Another Look at A Priori Relationships Among Traffic Flow Characteristics

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Past derivations of a priori relationships among speed, flow, and concentration (such as the fundamental relationship and the speed-flow-occupancy relationship) have involved unrealistic assumptions of uniformity in at least one traffic flow characteristic. Several relationships are derived for which these assumptions of uniformity are relaxed. For relationships involving both time- and distance-based variables, this requires that the relationship be understood in probabilistic terms; where all variables are time-based, deterministic relationships are also possible. The fundamental relationship can be shown to be strictly true in the limit where the time and distance intervals over which measurements are taken approach 0. Where the order of arrival of vehicles with particular speeds is random, the fundamental relationship is found to hold for average values of the variables in question; this is also true if the section over which density is measured is empty at the beginning and end of the time interval used for averaging. Relationships derived under various other assumptions involve covariance terms, so that if particular variables are not correlated, simple relationships continue to hold. Where these variables are correlated, biases may be expected. Under certain conditions, these biases may be quite serious. Comparison of the relationships derived here with those of past empirical studies results in good agreement for the relationship between density, as estimated from the fundamental relationship, and occupancy. On the other hand, previously reported discrepancies between measured speeds and speeds calculated from flows and occupancies cannot be explained fully by the covariance terms in the relationships derived here.

The study of relationships among the traffic flow characteristics speed, flow, and concentration (either density or occupancy) has long been a fundamental part of traffic research. In general, two types of relationships among these variables are possible: a priori relationships, which proceed from the definitions of the various measures, and empirical relationships, which can be discovered only by observing actual traffic flow.

Not surprisingly, the bulk of the literature focuses on empirical relationships. The major a priori relationships were worked out early in the history of traffic flow research and have been little examined since. They have often been taken for granted and used freely to transform data from one form to another or to move back and forth among the three possible bivariate relationships involving speed, flow, and concentration. Nevertheless, several recent studies by Hall et al. have raised questions about the accuracy and applicability of these relationships (*1–4*). These studies have also presented data that appear to contradict them to some extent and have suggested using three-dimensional empirical models that are independent of them.

The relationships in question include the so-called *fundamental relationship:*

$$q = uk \tag{1}$$

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where

q = flow,
u = speed, and
k = density.

A similar relationship exists among speed, flow, vehicle length, and occupancy:

$$u = \frac{qL}{H} \tag{2}$$

where L represents vehicle length and H occupancy, defined as the fraction of time that vehicles are present at a point. The classical derivation of Equation 1 is that of Wardrop (*5*). Equation 2, which most commonly has been used to estimate speeds from flow and occupancy data, was proposed by Athol (*6*).

Since the use of these relationships (especially Equation 1) has been pervasive in traffic flow theory, confirming major inaccuracies in them could have far-reaching consequences. The purpose of this paper is to reexamine the validity of these relationships, extend their derivations to address certain oversimplifications, and consider the possible reasons for apparent discrepancies between them and actual data, particularly those reported by Hall and Persaud (*1*).

FUNDAMENTAL RELATIONSHIP

Theoretical objections to the fundamental relationship (Equation 1) arise from the distinction between relationships that hold true for uniform traffic streams (those with constant, identical speed and spacing for all vehicles) and those that hold for averages of the characteristics of nonuniform traffic streams. In addition, the fundamental relationship involves both time- and distance-based variables, which may be incompatible with one another in nonuniform traffic streams.

Point Relationships Among Traffic Flow Characteristics

The version of Equation 1 presented here implicitly assumes a uniform traffic stream and under that assumption can be derived easily by means of dimensional analysis. It can also be shown to be true at a point, if all measures are regarded as continuous variables. This approach to its derivation makes use of a three-dimensional surface proposed by Makigami et al. (*7*). If vehicle trajectories are plotted and numbered (with some adjustments in cases in which vehicles pass one another), the trajectories may be considered as the contour lines of a surface of cumulative flow versus time and distance. For

such a surface to exist, it must be possible to smooth out the discrete steps in the actual cumulative vehicle function, so as to treat it as continuous. Where this is a reasonable simplification, the partial derivative of the cumulative vehicle function $A(x,t)$ with respect to time represents flow, that of A with respect to distance represents density, and that of distance with respect to time represents the speed of a vehicle at an instant of time. That is,

$$q = \frac{\partial A}{\partial t} \tag{3}$$

$$k = \frac{\partial A}{\partial x} \tag{4}$$

and

$$u = \frac{\partial x}{\partial t} \tag{5}$$

Then, since

$$\frac{\partial A}{\partial t} = \frac{\partial x}{\partial t}\frac{\partial A}{\partial x} \tag{6}$$

it follows that

$$q = uk$$

Nonuniform Flow over Long Time Intervals

Obvious problems with these derivations are that (*a*) real traffic streams are never truly uniform and, under some conditions (congested flow, for instance), are far from being even approximately uniform; and (*b*) point measures of traffic characteristics such as flow and density have theoretical meaning only. To apply to more-realistic models of the traffic stream, derivations of the fundamental relationship must be able to relate average values of the traffic flow variables, measured over more extended times and distances.

A possible way of doing so is to further consider the traffic flow surface proposed by Makigami et al. (*7*). Consider two locations x_1 and x_2 such that there are no entrances or exits between them. Plots of the cumulative numbers of vehicles passing these two points result in the functions $A(x_1, t)$ and $A(x_2, t)$. Now consider a time interval T that begins when the section between x_1 and x_2 is empty, continues so long as vehicles are present in the section, and ends as soon as the section is empty again. Figure 1 shows plots of $A(x_1, t)$ and $A(x_2, t)$ for time interval T. At any given time $N(t)$ vehicles are present in the section. The average number of vehicles in the section at any time is

$$\overline{N} = \frac{\int_0^T N(t)dt}{T} \tag{7}$$

and the average density is

$$\overline{k} = \frac{\int_0^T N(t)dt}{(x_2 - x_1)T} \tag{8}$$

The total flow exiting the section during time T is $A(x_2, T)$, which, under the preceding assumptions, is also the total flow entering, or $A(x_1, T)$. The average flow, then, is

$$\overline{q} = \frac{A(x_2, T)}{T} \tag{9}$$

Finally, the total time consumed by all vehicles in the section is

$$\int_0^T N(t)dt$$

so that the average time required for a vehicle to cross the section is

$$\overline{t} = \frac{\int_0^T N(t)dt}{A(x_2, T)} \tag{10}$$

and the harmonic mean or space mean speed is

$$\overline{u}_s = \frac{x_2 - x_1}{\overline{t}} = \frac{A(x_2, T)(x_2 - x_1)}{\int_0^T N(t)dt} \tag{11}$$

Substituting $\overline{k}T$ for

$$\frac{\int_0^T N(t)dt}{x_2 - x_1}$$

and $\overline{q}T$ for $A(x_2, T)$ results in

$$\overline{u}_s = \frac{\overline{q}T}{\overline{k}T} = \frac{\overline{q}}{\overline{k}} \tag{12}$$

Note that the assumption that the section is empty at both the beginning and end of period T is necessary for this relationship to be strictly true. In the absence of this assumption, the total travel time does not represent the sum of the travel times across the entire section for any particular group of vehicles; consequently, $\overline{u}_s$ would be at best an approximation. In cases in which T is long relative to $\overline{t}$, this may not be important, but in the extreme case in which T approaches 0, the average speed used in Equation 11 would be meaningless. Also, $A(x_1,T)$ would not equal $A(x_2,T)$, so that $\overline{q}$ would be defined ambiguously. Again, if $A(x_1,T)$ is approximately equal to $A(x_2,T)$, this may not be important, but over short time intervals the difference is apt to be fairly large, especially in congested flow.

Wardrop's Derivation

A second approach to deriving the fundamental relationship for nonuniform flow is that of Wardrop. This classical derivation relaxes the assumptions of uniform speeds and densities by assuming instead that the traffic stream is composed of a set of subsidiary traffic streams. Within each stream, speeds of all vehicles are identical and constant with respect to time and distance, but vehicle spacings are random. Wardrop's subsidiary streams thus represent a discrete approximation of the speed distribution. From these assumptions, Wardrop shows that if each traffic stream i has speed u_i and flow q_i, the characteristics of the traffic streams may be combined to give

$$q = \overline{u}_s k \tag{13}$$

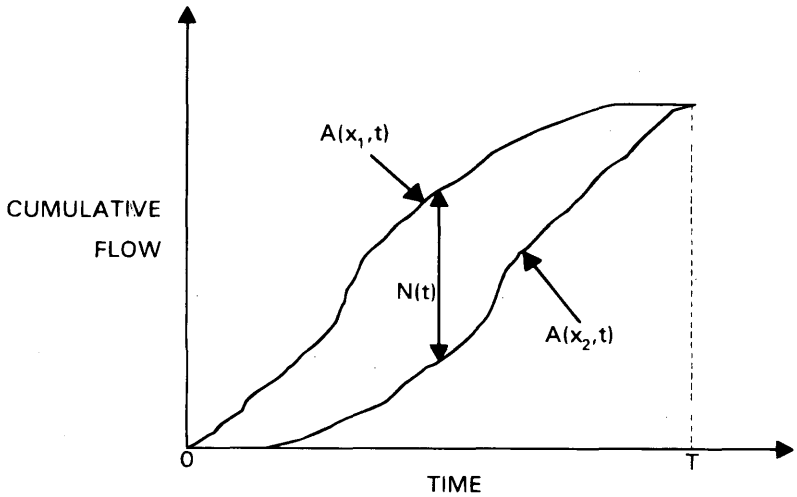

FIGURE 1 Cumulative flow versus time at two locations.

where q and k are the overall flow and density of the traffic stream and $\bar{u}_s$ is the harmonic mean or space mean speed.

Uncongested Flow: Speed Independent of Flow

A close look at Equation 13 shows that it is not exactly correct, since q and k cannot be uniform quantities if vehicles are spaced randomly. Instead, q and k must be intended to be averages or expected values. More important, the assumption of a discrete speed distribution is unrealistic. To relax this assumption, however, it is necessary to confront a fundamental difficulty that arises because of the combination of time and distance-based variables in the fundamental relationship.

Density is most properly measured by counting the number of vehicles present in a section of known length at an instant of time. The presence of any given vehicle in the section at this instant, however, is dependent on the exact time that it entered the section as well as its average speed across it. There can be no deterministic relationship among speed, flow, and density for nonuniform traffic streams because flow conveys only the average time between the arrival of successive vehicles at a point, not the exact time that each vehicle arrived.

Under certain assumptions of randomness, this difficulty can be circumvented by formulating the relationship in probabilistic terms. The resulting relationships must be understood to hold only among the average or expected values of the variables when repeated samples are taken, not for the measured values of any given sample. In the case of uncongested flow, it may be reasonable to assume a continuous speed distribution that is nearly independent of flow, especially over short periods of time, and for which the order of arrival of vehicles with given speeds is random.

Consider a traffic stream with an average flow rate of $\bar{q}$ over some period of time. Flow during this same period is also characterized by a speed distribution with a probability distribution function $p(u)$. If vehicles arrive at random, and the speeds of individual vehicles are independent of one another, the probability that a vehicle with speed u' is present in a section of length X at any instant is proportional to X/u', the amount of time that the vehicle spends in the section. Note that, to be strictly correct, the speed in question should be the average speed of the vehicle across the section, not its spot speed at any point. Thus the probability of detecting a vehicle with any given speed is

$$p = \bar{q}\left(\frac{X}{u}\right)p(u)du \tag{14}$$

For the entire traffic stream, the expected number of vehicles present in a section of length X is

$$E(N) = \bar{q}X\int_0^\infty \frac{p(u)}{u}du \tag{15}$$

The lower bound of the integral is shown as 0 on the assumption that negative speeds do not occur; this is not important, however, as long as it is understood that the integral of $p(u)$ from 0 to infinity is 1.0. Expected density, in turn, is the expected number of vehicles in the section divided by the length of the section, or

$$\bar{k} = \frac{E(N)}{X} = \bar{q}\int_0^\infty \frac{p(u)}{u}du \tag{16}$$

For a continuous speed distribution, however, space mean speed is defined as

$$\bar{u}_s = \frac{1}{\int_0^\infty \frac{p(u)}{u}du} \tag{17}$$

so that Equation 6 becomes

$$\bar{k} = \frac{\bar{q}}{\bar{u}_s} \tag{18}$$

Consequently, it can be shown that the fundamental relationship also holds between space mean speed and the arithmetic means of flow and density in cases in which speed distributions are continuous, so long as the order of arrival of vehicles with given speeds is random.

Congested Flow: Cyclic Speed-Flow Variation

In congested flow, the assumption of random arrivals is unlikely to be valid. Instead, what is usually observed is a pattern of waves moving upstream. Flow moving through these waves is characterized by alternating periods of acceleration and deceleration and is often called stop-and-go traffic. The behavior of such waves is not very well understood, although there is literature, both theoretical and empirical, related to them (*8–12*). In any case, however, speeds, flows, and densities in congested flow appear to be strongly correlated with one another.

At one extreme, the wave pattern might be assumed to consist of a series of identical waves with periods T and wave lengths X. Under that assumption, the relationship among speed, flow, and density is affected by the length of the section over which density is measured. Suppose, for instance, density is defined over the waves' length X and flow over their period T. In this case, both the number of vehicles present in the section and the amount of time it takes each vehicle to cross the section are constants, since flow is always identical at both boundaries of the section. The density in this case is $k = \bar{q}T/X$ and the speed is $u = X/T$, so that

$$\bar{k} = \frac{\bar{q}}{u} \tag{19}$$

Note, however, that k and u are constants only if measured over X (or some integral multiple thereof). For distances less than X, they vary, and for distances much less than X, they fluctuate widely. Under these conditions, Equation 19 is no longer valid.

Consider a traffic stream consisting of a series of identical waves with period T. At any instant t, there is an instantaneous flow rate $q(t)$ arriving at some point in a section of length X (which is less than the wave period) and an average speed across the section of $u(t)$. The speed $u(t)$ (as measured over distance X) varies less than the spot speed measured at any point in the section, but it still varies and is correlated with $q(t)$, the flow passing the point at time t. The probability of detecting a particular vehicle in a sample taken at a random instant is proportional to the amount of time that the vehicle spends in the section. In this case, this time is given by

$$\frac{X}{u(t)} \tag{20}$$

and, since the cycle repeats itself over T, the probability of detecting the vehicle that passed the point at t is

$$\frac{X}{Tu(t)} \tag{21}$$

Since the number of vehicles passing the point at t is given by $q(t)dt$, the expected number of such vehicles to be detected is

$$\frac{X}{T}\frac{q(t)dt}{u(t)} \tag{22}$$

Note that in any given sample, the $q(t)dt$ vehicles passing the point t are either in the section or not; Expression 22 gives the average number of such vehicles that would be detected in repeated samples. The expected *total* number of vehicles detected in any given sample may be found by integrating over t:

$$E(N) = \frac{X}{T}\int_0^T \frac{q(t)}{u(t)}dt \tag{23}$$

Once again, the expected density is the expected number of vehicles in the section divided by X, or

$$\bar{k} = \frac{1}{T}\int_0^T \frac{q(t)}{u(t)}dt \tag{24}$$

Now let Λ represent the reciprocal of speed, so that $\Lambda(t) = 1/u(t)$ and $\bar{\Lambda} = 1/\bar{u}_s$. Also, let $q(t)$ be replaced by $\bar{q} + [q(t) - \bar{q}]$ and $\Lambda(t)$ by $\bar{\Lambda} + [\Lambda(t) - \bar{\Lambda}]$, where $\bar{q}$ and $\bar{\Lambda}$ are the mean values of q and Λ. Equation 24 may now be rewritten as

$$\bar{k} = \frac{1}{T}\int_0^T \{\bar{q} + [q(t) - \bar{q}]\}\{\bar{\Lambda} + [\Lambda(t) - \bar{\Lambda}]\}dt \tag{25}$$

Expanding Equation 25 results in

$$\bar{k} = \frac{\int_0^T \bar{q}\bar{\Lambda}dt}{T} + \frac{\bar{\Lambda}\int_0^T [q(t) - \bar{q}]dt}{T} + \frac{\bar{q}\int_0^T [\Lambda(t) - \bar{\Lambda}]dt}{T} + \frac{\int_0^T [q(t) - \bar{q}][\Lambda(t) - \bar{\Lambda}]dt}{T} \tag{26}$$

By definition, however,

$$\int_0^T [q(t) - \bar{q}]dt = 0 \tag{27}$$

$$\int_0^T [\Lambda(t) - \bar{\Lambda}]dt = 0 \tag{28}$$

and

$$\sigma_{q\Lambda} = \frac{1}{T}\int_0^T [q(t) - \bar{q}][\Lambda(t) - \bar{\Lambda}]dt \tag{29}$$

where $\sigma_{q\Lambda}$ is the covariance of flow and the reciprocal of speed. Equation 26 may now be rewritten as

$$\bar{k} = \frac{\bar{q}\bar{\Lambda}T}{T} + \sigma_{q\Lambda} = \frac{\bar{q}}{\bar{u}_s} + \sigma_{q\Lambda} \tag{30}$$

Let the estimated density that would be calculated by dividing flow by space mean speed be represented by $\hat{k} = \bar{q}/\bar{u}_s$. Then, from Equation 30,

$$\bar{k} = \hat{k} + \sigma_{q\Lambda} \tag{31}$$

In congested flow, speeds and flows tend to be correlated positively; consequently, the covariance of flow and the reciprocal of speed should be negative. This means that in congested flow characterized by a uniform wave pattern, the actual expected density should be less than that estimated by dividing average flow by space mean speed, where $\bar{u}_s$ is defined over a distance less than the wave length.

The assumption of identical waves is, of course, not very realistic [see, for instance, the wave plots by Koshi et al. (*10*)]. It is far more likely that wave periods, wave lengths, and amplitudes (in terms of speed, flow, and density) vary in some irregular pattern. This affects the preceding derivation primarily in that it is no longer sufficient to integrate over the period of a single wave, as in Equa-

tion 23; instead, if the traffic stream is assumed to exhibit long-term averages of speed and flow, the period of integration should be long enough to allow these averages to be approached. The fundamental relationship thus appears to hold approximately for nonidentical waves, provided traffic flow characteristics tend toward long-term averages and are averaged over sufficiently long periods.

Density Calculated from Measured Speed and Flow

Equation 31 gives a relationship between the expected value of density, as measured over an extended section of roadway, and density calculated as the ratio of flow to space mean speed. Because true density data are hard to obtain, one common use of the fundamental relationship has been to calculate density from measured speeds and flows (*13*). In a number of empirical studies of speed-density or flow-density relationships, the "density" data were actually the estimate $\hat{k}$ rather than measured densities.

It is clear such densities are not based on conditions measured over extended sections of roadway; rather, this type of density may more nearly represent the reciprocal of the average distance headway between successive vehicles in the vicinity of a point. This quantity will be referred to as "inverse-spacing density" and the symbol k_s used to designate it.

The inverse-spacing density k_s, unlike k, is a time-based variable. That is, it is measured over time at a point in space or, more literally, over a comparatively short distance. Consequently, relationships involving speed, flow, and inverse-spacing density avoid the difficulties that arise from combining time-based and distance-based variables. As a result, it is possible to derive relationships that hold for the measured values of the variables for particular samples, rather than just for expected values obtained in repeated samples.

The relationship between $\hat{k}$ and inverse-spacing density may be derived as follows. Let x_i be the distance that vehicle i has traveled from some point at the instant vehicle $i + 1$ reaches the point, and t_i be the time elapsed between the time vehicle i passes the point and the time vehicle $i + 1$ passes it. Time t_i for vehicle i is given by

$$t_i = \frac{x_i}{u_i} = x_i \Lambda_i \tag{32}$$

For a total of N vehicles passing the point, the average flow is defined as

$$\bar{q} = \frac{N}{\sum t_i} \tag{33}$$

Then the estimated density $\hat{k}$ is given by

$$\hat{k} = \bar{q}\bar{\Lambda} = \frac{N\bar{\Lambda}}{\sum x_i \Lambda_i} \tag{34}$$

Now, in a procedure similar to that used to derive Equation 20, let x_i be replaced by $\bar{x} + (x_i - \bar{x})$ and Λ_i be replaced by $\bar{\Lambda} + (\Lambda_i - \bar{\Lambda})$. Then

$$\begin{aligned}\hat{k} &= \frac{N\bar{\Lambda}}{\sum\left[\bar{x}+(x_i-\bar{x})\right]\left[\bar{\Lambda}+(\Lambda_i-\bar{\Lambda})\right]} \\ &= \frac{N\bar{\Lambda}}{\sum\left[\bar{x}\bar{\Lambda}+\bar{x}(\Lambda_i-\bar{\Lambda})+\bar{\Lambda}(x_i-\bar{x})+(x_i-\bar{x})(\Lambda_i-\bar{\Lambda})\right]} \\ &= \frac{N\bar{\Lambda}}{N\bar{x}\bar{\Lambda}+\bar{x}\sum(\Lambda_i-\bar{\Lambda})+\bar{\Lambda}\sum(x_i-\bar{x})+\sum(x_i-\bar{x})(\Lambda_i-\bar{\Lambda})}\end{aligned} \tag{35}$$

In this case, $\Sigma(\Lambda - \bar{\Lambda}_i) = 0$, $\Sigma(x_i - \bar{x}) = 0$, and $\sigma_{x\Lambda} = \Sigma[(x_i - \bar{x}) \times (\Lambda_i - \bar{\Lambda})]/N$, so Equation 35 may be rewritten as

$$\hat{k} = \frac{\bar{\Lambda}}{\bar{x}\bar{\Lambda} + \sigma_{x\Lambda}} \tag{36}$$

If $\sigma_{x\Lambda} = 0$,

$$\hat{k} = \frac{1}{\bar{x}} \tag{37}$$

This indicates that if there is no correlation between the vehicle spacing and the speed of the individual vehicles (so that the covariance of x and Λ is 0), the estimated density is indeed the reciprocal of the distance spacing of the vehicles in the vicinity of the point of measurement. In reality, however, vehicle spacing and the reciprocal of speed are expected to have a negative correlation, especially in congested flow; consequently, $\hat{k}$ tends to overestimate inverse-spacing density as well as density measured over a section.

This tendency is somewhat counteracted in cases in which the average speed used in the calculation is the time mean speed rather that the space mean speed. In that case

$$\hat{k} = \left(\frac{\bar{u}_s}{\bar{u}_t}\right)\frac{\bar{\Lambda}}{\bar{x}\bar{\Lambda} + \sigma_{x\Lambda}} \tag{38}$$

Since space mean speed is always less than time mean speed, the first term is always less than 1.0; however, the extent to which this counteracts the negative covariance term in the denominator is uncertain.

OCCUPANCY-BASED RELATIONSHIPS

The relationship among speed, flow, occupancy, and vehicle length given by Equation 2 most commonly has been used to calculate estimated speeds from flows and occupancies. For uniform traffic streams, Equation 2 (like Equation 1) can easily be derived by dimensional analysis. In his classical derivation of the speed-flow-occupancy relationship, Athol assumes a uniform vehicle length $\bar{L}$ and proceeds to show that under this assumption,

$$H = \frac{q\bar{L}}{\bar{u}_s} \tag{39}$$

or, in the more familiar form used in Equation 2,

$$\bar{u}_s = \frac{q\bar{L}}{H} \tag{40}$$

Hall and Persaud (*1*) question the validity of this relationship for the realistic case in which vehicle lengths vary within the traffic stream; they also present data that are incompatible with it, although the speeds in the data in question are time mean speeds rather than space mean speeds.

The effect of nonuniform vehicle lengths may be incorporated in the derivation of the speed-flow-occupancy relationship as follows.

Let the speed estimate calculated from flow and occupancy be $\hat{u}$. Then Equation 40 becomes

$$\hat{u} = \frac{q\bar{L}}{H} \tag{41}$$

where $\bar{L}$ now represents the average effective vehicle length, consisting of the sum of the detector length and the vehicle's electrical length, which is related, but not identical, to its physical length. It is assumed that the effective length of an individual vehicle L_i is independent of the speed of the vehicle as it passes the detector, although it may, of course, vary from vehicle to vehicle. Under this assumption τ_i, the time that vehicle i "occupies" the detector, is given by

$$\tau_i = \frac{L_i}{u_i} = L_i \Lambda_i \tag{42}$$

and occupancy, if measured over time interval T, by

$$H = \frac{\sum \tau_i}{T} = \frac{\sum L_i \Lambda_i}{T} \tag{43}$$

Flow, meanwhile, is defined as

$$q \equiv \frac{N}{T} \tag{44}$$

where N is the total number of vehicles passing the detector during time T. Equation 41 may now be written as

$$\hat{u} = \frac{N\bar{L}}{HT} = \frac{N\bar{L}}{\sum \tau_i} = \frac{N\bar{L}}{\sum L_i \Lambda_i} \tag{45}$$

By a derivation similar to that of Equation 36, the denominator of Equation 45 can be shown to equal $N(\bar{L}\bar{\Lambda} + \sigma_{L\Lambda})$, so that it may be rewritten as

$$\hat{u} = \frac{1}{\bar{\Lambda} + \frac{\sigma_{L\Lambda}}{\bar{L}}} \tag{46}$$

If effective vehicle lengths are not correlated with the reciprocal of vehicle speeds, the covariance term is 0 and Equation 46 reduces to

$$\hat{u} = \frac{1}{\bar{\Lambda}} = \bar{u}_s \tag{47}$$

In cases in which speeds and vehicle lengths are correlated, however, the speed estimate calculated from flow and occupancy is not the same as the space mean speed. If there is a correlation between L and Λ, it should be positive, since larger vehicles normally would be assumed to have smaller speeds and hence larger values of Λ. Consequently, $\hat{u}$ may be an underestimate of the space mean speed.

A rough idea of the bias resulting from the covariance term may be gained by assuming a traffic stream composed of two distinct types of vehicles, one large and slow and the other small and fast. By this means it can be shown that for conditions typical of noncongested urban rush hour traffic on relatively flat roads (small percentage of trucks, relatively small difference in speed between trucks and other vehicles) the bias should be small, but that on steep grades with substantial truck traffic it should be quite significant. For instance, for a traffic stream consisting of 95 percent passenger cars with effective lengths of 7 m and speeds of 85 km/hr and 5 percent trucks with effective lengths of 22 m and speeds of 70 km/hr, $\bar{u}_t$ = 84.25 km/hr, $\bar{u}_s$ = 84.09 km/hr, and $\hat{u}$ = 82.45 km/hr. On the other hand, for a traffic stream consisting of 80 percent passenger cars with effective lengths of 7 m and speeds of 85 km/hr and 20 percent trucks with effective lengths of 22 m and speeds of 40 km/hr, $\bar{u}_t$ = 76.00 km/hr, $\bar{u}_s$ = 69.39 km/hr, and $\hat{u}$ = 56.86 km/hr.

DENSITY-OCCUPANCY RELATIONSHIP

Another relationship of interest is that between density and occupancy. Athol (*6*) shows that if it is assumed that vehicles are of uniform length and that the fundamental relationship holds, this relationship is

$$H = k\bar{L} \tag{48}$$

If Athol's assumptions are relaxed, the relationship may be derived as follows for occupancy and point density. From Equation 43,

$$H = \frac{\sum L_i \Lambda_i}{T} \tag{49}$$

Meanwhile,

$$T = \sum \tau_i = \sum x_i \Lambda_i \tag{50}$$

Substituting Equation 50 into Equation 49,

$$H = \frac{\sum L_i \Lambda_i}{\sum x_i \Lambda_i} \tag{51}$$

By logic similar to that used in deriving Equations 36 and 46, it can be shown that

$$H = \frac{\bar{L}\,\bar{\Lambda} + \sigma_{L\Lambda}}{\bar{x}\bar{\Lambda} + \sigma_{x\Lambda}} = \frac{k_s(\bar{L}\,\bar{\Lambda} + \sigma_{L\Lambda}}{\bar{\Lambda} + k_s \sigma_{L\Lambda}} \tag{52}$$

If both $\sigma_{L\Lambda}$ and $\sigma_{x\Lambda}$ equal 0,

$$H = \frac{\bar{L}}{\bar{x}} = k_s \bar{L} \tag{53}$$

which is identical to Athol's result. As argued previously, if the covariances are not 0, $\sigma_{L\Lambda}$ should be positive and $\sigma_{x\Lambda}$ should be negative; consequently, where either covariance is not 0, H should be greater than $k_s\bar{L}$, and the relationship between H and k_s should be nonlinear.

Equation 52 gives the relationship between occupancy and inverse-spacing density. A more interesting comparison may be that between density estimate $\hat{k}$ and occupancy, since these have commonly been the concentration measures used in empirical studies of speed-concentration and flow-concentration relationships. By definition, the estimated density is $\hat{k} = q\bar{\Lambda}$. Meanwhile, combining Equations 41 and 46 leads to

$$\frac{q\bar{L}}{H} = \frac{1}{\bar{\Lambda} + \frac{\sigma_{L\Lambda}}{\bar{L}}} \tag{54}$$

Cross-multiplying,

$$H = q\bar{L}\left(\bar{\Lambda} + \frac{\sigma_{L\Lambda}}{\bar{L}}\right) = \hat{k}\bar{L} + q\sigma_{L\Lambda} \tag{55}$$

Since $\sigma_{L\Lambda}$ is assumed to be positive, H should also normally be greater than $\hat{k}\bar{L}$. Of the two covariances, however, $\sigma_{L\Lambda}$ is more likely to be negligible (except in certain obvious situations such as steep upgrades) than is $\sigma_{x\Lambda}$; hence, the relationship between H and $\hat{k}$ is more likely to be nearly linear than is that between H and k_s.

EMPIRICAL EVIDENCE

A few studies have attempted to verify some of the a priori relationships among speed, flow, and concentration. These include comparisons of occupancy with density by Koshi et al. (*10*) and Athol (*6*) and comparisons of measured speeds with speeds estimated from flows and occupancies by Hall and Persaud (*1*).

Koshi et al. compare occupancies with densities calculated from double-loop data from Tokyo expressways. These densities were thus (presumably) calculated as measured flow divided by measured average speed, which would result in what has been referred to here as $\hat{k}$, provided that the speed data were reduced as space mean speed. They found the relationship to be slightly nonlinear, with a negative second derivative of H with respect to k.

Athol compares occupancy with what he calls accumulation, which also turns out to be $\hat{k}$, and both occupancy and accumulation with what he calls aerial density (density defined over a section, k in the present notation; Athol calls this aerial density because it was measured from aerial photographs). Athol found good agreement between H and $\hat{k}$ and plotted a linear relationship between them, although the data could possibly indicate a nonlinear one. Plots of either H or $\hat{k}$ versus k were badly scattered, however, especially that of $\hat{k}$ versus k.

Hall and Persaud compare measured speed with speed calculated from flow and occupancy ($\hat{u}$ in the present notation) and find major discrepancies between the two, especially for very high and very low occupancies. In general, their data show that if reasonable values of $\bar{L}$ are assumed, $\hat{u}$ will significantly overestimate true speed at very low occupancies and underestimate most of the range representing congested flow.

The experiment actually performed by Hall and Persaud was to calculate a term g, defined as

$$g = \frac{q}{uH} \tag{56}$$

This term corresponds to $1/\bar{L}$ in Equation 41; however, since speeds in their data are measured in kilometers per hour and occupancies in percent, the equivalent average effective vehicle lengths in meters are given by $\bar{L} = 10/g$. In addition, the speeds used in the calculation were time mean rather than space mean speeds.

Hall and Persaud present plots of mean values of g versus H for various locations; Figure 2 is a reproduction of one of these plots. The plots indicate that mean values of g for the very lowest occupancies ranged from 1.8 to 2.4, with the average being about 2.2 for the four locations. Values of g for most of the rest of the uncongested regime are near 1.4 or 1.5, and those for the very highest occupancies (between 70 and 80 percent) range from about 0.3 to 0.6, with an average of perhaps 0.5. The values of g for the higher-volume uncongested regime correspond to a value of $\bar{L}$ of about

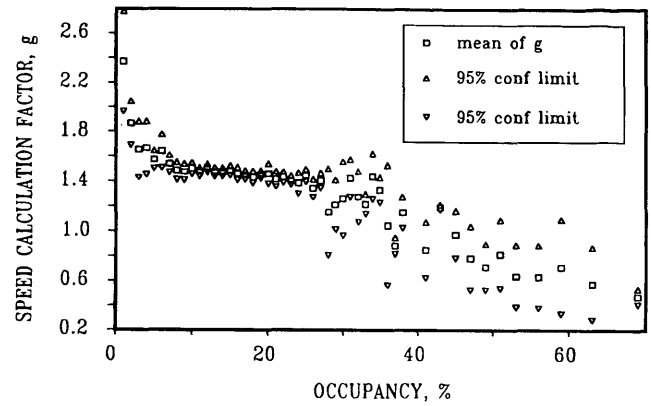

FIGURE 2 Reproduction of plot of mean values of *g* versus *H* (*1*, Figure 2).

7 m, which is credible; those for heavily congested flow, however, correspond to $\bar{L}$ of 20 m, which is not.

The overestimates of speed at very low occupancies are related to data reduction practices. The traffic management system in question reported occupancies in whole percentages and truncated these to the next lowest whole percent. This practice led to the large biases at very low flows (F. L. Hall, unpublished data).

For data taken at high occupancies, biases might result from either the covariance term or the difference between time mean and space mean speed. It is unlikely that the covariance term was very large, however, because the data were taken in the median lane at sites for which trucks were excluded from this lane. This was done to reduce the variance of both speeds and vehicle lengths.

The relationship between time mean and space mean speed was shown by Wardrop (*1*) to be

$$\bar{u}_t = \bar{u}_s + \frac{\sigma_s^2}{\bar{u}_s} \tag{57}$$

where

$$\sigma_s^2 = \sum_{i=1}^{N} \frac{\Lambda_i(\bar{u}_s - u_i)^2}{\sum \Lambda_i} \tag{58}$$

Substituting for σ_s^2 in Equation 57 and simplifying gives

$$\bar{u}_t = \bar{u}_s + \frac{1}{N}\sum \frac{(\bar{u}_s - u_i)^2}{u_i} \tag{59}$$

as a general relationship between the two. From Equation 59, it may be seen that the difference between space mean and time mean speed will increase as the dispersion of the speed distribution increases and that even a very small number of very low speeds could bias the relationship significantly. From this, it may be concluded that the bias will be greatest in heavily congested traffic, but it is difficult to tell how large it might be in any given case.

One past attempt to quantify the relationship between time mean and space mean speed is that of Drake et al. (*14*), who found the following relationship:

$$\bar{u}_s = -1.88960 + 1.02619\bar{u}_t \tag{60}$$

where speeds are in miles per hour. Drake goes on to comment that the maximum difference in the two averages is about 3 km/hr (1.9

mph) at zero speed, a conclusion that proceeds directly from the regression equation. This is certainly much smaller than the bias reported by Hall and Persaud, although, since the relationship is not linear, Drake may have understated the bias at very low speeds. In any case, the discrepancies found by Hall and Persaud appear larger than can be accounted for by the known biases. This raises the possibility that there may have also been counting or data reduction errors under congested conditions, although it is not clear what these might have been.

CONCLUSIONS

Previous derivations of a priori relationships among speed, flow, and concentration variables have assumed that certain features of the traffic stream are uniform. In the case of relationships among speed, flow, and density, either speeds or vehicle spacings (sometimes both) have been assumed to be uniform. It has been shown that, for the fundamental relationship, these assumptions can be relaxed provided that the relationship is understood in probabilistic terms. For relationships involving only time-based terms, these assumptions can also be relaxed in deterministic cases.

Where the order of arrival of vehicles with particular speeds is random, it has been shown that the fundamental relationship applies to average values of the variables in question. Where there are cyclic variations in speeds and flows resulting from waves in congested flow, the expected relationship among speed, flow, and density involves the covariance of flow and the reciprocal of speed and should vary depending on the relationship between the length of the wave and of the section over which density is measured. For the relationship among speed, flow, and what has been called inverse-spacing density (the reciprocal of the average distance separations of vehicles in the vicinity of a point), relaxation of the assumptions of uniform speed or uniform spacing leads to a relationship involving the covariance of vehicle spacing and the reciprocal of speed. For the relationship among speed, flow, vehicle length, and occupancy, relaxation of the assumption of uniform vehicle length leads to a relationship involving the covariance of vehicle length and the reciprocal of speed. It has further been shown that the relationship between inverse-spacing density and occupancy contains both of these covariance terms, but that the relationship between occupancy and density estimated from flow and measured space mean speed contains only the covariance of vehicle length and the reciprocal of speed.

These findings imply that simple relationships among speed, flow, and concentration variables hold a priori not only in cases in which particular variables are uniform (which is almost always unrealistic), but also in cases in which certain variables are not significantly correlated with one another. For the relationship among speed, flow, and occupancy, this is an important advantage, since the covariance of vehicle length and the reciprocal of speed is not likely to be of practical significance except in certain easily identified circumstances such as steep grades with considerable truck traffic. For relationships including inverse-spacing density, the covariance term involving spacing and the reciprocal of speed is likely to be fairly large, especially in congested traffic. This need not be a major problem, however, since inverse-spacing density is not a very useful measure otherwise, and it can be shown that density estimated from flow and measured speed should agree closely with occupancy, except in cases in which speeds and vehicle lengths are strongly correlated.

Comparison of the relationships derived here with empirical studies of the relationship between occupancy and density estimated from flow and measured speed indicates good agreement. In the case of the empirical data in Hall and Persaud's study of speeds calculated from flows and occupancies, on the other hand, there are large discrepancies, both in cases in which occupancies were very low and in most of the congested-flow regime, where they were relatively high. Those involving very low occupancies were due to data reduction techniques. Those involving data from the congested-flow regime may to some extent be due to the difference between time mean and space mean speed or to the covariance term identified here; however, the magnitude of the discrepancy appears to be too large to be explained by the combined effects of these two sources of bias. This situation raises the possibility that there may have also been counting or data reduction errors under congested conditions, although it is not clear what these might have been.

The a priori relationships examined here have commonly been used for calculating speed estimates by traffic management systems and as a basis for studying empirical relationships among traffic flow characteristics. Given the nature of the relationships for nonuniform flow, it appears that the use of the flow-speed-occupancy relationship to estimate speeds and transform variables in empirical studies should be valid in all but a few cases—provided that flows and occupancies are measured accurately. This is certainly true for uncongested conditions. For heavily congested conditions, it should also be true provided speeds are reduced consistently as space mean and the correlation between vehicle lengths and speeds is small. Given the results of Hall and Persaud's study, however, the accuracy of the measurements should not be taken for granted.

ACKNOWLEDGMENT

The author would like to thank Fred Hall for encouraging him to pursue this topic. It was originally suggested by his published work questioning the validity of the fundamental relationship. In addition to the contribution made by Hall's published work, several points addressed in this paper were originally suggested by Hall either in personal communications or in drafts of work that is not yet published.

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DISCUSSION

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This paper is based on the premise that the equation $q = vk$ is valid only for a limited range of flow conditions such as stationary conditions where all families of vehicle trajectories are parallel, equidistant, and of constant speed. Yet, Edie (*1,2*) provided definitions of flow, speed, and density so as to guarantee the validity of this equation for all traffic conditions. To resolve this apparent contradiction, we apply Edie's definitions in a manner consistent with earlier experiments performed by Hall and Persaud (*3*), which served as motivation for the present paper.

We visualize on a time-space plane a rectangular region A of spatial dimension L, the distance separating paired loop detectors (6 m), and temporal duration T, the count interval (30 sec). An instant at time t within rectangular region A is itself a region (i.e., a "slice") of spatial dimension L and elemental time duration dt. Density at time t is conventionally defined as n/L, the number of vehicles within region A at time t divided by the segment length. Equivalently, density can be expressed as $(n \cdot dt)/(L \cdot dt)$, the ratio of the total time spent by all vehicles in the slice corresponding to time t to the "area" of the slice. As our original rectangular region A is composed of elementary slices, it makes sense to define density in region A as $t(A)/(L \cdot T)$, where the numerator is the total time spent by all vehicles in A. This is Edie's generalized definition of density. Given that region A is a rectangle composed of elementary slices of fixed "area" $L \cdot dt$, this generalized definition is simply the average density over time.

We exchange the roles of space and time and next visualize a single point at location x within rectangular region A. This point defines a region of duration T and elemental spatial dimension dx so that flow can be expressed as $(n \cdot dx/T \cdot dx)$, the total distance traveled by all vehicles crossing point x divided by the "area" of the slice corresponding to this point. As region A is composed of elementary slices of "area" $T \cdot dx$, it makes sense to define flow in A as $d(A)/(L \cdot T)$, where the numerator is the distance traveled by all vehicles in A. Again, this is Edie's generalized definition of flow; it reduces to the conventional definition when A is taken to be a slice of spatial dimension dx and temporal duration T.

As we have now defined these measures, dividing flow by density results in $d(A)/t(A)$, which can be taken to be a definition of an average velocity in region A. This was proposed by Edie, and with his definitions, the equation $q = vk$ is always valid.

The expression evaluated by Hall and Persaud (*3*) is

$$\text{occupancy}/\bar{\ell}_r = \text{flow}/\bar{v}$$

where $\bar{\ell}_r$ is mean effective vehicle length and $\bar{v}$ is mean speed. This can likewise be shown to be true by definition provided measurements of $\bar{v}$ are the generalized ones because the left-hand side can be shown to be Edie's generalized definition of density and the conventional definition of flow appearing in the right-hand side coincides with the generalized one. [When correlations exist between vehicle length and speed, $\bar{\ell}_e$ must be an average value occurring over space. When traffic is not stationary, occupancy measured at a point in space over time (i.e., by a detector) is not equivalent to a spatial measure of road occupancy. We anticipate addressing these issues at some future date.] Reported discrepancies can be explained by methods used for averaging observations. Edie's definition of average speed can be computed using the arithmetic mean of trip times between two points (e.g., paired detectors) or as the harmonic mean speed of vehicles passing a single point (e.g., detector) when conditions are stationary. There is no reason to expect a $\bar{v}$ calculated in a different manner to satisfy the relation.

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AUTHOR'S CLOSURE

I wish to thank Mr. Cassidy for calling attention to work by Edie, which I probably should have mentioned. It is, however, just one more in a long series of efforts to make the fundamental relationship work by imposing special conditions or (in this case) by adopting unnatural definitions of the variables. Strictly speaking, none of Edie's variables can actually be measured, although they can be fairly closely approximated under the conditions Cassidy outlines. Certainly, they are not the conventional definitions of the variables in question. The thrust of my paper was to acknowledge (and in some cases elaborate on) these special cases while at the same time determining the nature of the discrepancies that result when conventional definitions of the variables are used.

In the case of Hall and Persaud, the special case proof is interesting, but it does not address the practical concern underlying their work. The occupancies they were concerned with were measured over time, and the traffic flow was not stationary. Also, given that the flows were not stationary, none of the possible measures of speed available to them really conformed to Edie's definition. Given that situation, I believe that it makes sense to ask whether the biases introduced by the nonstationary traffic stream account for the discrepancies they observed. I am not so sure that it makes sense to try to define a relationship that is a priori true but involves variables that are unlikely ever to be measured in practice.

DISCUSSION

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The paper deals with the fundamental traffic flow relationships and the problems that basically local measurement techniques cause in the analysis of these relationships with some sectionwide variables.

I have three main comments on the paper. The first is that it is hard for me to understand why the writer, who clearly wants to challenge some basic traffic flow theory paradigms, fails to give precise definitions for his variables in each case of analysis. This vagueness, I believe, is also the reason for some errors that can be found in the equations that he derives. In addition to that, the vague definitions can cause misinterpretations of correct results.

My second comment is that the writer does not at all describe the most general definitions of the three basic traffic flow variables (*1,2*), which guarantee that the fundamental flow relationship is valid for any kind of traffic (congested, uncongested, random, uniform, etc.) in any time-space domain *XT* (*X* being the space axis and *T* being the time axis).

These definitions are as follows:

- Traffic flow q = the amount of vehicle-kilometers of travel (S) in the domain divided by the area of the domain, that is,

$$q = \frac{S}{X \times T}$$

- Traffic density k = total vehicle-hours of travel (T_{Tot}) in the domain divided by the area of the domain, that is,

$$k = \frac{T_{\text{Tot}}}{X \times T}$$

- Space mean speed of traffic u = total vehicle-kilometers of travel in the domain divided by the total vehicle-hours of travel in the domain, that is,

$$u = \frac{S}{T_{\text{Tot}}}$$

From these definitions it can easily be seen that the fundamental flow relationship $q = u \times k$ is valid for the time-space domain in question. The derivation of these relationships is given for example by Leutzbach (*2*).

So, in theory, traffic flow variables can be measured in a way that is in accordance with the fundamental flow relationship. The problems arise from our inability to measure the variables simultaneously in time and space.

My third comment is related to Equations 20 through 31 in the paper. In this part of the paper the writer develops equations for a time-space domain *TX* in a cyclic flow situation. His derivations leading to Equation 24 are correct. This equation gives the definition of traffic density averaged over the time-space domain. It can easily be seen that his result is in accordance with the above given general definition, that traffic density equals the amount of vehicle hours in the time-space domain divided by the area of the domain. In Equation 24 only division by time is needed, because the derivation of the equation already averaged the value over the space axis.

In Equations 25 and 26 the writer makes some basically correct mathematical manipulations of Equation 24 to develop it further. But then he makes a major error in the definition of space mean speed, given in the form of the reciprocal of speed, in Equation 28. According to that equation the mean travel time in the domain is the average value of travel time $\Lambda(t)$ over the time axis without consideration of the number of vehicles traveling within the travel time in question. The error can be seen from the following equations, the first being the one given in the paper and the second the correct one (in two equivalent forms).

$$\int_0^T [\Lambda(t) - \overline{\Lambda}]dt = 0 \qquad \text{(Equation 28)}$$

$$\overline{\Lambda} = \frac{\int_0^T q(t)\Lambda(t)dt}{\int_0^T q(t)dt} \text{ or } \frac{1}{T}\int_0^T [q(t)\Lambda(t) - \overline{q}\overline{\Lambda}]dt = 0 \qquad \text{(correct form)}$$

When Equation 28 is replaced with the correct one, the calculations based on Equation 26 result in a simple identity (i.e., Equation 25).

On the basis of Equations 25 and 27 and the correct equation for mean travel time given above, one can easily see that the fundamental flow relationship holds for this situation and no correction term is needed in the calculation of density.

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AUTHOR'S CLOSURE

I wish to thank Pursula for pointing out the mistake in Equation 28 and calling attention to work by Edie and Leutzbach, which I probably should have mentioned. In the case of Equation 28, his version is the correct one, and the consequence is indeed that the correction term disappears. This leads to the somewhat more satisfying conclusion that the fundamental relationship holds without bias for cyclic flow, regardless of the relationship between the wavelength and the distance over which density is measured.

The work by Edie and Leutzbach is of historical interest and should have been cited. It is, however, just one more in a long series of efforts to make the fundamental relationship work by imposing special conditions or (in this case) by adopting unnatural definitions of the variables. Strictly speaking, none of Edie's variables can be measured. Certainly, they are not the conventional definitions of the variables in question. The thrust of my paper was to acknowledge (and in some cases elaborate on) these special cases while at the same time determining the nature of the discrepancies that result when conventional definitions of the variables are used.

I am somewhat puzzled by the assertion that my definitions of the variables are vague. In most cases the exact mathematical meaning is stated. It is true that in the "special case" formulations there are variations in the precise definitions from one formulation to another, but these are a result of attempts to make the fundamental relationship work. Finally, I must deny that it was my intent to "challenge some basic traffic flow theory paradigms." On the contrary, they had already been challenged, most notably by Hall, and my intent was to try to limit the uncertainty by determining, where possible, the nature of any biases.