-Florian algorithm. Particularly in the case of routes with identical headways, the variance of probabilities conditional on the starting time is high. With the aid of a simulation study, it was shown that, in most cases,

TABLE 1 Comparison of Spiess-Florian with Exact Method

Stage No:	Line No:	Headways (minutes)	Spiess-Florian Method		Exact method	
			E [Wait]	Probability	E [Wait]	Probability
1	1	30	15	1.000	15	1.000
2	1	30	6	0.400	7.78	0.333
	2	20		0.600		0.667
3	1	30	3	0.200	4.24	0.160
	2	20		0.300		0.260
	3	12		0.500		0.580
4	1	30	1.2	0.080	1.60	0.055
	2	20		0.120		0.085
	3	12		0.200		0.150
	4	4		0.600		0.710

the variance of the probabilities caused by different starting times was low enough to minimize any concerns (Table 2). Moreover, the initial starting times for transit services over a large network are variable and hence the predicted probabilities, which are unconditional on the starting times, are the best estimates on average, especially during the planning process when assignments are used. For combinations of headways that do not share many common divisible factors (for example, headways of 5 and 12 min), the actual link probabilities, irrespective of the starting time, closely approximate the theoretical unconditional values provided that the time horizon is sufficiently long. A typical peak-hour operation of 3 hr is sufficient to expect the system to converge to the theoretical values of the probabilities, as confirmed by the simulation study.

IMPLICATIONS OF USING EXACT FORMULATIONS

The exact formulations derived for the probabilities and waiting time for constant headways and uniform random passenger arrivals were implemented in an operational version of the Spiess-Florian algorithm. This task was somewhat formidable given the complexity of both the derived expressions and the resulting algorithm. The Spiess-Florian algorithm performs efficiently largely because of the nature of the expressions for line probabilities and waiting times. This algorithm adds lines in the order of increasing link costs. Consider a simple example where there are n lines operating between a single O-D pair. Let the link costs be $t_{1,2}, \ldots, t_{n-1}, t_n$ such that $t_n > t_{n-1} > \ldots > t_1$. Let $H_1, H_2, \ldots, H_n$ be the corresponding headways. Two rules are presented next. The first rule is implicit in the Spiess-Florian algorithm and is used as the criterion for choosing an optimal strategy set. Because the optimal strategy set is different according to the exact expressions, the second rule is provided as a supporting rule to enumerate the optimal strategy set in this case.

Rule 1: Conventional Rule for Enumeration of Optimal Strategy Set

The Spiess-Florian algorithm adds Line 1 into the optimal strategy set and then Line 2 if $t_2 < E[\text{travel time}]$. For the general case, assume that $(n - 1)$ lines are already a part of the optimal strategy set. The expected waiting time for the $(n - 1)$ lines is

$$E[\text{wait}] = \frac{1}{2\sum_{i=1}^{n-1} \frac{1}{H_i}} \tag{13}$$

The probability of picking a line i is given as

$$P_i = \frac{\frac{1}{H_i}}{\sum_{i=1}^{n-1} \frac{1}{H_i}} \tag{14}$$

Using these expressions, the expected total travel time is given by

$$E[\text{total travel time}] = E[\text{wait}] + \sum_{i=1}^{n-1} P_i t_i \tag{15}$$

Define a and b as

$$a = \frac{1}{2} + \sum_{i=1}^{n-1} \frac{t_i}{H_i}$$

$$b = \sum_{i=1}^{n-1} \frac{1}{H_i} \tag{16}$$

It follows from these expressions that the expected total travel time is given by (a/b). If the nth line is added to the optimal strategy, then the new a' is given by $[a + (t_n/H_n)]$ and the new b' is given by $[b + (1/H_n)]$. The new expected travel time is given by a'/b'. This line would be a part of the optimal strategy only if $(a'/b') < a/b$ {i.e., if $[a + (t_n/H_n)]/[b + (1/H_n)] < [a/b]$}.

Cross-multiplying yields

$$ab + b(t_n/H_n < ab + a(1/H_n) \tag{17}$$

This expression readily simplifies to

$$t_n < a/b \tag{18}$$

Because a/b is the expected travel time with $(n - 1)$ lines, the inclusion of a new line has to satisfy the preceding criterion as employed by the Spiess-Florian algorithm, that is, the travel time on the new link is less than the current expected travel time to the destination. Any line that satisfies this criterion also becomes a line in the opti-

TABLE 2 Summary of Simulation Results for Two-Line Case

	Same Headways (min)		Different Headways (min)	
	10	10	12	5
Time Horizon (hours)	3		3	
Probability (Simulation)	0.485	0.515	0.209	0.791
Probability (Theoretical)	0.500	0.500	0.210	0.790
Expected Wait Time (Simulation)	3.907		2.149	
Expected Wait Time (Theoretical)	3.333		2.160	
Probability Variance	0.086		0.001	
Expected Waiting Time Variance	8.075		3.856	

mal strategy set because it also improves the expected total travel time. It is a crucial difference in the context of implementing the exact expressions because all the lines that satisfy this condition may not be in the optimal strategy set. It is not necessary, however, that an entirely new criterion be developed. The Spiess-Florian criterion can be used to eliminate unfavorable lines; the strategy set needs to be further reduced to make it optimal because the addition of a line that satisfies the above criterion may not improve the expected total travel time according to the exact expressions.

Rule 2: Supporting Rule To Enumerate Optimal Strategy Set in Exact Formulation

A supporting rule must be developed for the previous criterion to find the optimal set of lines at a node that minimizes the expected total travel time. The following rule is proposed:

> One by one, add the lines from the ordered set that satisfy the criterion stated in Case 1 and determine the expected total travel time as each line is added. Store the expected total travel time values at each stage (typically the dual variables in the algorithm). Once the set of lines is exhausted, pick the minimum and eliminate all the lines that were included after the minimum line.

This rule ensures that some of the lines included in the strategy set according to Rule 1 are removed once all the lines are considered. Only the lines up to and including the line that caused the minimum expected travel time remain in the final set. Thus Rule 1 is used to make an initial reduction of the strategy set, and Rule 2 is used to find the optimal set. It can be seen that because the strategy set between any two nodes is optimal, all the arguments presented by Spiess and Florian for the optimality of the algorithm apply here too.

Special Example of Implementation of Exact Expressions

Consider a two-line example with headways for Lines 1 and 2 of 20 and 6 min, respectively. The lines connect Nodes A and B directly. Applying the Spiess-Florian algorithm, Line 1 is added first to the optimal strategy set because it has the lowest link cost (the expected total travel time via Lines 1 and 2 are 13 and 15 min).

Considering Line 2, the selection criterion $u_B + C_a$ is 12 min and would be considered favorable by the Spiess-Florian algorithm because it is less than u_A (currently equal to 13 min). This is true because the expected total travel time from A to B is improved to 12.24 min. However, the exact expressions yield an expected total travel time of 13.35 min, which is inferior to the current value of u_A. The inclusion of Line 2 has therefore increased the expected total travel time despite satisfying the selection criterion.

- Spiess-Florian algorithm:
  - –Expected wait time = $1/2[1/(1/20+1/6)] = 2.31$
  - –Probability of picking Line 1 = $1/20/[1/20+1/6] = 0.23$
  - –Probability of picking Line 2 = $1/6[1/20 + 1/6] = 0.77$
  - –Expected total travel time = $2.31 + 0.23 * 3 + 0.77 * 12 = 12.24$ min.
- Exact expressions:
  - –Expected wait time = $1/120[(-1) * 2 * 6^3/3 + 1/2 * 6^2 * (20 + 6)] = 2.70$
  - –Probability of picking Line 1 = $1/120[1/2 * 6 * 6] = 0.15$
  - –Probability of picking Line 2 = $1/120[120 - 1/2 * 6 * 6] = 0.85$
  - –Expected total travel time = $2.7 + 0.15 * 3 + 0.85 * 12 = 13.35$ min

Line 2 cannot be excluded from further consideration at this stage because the addition of another line could make it favorable. The main implication in the application of the exact expressions is that the expected total travel times calculated at each stage must be stored and compared for the minimum value only after all lines at a node have been evaluated.

APPLICATION TO REAL TRANSIT NETWORK

To illustrate the implementation of the new transit assignment model, a small network of four lines has been extracted from the in transit network Orange County, California. The optimal strategy to reach the University of California, Irvine (Node B) from the Santa Ana Transit Terminal (Node A) is to be identified. Line 65 connects the O-D pair directly, whereas Line 61 requires a transfer at Node C to either Line 382 or Line 74 to reach the destination. Figure 1 illustrates the subnetwork along with the headways and link travel time (both in minutes). Figures 2 and 3 give the ridership probabilities according to the Spiess-Florian algorithm and according to the proposed algorithm using the exact expressions for expected waiting time and ridership probabilities, respectively. The Spiess-Florian algorithm was independently coded; the EMME/2 program was not used.

The network was simulated with an O-D demand of 500 person trips per hour and for a time horizon of 3 hr. To ensure that the resulting line probabilities reflected unconditional values with respect to the initial starting times, 200 initial line starting times were simulated. Figure 4 gives the ridership probabilities. The expected waiting time at the origin node, expected waiting time at the transfer node, and the minimum expected total travel time to reach the destination for all three cases are presented in Table 3. The results obtained from the exact expressions are clearly comparable to those of the network simulation. Table 3 also indicates that the Spiess-Florian algorithm underestimates the expected waiting time at a node and consequently the minimum total travel time to the destination. The expected waiting time at the transfer node validates the assumption that transfer passengers behave similarly to random uniform passenger arrivals, particularly if the simulation is run for a large number of combinations of initial line starting times. The O-D demand rate of 500 persons per hour is considerably heavy; the time horizon of 3 hr is a representative value for a peak period. In the case of the exact expressions, the simulation results are only a validation of their correctness because the simulations were performed under the same assumptions as those behind the expressions.

CONCLUSIONS AND DIRECTIONS FOR FUTURE RESEARCH

The expressions used in transit assignment models for ridership probabilities and expected waiting time are often based on approximations and assumptions, such as Poisson line arrivals. However, real-world situations possibly conform better to assumptions of constant-line headways and random uniform passenger arrivals,

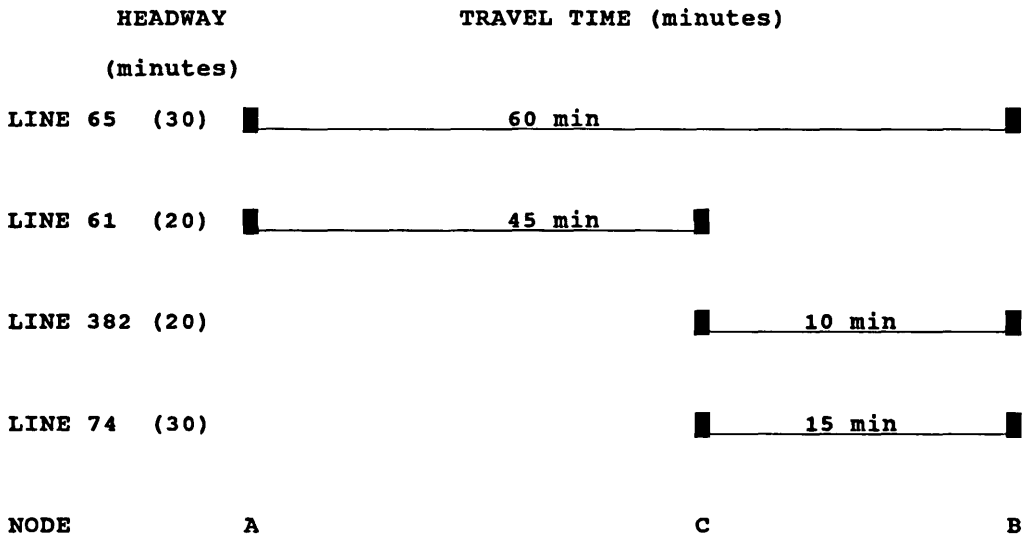

FIGURE 1 Selected lines from extracted network.

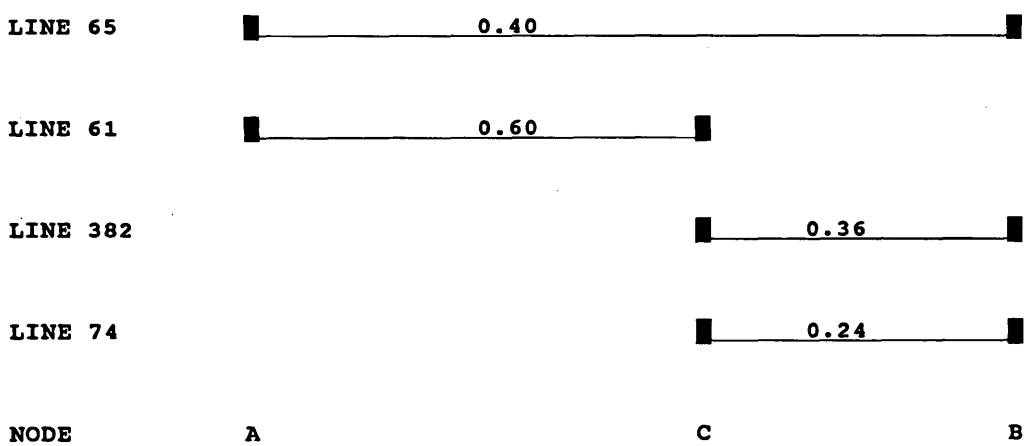

FIGURE 2 Ridership probabilities using Spiess-Florian algorithm.

which have been considered mathematically difficult. Exact expressions for these ridership probabilities and expected waiting times have been derived in this paper. The expressions are most applicable to assignment frameworks that enumerate the choice set of paths on the basis of travel times and expected waiting times, as well as to those that assign the ridership on the basis of line probabilities among a selected set of candidate lines (even if these are based on additional criteria besides travel and waiting times). The Spiess-Florian algorithm has both these characteristics, hence it is used as a benchmark to implement and compare the proposed expressions. The modified transit assignment model yields more robust values for line ridership than the original Spiess-Florian model in experiments based on a simulation of simple transit lines.

Research goals were not limited to developing the exact expressions in the algorithm; they also included the development of an efficient implementation procedure to minimize computational efforts. A recursive approach that avoids calculation of the expressions by brute force at each stage has been successfully implemented in the modified algorithm.

The assumption of uniform random passenger arrivals may not be justified in the presence of improved traveler information. A possible extension would be to develop a comparable model that considers the advantage of real-time information provided both at terminals and in-vehicle. This would result in a dynamic choice set of transit lines as a function of real-time information provided. Such models can then be used for the planning and evaluation of advanced public transit systems. This proposed extension to the model would essentially involve more complex strategies that change in real time, depending on the information provided to the user. The reliability of a transit service can be significantly

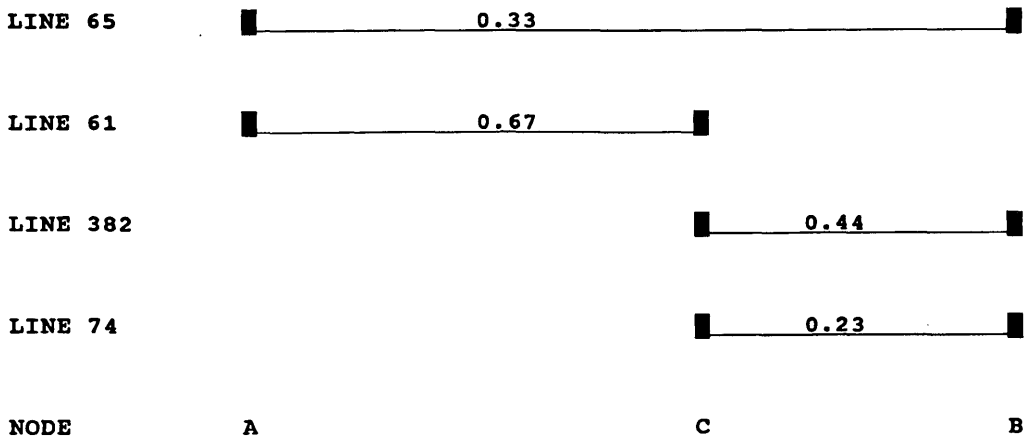

FIGURE 3 Ridership probabilities using exact expressions.

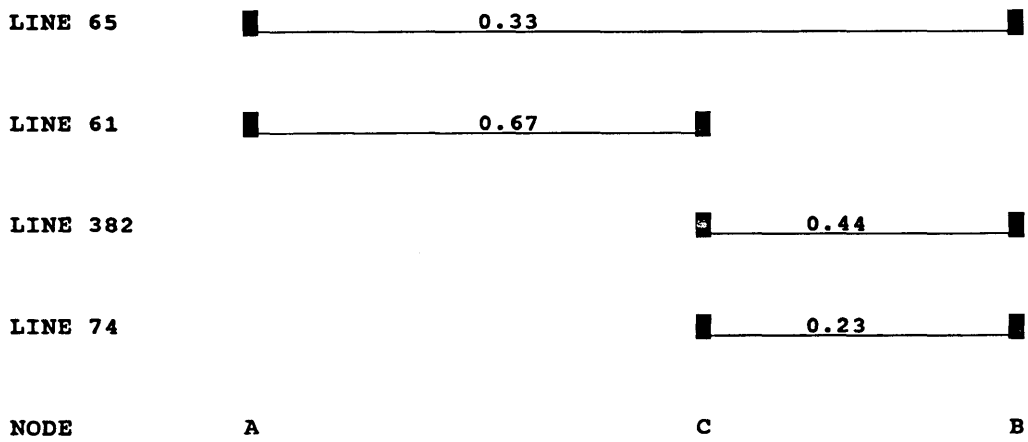

FIGURE 4 Ridership probabilities using network simulation.

improved if real-time information is provided; future research in this direction is needed.

The ability of the assumption that the passenger's choice set consists of multiple paths that give minimum expected travel times still needs to be examined via field validations. Data describing the nature of passenger arrival distributions are critical in evaluating the applicability of the proposed models to transit systems with real-time information availability.

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TABLE 3 Comparison of Results

	Spiess-Florian Algorithm	Exact Expressions	Network Simulation
Minimum Expected Travel Time (min)	67.80	70.74	70.83
Expected Wait at Origin (min)	6.00	7.78	7.80
Expected Wait at Transfer (min)	6.00	7.78	7.83

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