Contributions to Logit Assignment Model

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In the past, research in traffic assignment modeling has been directed primarily toward the deterministic model. Alternative, more behavioral principles were thought to be too demanding computationally. Two mathematical contributions that enable one to solve a logit assignment model with flow-dependent travel times at a reduced cost are presented. First, a convergence test for Fisk's minimization program is introduced on the basis of a duality gap principle. Second, a new definition of Dial's STOCH fixed-time logit assignment procedure is given, in which the set of available paths is defined only once and the computations are reinterpreted. A numerical experiment indicates that these tools make the logit assignment model competitive compared with the procedures conventionally used to solve the deterministic model.

Traffic assignment is the fourth and final step in the conventional travel demand forecasting scheme; by partitioning the origin-destination (O-D) trip rates between several paths, the assignment program attempts to duplicate the vehicular flows on the network. Most assignment models assume that travelers behave rationally. The most well-known assignment principle is that of Wardrop (*1*); that travelers strive to maximize the utility derived from their transportation choices—in other words they try to minimize their generalized travel time. Thus, a user-optimal equilibrium is achieved when no traveler may decrease travel time by unilaterally switching paths.

To account for errors in trip-makers' perception of travel time, Daganzo and Sheffi (*2*) defined the stochastic user principle, according to which all trip makers strive to minimize their stochastic generalized travel time. This rule allows for partitioning O-D trip rates between several alternative paths, even if their true travel times differ from each other.

Two stochastic models are of particular interest: the logit model (*3*) and the probit model (*2,4,5*). The latter, although behaviorally more appealing, is impractical because only Monte-Carlo procedures are available, unless all paths can be identified. The logit model however, is endowed with both an extremely efficient fixed-time assignment procedure (Dial's STOCH2) and a convex minimization formulation with a closed-form objective function (*6*). Nevertheless, computational difficulties have prevented the logit model from enjoying more widespread use. Among other drawbacks, Fisk's (*6*) objective function was thought difficult to evaluate. Only recently have heuristic methods been developed (*7,8*).

In this paper two developments that make computation of a logit user equilibrium competitive with its deterministic counterpart are presented. First, a theoretically sound convergence test for an equilibrium algorithm such as the method of successive averages (MSA) is designed; then it is possible to check whether an equilibrium has been reached. Second, the definition of the set of available paths in Dial's STOCH2 procedure is modified; the procedure is problematic if it is implemented within an equilibration scheme because the path set is likely to change from one iteration to the next. Some changes that remedy this flaw are suggested.

The organization of the paper is as follows. First the problem is stated formally. Second the convergence test for Fisk's model is introduced. Third a definition of efficient paths that does not depend on congestion phenomena is derived; it is inspired from Dial's STOCH2, and a related path loading procedure is provided, wherein it is easy to compute all the terms in Fisk's objective function. Fourth a numerical experiment is carried out to demonstrate that the MSA, combined with the proposed tools, is indeed an efficient algorithm when applied to the logit model. All proofs of the assertions presented here can be found in work by Leurent (*9*), in which elastic demand and capacity constraints are also considered and a dual solution scheme is proposed.

PROBLEM FORMULATION AND MODELING NEEDS

Logit Equilibrium Model

Let $r-s$ be an O-D pair with traffic flow q_{rs}, θ a nonnegative parameter, k a path from r to s with deterministic travel time T^k_{rs} and flow f^k_{rs}. In the logit assignment model (*3*), it is assumed that the path flow f^k_{rs} is proportional to a negative exponential function of the travel time T^k_{rs}:

$$f^k_{rs} = q_{rs} \frac{\exp(-\theta T^k_{rs})}{\Sigma_k \exp(-\theta T^k_{rs})} \tag{1}$$

Then it is automatically ensured that

$$q_{rs} = \Sigma_k f^k_{rs} \tag{2}$$

The travel time of path k is related to the travel times T_a of the links a that belong to it via

$$T^k_{rs} = \Sigma_{a \in k} T_a = \Sigma_a \delta^{ak}_{rs} T_a \tag{3}$$

where $\delta^{ak}_{rs} = 1$ if $a \in k$ or 0 if not.

Let x_a be the traffic flow on link a:

$$x_a = \Sigma_{rsk} \delta^{ak}_{rs} f^k_{rs} \tag{4}$$

Finally let t_a be the travel time function of link a (assumed to be continuous and nondecreasing):

$$T_a = t_a(x_a) \tag{5}$$

Then Equations 1 through 5 define a logit-based equilibrium. Figure 1 illustrates a logit split between two paths.

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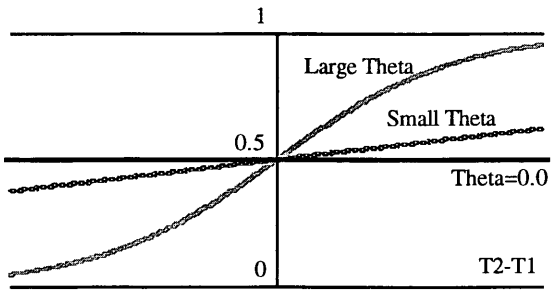

FIGURE 1 Proportion of travelers choosing Path 1 as function of Θ and time difference T_2-T_1 (binary case).

Fisk's Minimization Program

Fisk (*6*) characterized the logit equilibrium with variable travel times as the unique solution to the following convex minimization equation:

$$\min_f J_L(\mathbf{f}) = \Sigma_a \int_0^{x_a} t_a(x)\,dx + \frac{1}{\theta}\,\Sigma_{rsk} f_{rs}^k \log\left(\frac{f_{rs}^k}{q_{rs}}\right) \qquad (6)$$

subject to Equations 2 and 4 and of course to $f_{rs}^k \geq 0$. In Equation 6 Fisk's $\Sigma_{rsk}\ f_{rs}^k$ log (f_{rs}^k) was replaced with $\Sigma_{rsk}\ f_{rs}^k$ log (f_{rs}^k/q_{rs}) to facilitate the understanding of the relationship between Equation 6 and the computations in the STOCH algorithm. This does not alter the existence and uniqueness results obtained by Fisk.

Fisk did not address a crucial question: How should the available paths be defined? In the deterministic model of Beckmann et al. (*10*), all existing acyclic paths may be considered; however, in a logit model a specific definition is required because the conventional shortest-path routines do not automatically find suboptimal paths.

In Dial's paper (*3*), two alternative definitions of efficient paths are provided, namely STOCH and STOCH2. But these definitions are consistent only with respect to fixed travel times (i.e., with constant functions t_a in Equation 6 and cannot be used in a variable-time program. A definition that is consistent with flow-dependent travel times will be provided later (STOCH3 procedure). First, equilibration issues are addressed.

Method of Successive Averages

Powell and Sheffi (*11*) proved the convergence of the MSA applied to minimization programs as Fisk's (provided that the definition of available paths cannot vary).

Fixed-time assignment (FTA) is defined as a path-loading procedure that partitions the O-D flow according to the logit rule, based on a given set of available paths. An FTA yields a solution to Equation 6 with constant travel time functions and a given set of utilized paths. The MSA equilibration algorithm is composed of the following steps:

- Step 0: Initialization.
  –Set iteration counter $n = 0$.
  –Choose a sequence $\alpha^{(k)}$ of real numbers such that $[0 \leq \alpha^{(k)} \leq 1]$, $[\Sigma\alpha^{(k)} = +\infty]$ and $[\Sigma\alpha^{(k)2} < +\infty]$.
  –Find an initial feasible flow pattern $x_a^{(0)} = x_a[\mathbf{f}^{(0)}]$. It may be obtained through an FTA on the basis of link times $t_a^{(-1)} = t_a(0)$.
- Step 1: Link travel time update.
  –Set $t_a^{(n)} = t_a(x_a^{(n)})$.
- Step 2: Direction finding.
  –Compute an FTA of traffic of all O-D pairs on the basis of link travel times $t_a^{(n)}$: this yields a path flow solution $\mathbf{g}^{(n)}$ and also an auxiliary are flow pattern

  $$y_a^{(n)} = x_a[\mathbf{g}^{(n)}]$$
- Step 3: Link flow update.
  –Set $x_a^{(n+1)} = x_a(\mathbf{f}^{(n+1)}) = x_a^{(n)} + \alpha^{(n)}[y_a^{(n)} - x_a^{(n)}]$
- Step 4: Convergence test.
  –Apply a convergence test: either a maximum number of iterations or a test on the maximum value (over the arcs a of the network) of the change in $\Sigma_{k=1}^n \alpha^{(k)} x^{(k)}{}_a / \Sigma_{k=1}^n \alpha^{(k)}$ from the previous iteration $n - 1$ to the current one n. If the test is satisfied, then terminate; if not, increment the iteration counter $n = n + 1$ and go to Step 1.

The MSA has been widely applied to solve Fisk's program. However, the definition of efficient paths has not been adequately addressed. Thomas (*12*) wrote that "it seems likely that methods which incorporate definitions of acceptable paths similar to those of Dial and Gunnarsson are intrinsically non-convergent, though in practice users often claim them to be satisfactory in that respect." In the following section, a theoretically sound convergence test is provided for the equilibration algorithm that will be useful together with a formal definition of the efficient paths, as will be given later.

CONVERGENCE TEST FOR LOGIT MODELS

First the issue of designing a theoretically sound convergence test for an application of the MSA to Fisk's program is considered. It is based on a duality gap principle inspired from the deterministic model.

Duality Gap Principle in Deterministic Model

In the deterministic case, where only those paths whose travel times are minimal are used, the objective function reduces to $J_D(\mathbf{f}) = \Sigma_\alpha \int_0^{x_a} (\mathbf{f})\, t_\alpha(x)dx$. The usual convergence test is to evaluate a duality gap between the objective function $J_D[\mathbf{f}^{(n+1)}]$ and a lower bound estimate:

$$J_D[\mathbf{f}^{(n)}] + \nabla J_D[\mathbf{f}^{(n)}] \cdot [\mathbf{g}^{(n)} - \mathbf{f}^{(n)}]$$

where $\mathbf{g}^{(n)}$ is obtained in Step 2 of the MSA (or equivalently of the Frank-Wolfe method). Thus, the duality gap is given by

$$\begin{aligned} DG_D^{(n)} &= \Sigma_a\, \mathbf{t}_\alpha^{(n)}\{x_\alpha[\mathbf{f}^{(n+1)}] - x_\alpha[g^{(n)}] \\ &= \Sigma_{rsk} f_{rs}^{k(n+1)}[T_{rs}^{k(n)} - \min_k T_{rs}^{k(n)}] \end{aligned}$$

The duality gap $DG^{(n)}{}_D$ is always positive, except at equilibrium, at which point it is 0. Hence, a convergence test involves checking whether DG is close to 0.

Convergence Test for Fisk's Model

Application of the duality gap principle to the logit model is suggested. Denoting the entropic part of the logit objective function indicates the following:

$$J_E(\mathbf{f}) = J_L(\mathbf{f}) - J_D(\mathbf{f}) = \frac{1}{\theta} \Sigma_{rsk} f_{rs}^k \log\left(\frac{f_{rs}^k}{q_{rs}}\right) \tag{7}$$

Then the flow vector $\mathbf{g}^{(n)}$ considered in Step 2 of the MSA is the unique solution to the following auxiliary program:

$$\min_g J_f^{(n)}(\mathbf{g}) = J_D[\mathbf{f}^{(n)}] + \nabla J_D[\mathbf{f}^{(n)}] \cdot [\mathbf{g} - \mathbf{f}^{(n)}] + J_E(\mathbf{g}) \tag{8}$$

The duality gap associated with the logit objective function is $DG_L^{(n)} = J_L[\mathbf{f}^{(n+1)}] - \mathrm{LBE}^{(n)}$, where the lower bound estimate $\mathrm{LBE}^{(n)}$ is defined as

$$J_D[\mathbf{f}^{(n)}] + \Delta J_D[\mathbf{f}^{(n)}] \cdot [\mathbf{g}^{(n)} - \mathbf{f}^{(n)}] + J_E[\mathbf{g}^{(n)}]$$

When applying the MSA algorithm to the logit model, generally it is not possible to compute $J_E(\mathbf{f})$, unless all paths are identified. However, for some models such as the one that will be described later, it is easy to compute $J_E(\mathbf{g})$. The trick is to evaluate the duality gap with respect to $\mathbf{g}^{(n)}$ and not with respect to $\mathbf{f}^{(n+1)}$. The following convergence test is also suggested on the basis of functions related to $\mathbf{g}^{(n)}$ rather than to $\mathbf{f}^{(n+1)}$:

if

$$J_L[\mathbf{g}^{(n)}) - \mathrm{LBE}^{(n)} \leq \epsilon \{|J_L[\mathbf{g}^{(n)}]| + |LBE^{(n)}|\}$$

then

terminate and let $\mathbf{g}^{(n)}$ be the solution to the minimization equation (Equation 6) or else return to Step 1.

If true, the test gives a vector that solves the minimization equation on the basis of the convexity of J_L. Conversely, if the path flow vector $\mathbf{f}^*$ solves the program, then auxiliary vector $\mathbf{g}^*$ that corresponds to $\mathbf{f}^*$ is in fact equal to it and thus the convergence test is satisfied (*9*). If only a relative measure $J_L - \mathrm{LBE}$ of the duality gap is needed, then it is not necessary to compute J_E: the test can reduce to check if $J_L[\mathbf{g}^{(n)}] - \mathrm{LBE}^{(n)} \leq \epsilon$; in other words to check if $J_D[\mathbf{g}^{(n)}] - J_D[\mathbf{f}^{(n)}] - \Delta J_D[\mathbf{f}^{(n)}] \cdot [\mathbf{g}^{(n)} - \mathbf{f}^{(n)}] \leq \epsilon$.

DEVELOPMENT OF STOCH3 PROCEDURE

The results obtained so far apply to any set of utilized paths under the sole constraint that no path may include a given node more than once. Now a set of efficient paths that enable one to benefit from the efficiency of Dial's STOCH2 is defined.

Most previous logit assignment models have used Dial's second definition of efficient paths, according to which "a path is efficient (reasonable) if every link in it has its initial node closer to the origin than is its final node." The word "closer" refers to the travel time measured from the origin with respect to a current travel time vector that may change from one iteration to the next. Therefore, there was no use trying to compute an objective function for the logit assignment model.

Three problems had to be tackled:

1. Restrict Dial's set of efficient paths so as to limit its size and for each reasonable path not to be much longer than the shortest one,
2. Stabilize the definition of efficient paths so that it depends neither on congestion nor on the iteration number, and
3. Find a way to compute the entropic part of the objective function, so as to measure the convergence rate.

The first two problems are discussed first on the basis of previous work by Tobin (*13*). Then the STOCH3 procedure, which offers a practical way to perform a fixed-time logit assignment on the efficient paths defined formerly, will be introduced. Finally, a way to evaluate $J_E(\mathbf{g})$ in the STOCH3 model will be described.

Definition of Stable Set of Efficient, Not-Too-Long Paths

A path is considered "STOCH3 efficient" (or reasonable, or available) if it does not include the same node more than once, if every link has its initial node closer to the origin than its final node, if every link is "reasonable enough" compared with a reference shortest path.

More precisely, let

T_a^0 = a reference generalized travel cost for Link a;
$C_r^0(n)$ = a reference shortest generalized travel cost from origin r to node n, based on link costs T_a^0;
h_r^a = a maximum "elongation ratio" for link a origin r;
B_a, E_a = the beginning and end nodes respectively, of Link a.

Definition 1; a path k from origin r to destination s is STOCH3 efficient if (*a*) it does not comprise a given node more than once, (*b*) $C_r^0(E_a) > C_r^0(B_a)\ \forall\ a \in k$; and (*c*) $(1 + h_r^a)[C_r^0(E_a) - C_r^0(B_a)] \geq T_a^0$, with $h_r^a \geq 0$, $\forall\ a \in k$.

A link a that satisfies the last two conditions is called STOCH3-reasonable wrt. origin r.

The last condition in Definition 1 limits the number of efficient paths by limiting their total reference generalized travel cost: defining $H_r = \max_a h_r^a$, summing over all links a that are incident to an efficient path k yields that

$$\begin{aligned}\text{Length}(k) &= \sum_{a \in k} T_a^0 \leq (1 + H_r)(C_r^0(s) - C_r^0(r)) \\ &= (1 + H_r) \min_{k'} \text{length}(k')\end{aligned}$$

Conversely, if k satisfies length $(k) \leq (1 + H_r) \min_{k'}$ length (k'), it may not be efficient because the first two conditions in Definition 1 must hold as well.

Definition 1 is inspired from Dial's specification STOCH2 (*3*), with respect to the second condition, and from Tobin (*13*) with respect to the third. The contribution of the author is to impose fixed reference travel costs, thus ensuring a stable definition of the efficient paths, whatever the congestion phenomena.

STOCH3 Procedure

Recall that in the STOCH3 procedure it is necessary to consider, on the one hand, the reference generalized travel costs to enumerate the available paths and, on the other hand, the "actual" travel times according to which the O-D flows are partitioned between the paths.

Equation Variables

The following variables apply.

n = node with reference travel cost $C_r^0(n)$ from origin r,
$O_r(i)$ = the ith node in order of increasing access cost $C_r^0(n)$ from r,
Ω_r^a = an indicator variable of 1 if link a is reasonable from r and 0 otherwise,
T_a = current travel time on link a,
$A(a)$ = impedance of link a,
$W_A(a)$ = link weight that accounts for importance of a in contributing to a reasonable path,
$W_N(n)$ = node weight,
$X_A(a)$ = flow on link a from current origin r,
$X_N(n)$ = flow passing through node n from current origin r, and
$F(a)$ = total current flow on link a (over all origins).

Index r can be omitted when writing variables A, W_A, W_N, X_A, and X_N because these variables do not need to be stored after dealing with origin r.

Algorithm STOCH3

- Step 0: Overall preliminaries: calculation of reasonable path.
  –From every origin node r, compute the shortest paths to all nodes n, on the basis of the reference link travel costs T_a^0, yielding the reference access costs $C_r^0(n)$ and a labeling $O_r(i)$ of the nodes n in the order of increasing access cost from r.
  –For each link a, set $\Omega_r^a := 1$ if $(1 + h_r^a)[C_r^0(E_a) - C_r^0(B_a)] \geq T_a^0 > 0$, $\Omega_r^a := 0$ otherwise.

- Step 1: Preliminaries for a standard iteration.
  –Initialize the total link flow variables $F(a)$ to 0.
  –Set the link impedances $A(a) = \exp(-\theta T_a)$.

Steps 2, 3 and 4 are to be run for each origin node r.

- Step 2: Forward pass.
  –Set all $W_A(a)$ and $W_N(n)$ to 0. Set $W_N(r) = 1$.
  –For each node n taken in the order of increasing reference cost $C_r^0(n)$ (the ith node to be considered is indicated by $O_r(i)$), for each link a with beginning node $B_a = n$; if $\Omega_r^a = 1$, then compute $W_A(a) = A(a)W_N(n)$ and add $W_A(a)$ to $W_N(E_a)$ or else do nothing.

- Step 3: Backward pass.
  –For each node n, set $X_N(n) = q_{rn}$ if n is a destination node, 0 otherwise.
  –For each node n taken in the order of decreasing reference cost $C_r^0(n)$ (use the labeling $O_r(i)$ in decreasing order), for each link a with end node $E_\alpha = n$, if $\Omega_r^a = 1$ then compute $X_A(a) = X_N(n)W_A(a)/W_N E_a)$ and add $X_A(a)$ to $X_N(B_a)$ or else set $X_A(a) = 0$.

- Step 4: Contribution to total link flows.
  –$\forall$, $F(a) = F(a) + X_A(a)$

At the end of the procedure, the vector **F** gives the fixed-time logit assignment on the basis of link travel times T_a.

Computation of Entropic Part of Objective Function in STOCH3 Model

It is shown by Leurent (*9*) that, at the end of the forward pass from origin r, it holds that

$$W_N(s) = \Sigma_k \exp(-\theta T_{rs}^k) \tag{10}$$

$$g_{rs}^k = q_{rs} \exp(-\theta T_{rs}^k) / \Sigma_k \exp(-\theta T_{rs}^k)$$

therefore

$$\frac{1}{\theta} \Sigma_k g_{rs}^k \log\left(\frac{g_{rs}^k}{q_{rs}}\right) = -\Sigma_k g_{rs}^k T_{rs}^k - \frac{q_{rs}}{\theta} \log[\Sigma_k \exp(-\theta T_{rs}^k)]$$

By summing over all O-D pairs $r - s$,

$$J_E(\mathbf{g}) = \frac{1}{\theta} \Sigma_{rsk} g_{rs}^k \log\left(\frac{g_{rs}^k}{q_{rs}}\right) = -\Sigma_a x_a(\mathbf{g}) \cdot T_a - \frac{1}{\theta} \Sigma_{rs} q_{rs} \log[\Sigma_k \exp(-\theta T_{rs}^k)]$$

Then, the convergence test designed earlier can be applied to the STOCH3 set of available paths.

COMPUTATIONAL EVIDENCE

In this section, a numerical example to compare the performance of the STOCH3 logit model using the MSA is compared with that of the deterministic model using both the Frank-Wolfe algorithm and the MSA.

Case Study

The application is related to the western part of the Paris metropolitan area during the evening peak period, with a typical trip travel time of 1 hr. The test network is composed of 2,000 directed links. There are 141 O-D zones.

The dispersion parameter θ is set to 0.233 mn^{-1} so that when two routes compete with each other, the first one with a travel time 5 min shorter than the second one, approximately three out of four drivers choose the first road. Because only the rate of convergence is of interest here, the elongation ratios h^a_r are set to $+\infty$ [as noted from previous surveys they may be set to $h^a_r = 1.6$ for interurban studies (*14*) or $h^a_r \in [1.3; 1.5]$ for urban studies (*15*)].

Results

Figure 2 depicts the performance of the three algorithms, showing the evolution of

$$\log\left|\frac{X^{(n)}}{J^*} - 1\right|$$

where (*a*) in the logit model, J^* is the optimal value of the objective function in Equation 6, and $X^{(n)}$ is the value of $J_L[\mathbf{g}^{(n)}]$. In the MSA, the step size $\alpha^{(n)}$ is set to $1/(4 + n/10)$; (*b*) for the deterministic model, J^* is the optimal value of the deterministic objective func-

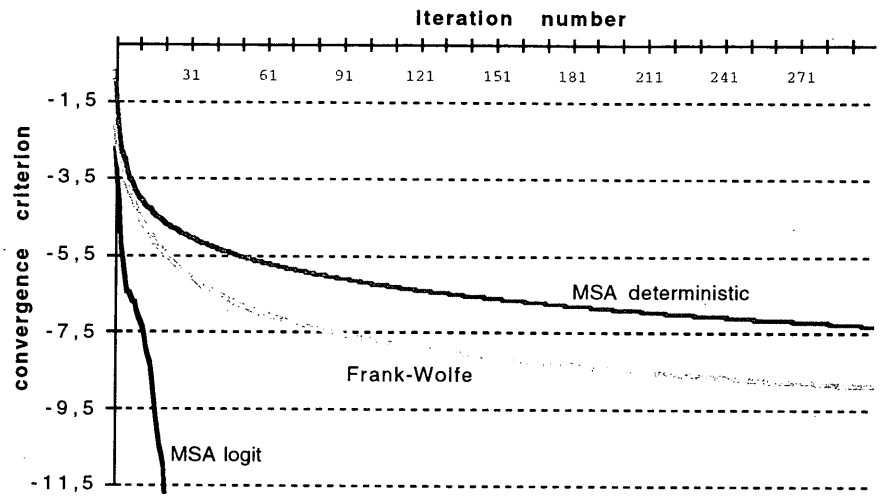

FIGURE 2 Convergence rates of three algorithms.

tion, and $X^{(n)}$ is the value of $J_D[\mathbf{f}^{(n+1)}]$. In the MSA, the step size $\alpha^{(n)}$ is set to $1/(1 + n)$. The convergence rate is much better in the case of the logit model, notably because the descent direction includes information about all of the available paths—not only about the shortest path in each iteration.

COMMENTS AND CONCLUSION

Intelligent Vehicle-Highway System Implications

In an intelligent vehicle-highway system context, the logit model may be of particular interest for assessing the level of information provided to motorists by a route guidance system (*16*). One way to evaluate the effects of such a system is to model two classes of motorists, the first equipped with a route guidance device and characterized by a large dispersion parameter θ, and the other class of nonequipped drivers characterized by a small θ.

Model Extensions

The case of elastic demand and capacity constraints is addressed by Leurent (*9*). A dual solution scheme is also introduced, but for large-scale applications it is not efficient. The computational efficiency of the MSA applied to the logit assignment model facilitates the following possible extensions of the model:

- Diagonalization schemes, for example with travel time functions that depend on flows of several links (it is easy to derive a variational inequality formulation of Equation 6), and
- Simultaneous models that capture more than one step in the conventional transportation planning process.

Path Identification

- *It is useful to identify paths.* The STOCH procedure is a way to consider all available paths at a reduced cost. The numerical experiment demonstrates here, above all, that path-based equilibration algorithms are much more efficient than link-based algorithms. This conclusion is also supported by recent work (*17,18*).
- *Algorithms that identify paths should better address more behavioral models.* In a fixed-time path-loading procedure like STOCH, the O-D flow is partitioned between the paths according to a behavioral rule. Other available behavioral rules are the probit model [(*4,5*); see the work of Daganzo and Sheffi (*2*) and Powell and Sheffi (*11*) for a mathematical foundation], and the bicriterion, cost-versus-time model [(*19*); Leurent (*20*) gives a mathematical foundation]. By applying a behavioral rule, the need to search for an effective step size in the descent is bypassed. It is thus remarkable that, by the identification of paths, the computational process is greatly facilitated, especially in the case of behavioral models.

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