



THE UNIVERSITY OF TEXAS AT AUSTIN  
**CENTER FOR TRANSPORTATION RESEARCH**

# A Spatial Multiple Discrete-Continuous Model

**Chandra R. Bhat<sup>1,2</sup> and Sebastian Astroza<sup>1,3</sup>**

1: The University of Texas at Austin

2: The Hong Kong Polytechnic University

3: Universidad de Concepción

# Outline

- ❑ **Introduction**
- ❑ **Research objectives**
- ❑ **Methodology**
  - The multivariate skew normal distribution
  - The aspatial skew MDC model
  - The spatial skew MDC model
- ❑ **Simulation exercises**
- ❑ **Empirical application**
  - Land use change models
  - Estimation results
  - Policy implications
- ❑ **Conclusions**



THE UNIVERSITY OF TEXAS AT AUSTIN  
**CENTER FOR TRANSPORTATION RESEARCH**

# Introduction

# MDC models

## ❑ Multiple discrete-continuous (MDC) choice:

Consumers choose an alternative from a set and then determine the amount of the chosen alternative to consume.

## ❑ Classical discrete choice models

- alternatives are mutually exclusive
- only one alternative can be chosen

## ❑ Multiple discrete-continuous (MDC) models

- allow consumers to choose multiple alternatives at the same time,
- along with the continuous dimension of the amount of consumption

# MDC applications



Consumer brand choice and purchase quantity



Activity participation and time allocation



Household vehicle types and usage



Recreational destination choice and number of trips



Land-use type and intensity



Stock portfolio selection choice and investment amounts

# MDC model formulation

- ❑ Based on a **utility maximization framework**
  - non-linear (but increasing and continuously differentiable) utility function
  - relationship between the decreasing marginal utility (satiation) and the increasing investment in an alternative.
  - budget constraint
  - **Karush-Kuhn-Tucker (KKT) first-order conditions** → Optimal consumption quantities
  
- ❑ Bhat (2008): very general utility form for this KKT approach
  - **Stochasticity** in the **baseline preference for each alternative** (unobserved factors)
  
- ❑ The **most common distributions** used for the **kernel stochastic error term**:
  - The generalized extreme value (GEV) distribution → MDC GEV model structure
  - The multivariate normal distribution → **MDC probit (MDCP)** model structure

# Evolution of MDC models

- ❑ Inclusion of **unobserved heterogeneity** in the coefficients of the exogenous variables.
  - Usually, response coefficients are assumed to be distributed in a multivariate normal fashion, as is the vector of alternative kernel error terms.
  - **Normality assumptions** can lead to severe misspecifications when non-normality is in play.

# Evolution of MDC models

- ❑ Inclusion of **spatial dependence** among observation units:
  - The dependent variable at one point in space is explicitly influenced by observed covariates and/or by unobserved factors at another point in space.
  - Only two studies (Kaza et al., 2012 and Bhat et al., 2015), to our knowledge, have included spatial dependency within the MDC framework.





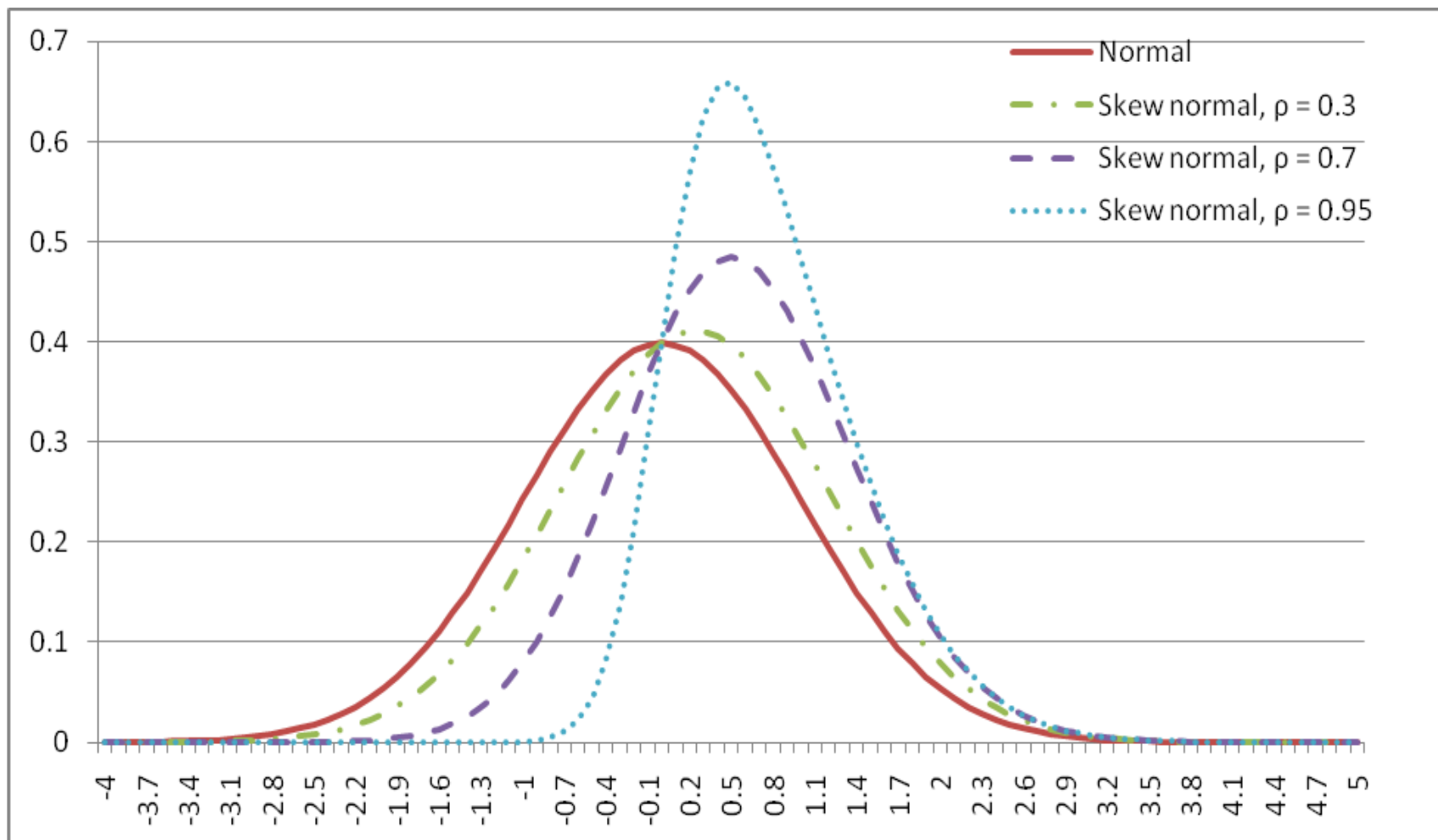
THE UNIVERSITY OF TEXAS AT AUSTIN  
**CENTER FOR TRANSPORTATION RESEARCH**

# Research Objectives

# The current research

- ❑ New **spatial MDC model** with skew-normal kernel error terms and skew-normal distributed random response coefficients.
- ❑ The **multivariate skew-normal (MVSN) distribution**:
  - Is a flexible distribution which has the multivariate normal as a particular case.
  - Does not require the analyst to make any a-priori assumptions
  - Makes the incorporation of spatial dependency somewhat “easier”

## Shape of the SSN density function for a number of positive values of $\rho$



# The current research

- ❑ Proposed estimation method using Bhat's (2011) maximum approximate composite marginal likelihood (MACML) inference approach.
- ❑ **Simulation exercises** are undertaken to examine the ability of proposed method to recover parameters from finite samples.
- ❑ Empirical demonstration:
  - **Land-use-change decisions** using the city of Austin's parcel-level land-use data.
  - Model land-use in multiple discrete states
  - Along with the area invested in each land-use discrete state, within each spatial unit in an entire urban region



THE UNIVERSITY OF TEXAS AT AUSTIN  
**CENTER FOR TRANSPORTATION RESEARCH**

# Simulation Exercises

# Simulation Plan

- ☐ 3 alternatives, 4 coefficients: 1 fixed, 3 random
- ☐ Two sets of spatial auto-correlation parameters (high and low correlation)
- ☐ Two sets of skew parameters (high and low skewness)
- ☐ Gamma profile
- ☐ 750 observations, 30 datasets with 10 permutations (a total of 300 runs)
- ☐ 50x15 rectangular grid
- ☐ Proximity: inverse of distance
- ☐ Comparison with additional restrictive models.

Effects of ignoring spatial autocorrelation and skewness when present (for the high spatial dependence and high skewness case)

Parameters	True Value	SSN-MDC		ASN-MDC		S-MDCP		A-MDCP	
		Mean Est.	Absolute Percentage Bias (APB)	Mean Est.	Absolute Percentage Bias (APB)	Mean Est.	Absolute Percentage Bias (APB)	Mean Est.	Absolute Percentage Bias (APB)
$b_1$	0.50	0.472	5.60%	0.514	2.80%	0.452	9.60%	0.451	19.48%
$b_2$	1.00	1.037	3.70%	0.993	0.70%	1.092	9.20%	1.003	0.62%
$b_3$	-1.00	-0.960	4.00%	-0.997	0.30%	-0.872	12.80%	-0.912	17.64%
$b_4$	0.80	0.769	3.88%	0.756	5.50%	0.816	2.00%	0.703	24.31%
$\gamma_1$	1.00	0.959	4.10%	0.893	10.70%	0.911	8.90%	0.974	5.18%
$\gamma_2$	1.00	0.970	3.00%	1.001	0.10%	1.016	1.60%	0.999	0.23%
$\omega_1$	1.00	1.041	4.10%	1.122	12.20%	0.993	0.70%	1.200	40.05%
$\omega_2$	1.00	0.951	4.90%	0.991	0.90%	0.885	11.50%	0.799	40.29%
$\omega_3$	1.25	1.188	4.96%	1.178	5.76%	1.121	10.32%	1.142	17.34%
$\delta_1$	0.70	0.676	3.43%	—	—	0.727	3.86%	—	—
$\delta_2$	0.80	0.774	3.25%	—	—	0.846	5.75%	—	—
$\rho_1$	0.70	0.671	4.14%	0.638	8.86%	—	—	—	—
$\rho_2$	0.70	0.729	4.14%	0.718	2.57%	—	—	—	—
$\rho_3$	0.70	0.740	5.71%	0.800	14.29%	—	—	—	—
Overall mean value across parameters		0.716	3.64%	0.717	5.39%	0.726	6.93%	0.707	18.35%
Mean composite log-likelihood value at convergence				-94,720.06		-96,995.86		-99,574.42	
Number of times the adjusted composite likelihood ratio test (ADCLRT) statistic favors the SSN-MDC model		NA		All thirty times when compared with $\chi^2_{2,0.99} = 9.21$ value (mean ADCLRT statistic is 18.5)		All thirty times when compared with $\chi^2_{3,0.99} = 11.34$ value (mean ADCLRT statistic is 39.1)		All thirty times when compared with $\chi^2_{5,0.99} = 15.09$ value (mean ADCLRT statistic is 41.7)	

SSN-MDC: Spatial Skew Normal MDC, ASN-MDC: Aspatial Skew Normal MDC, S-MDCP: Spatial MDCP, A-MDCP: Aspatial MDCP



THE UNIVERSITY OF TEXAS AT AUSTIN  
**CENTER FOR TRANSPORTATION RESEARCH**

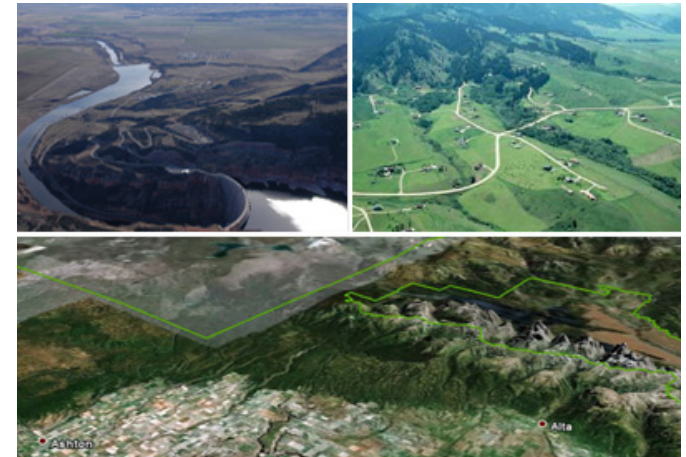
# Empirical Application



# Land-use modeling

❑ Land-use models are used in many fields:

- Planning,
- Urban science,
- Ecological science,
- Climate science,
- Watershed hydrology,
- Environmental science,
- Political science, and
- Transportation



# Land-use modeling

- ❑ Used to examine future land-use scenarios
- ❑ Evaluate potential effects of policies
- ❑ Recently, substantial attention on
  - Biodiversity loss,
  - Deforestation consequences, and
  - Carbon emissions increases caused by land-use development
- ❑ Land-use patterns constitute one of the most important “habitat” elements characterizing Earth’s terrestrial and aquatic ecosystem

# Earlier Literature

- Three Modeling Approaches
  - Pattern-based Models
  - Process-based Models
  - Spatial-based Models

# Pattern Based Models

- Developed by geographers and natural scientists
- Well suited for land-use modeling over relatively large geographic extents (such as urban regions or entire states or even countries)
- Unit of analysis: Aggregated Spatial Unit (Large grid, TAZ, Census Tract, County or State)
- Two types
  - Cellular automata-based Models
  - Empirical models at aggregated spatial unit level

# Cellular Automata-based Models

- Hypothesizes the nature of the deterministic or probabilistic updating functions
- Simulates the states of cells over many “virtual” time periods,
- Aggregates up the states of the cells at the end to obtain land-use patterns
- Limitations
  - Updating functions not based on actual data → no direct evidence linking the updating mechanism at the cell level to the spatial evolution of land-use patterns at the aggregate spatial unit level
  - Do not use exogenous variables such as socio-demographic characteristics of spatial units, transportation network features, and other environmental features → Policy value is extremely limited

# Empirical models at Aggregated Spatial Unit Level

- Relates transportation network, pedoclimatic, biophysical and accessibility variables to land-use patterns
- Can be used in a simulation setting to predict land-use patterns in response to different exogenously imposed policy scenarios
- Not formulated in a manner that appropriately recognizes the multiple discrete-continuous nature of land-use patterns in the aggregated spatial units
- Do not adequately consider population characteristics of spatial units in explaining land-use patterns within that unit

# Process-based Models

- Developed by economists
- Well suited for modeling landowners' decisions of land-use type choice for their parcels
- Unit of analysis: Land-owner is considered as an economic agent
- Considers the human element in land-use modeling
- Forward-looking inter-temporal land use decisions based on profit-maximizing behavior

- Difficulties incorporating spatial considerations at this micro-level
- High data and computing demands when analysis is being conducted at the level of entire urban regions or states
- Presence of land-use and zoning regulations → Individual landowners may not have carte blanche authority
- Multiple parcels under the purview of a single decision-making agent → Multiple parcels in close proximity tend to get similarly developed



# Spatial-based models

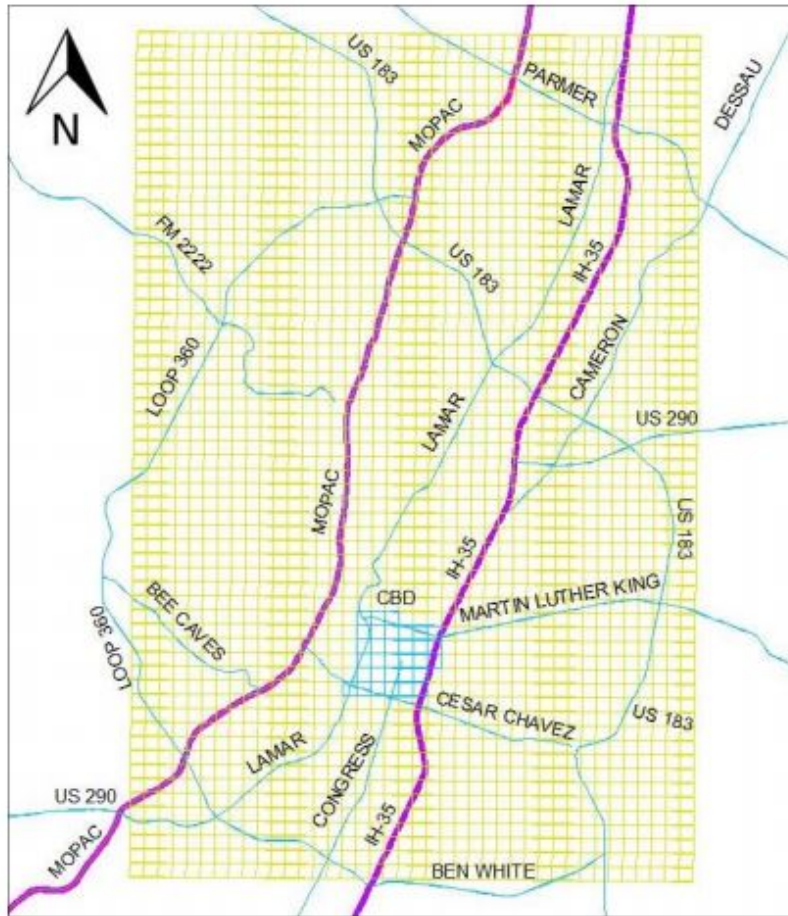
- Emphasis on spatial dependence among spatial units (in pattern-based models) or among landowners (in process-based models)
- Caused by diffusion effects, or zoning and land-use regulation effects, or social interaction effects, or observed and unobserved location-related influences
- Dominant formulations → Spatial lag and spatial
- Spatial lag structure
  - Considers spillover effects caused by exogenous variables
  - Generates spatial heteroscedasticity.

- Spatial heterogeneity → Differences in relationships between the dependent variable and the independent variables across decision-makers or spatial units in a study region
- Essential to accommodate local variations (*i.e.*, recognize spatial non-stationarity) in the relationship across a study region rather than settle for a single global relationship

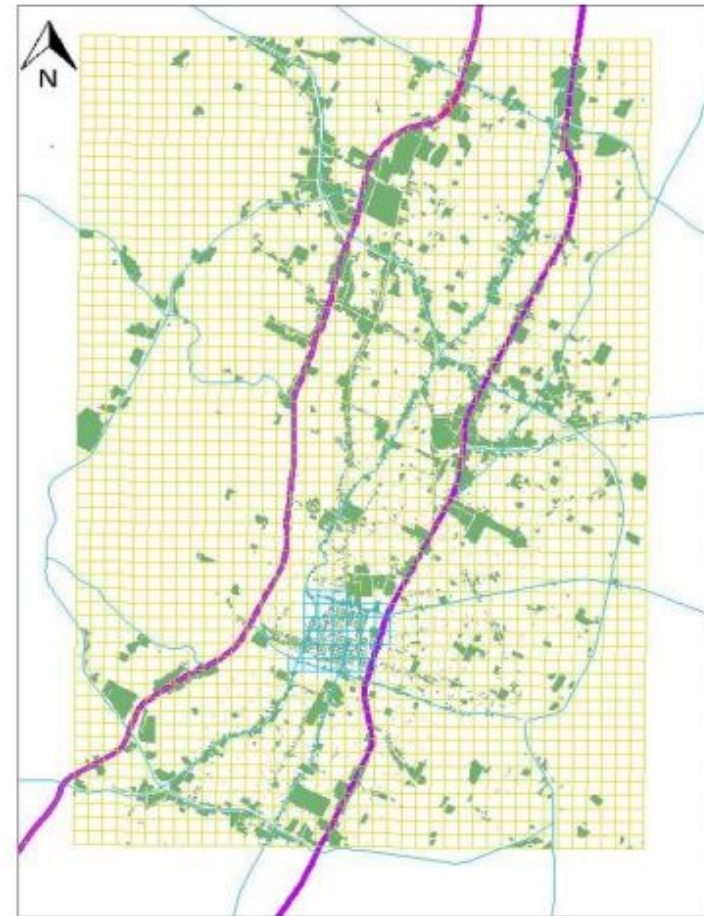
# Data description

- ❑ Parcel level land-use inventory data for City of Austin, Texas, year 2010.
- ❑ Land-use types were aggregated into:
  - commercial,
  - industrial,
  - residential and
  - undeveloped (outside alternative).
- ❑ Size of analysis area : 145.91 square miles.
- ❑ Size of analysis grid : 0.25 X 0.25 miles.
- ❑ **Explanatory Variables:** Road access measures (distance to highways and thoroughfares), distance to nearest school and hospital, fraction of area under floodplain, average elevation of the grid.
- ❑ **Four models compared:** A-MDCP, ASN-MDC, S-MDCP, and SSN-MDC

# Area description

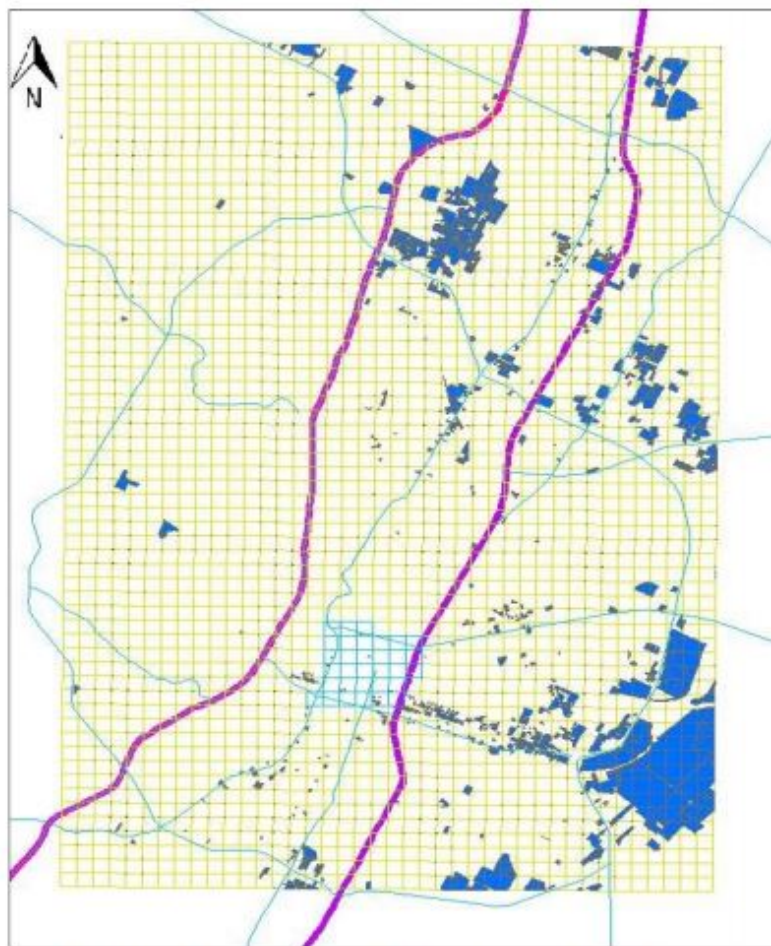


**FIGURE 1a: Highways, thoroughfares, and CBD location in the analysis area**

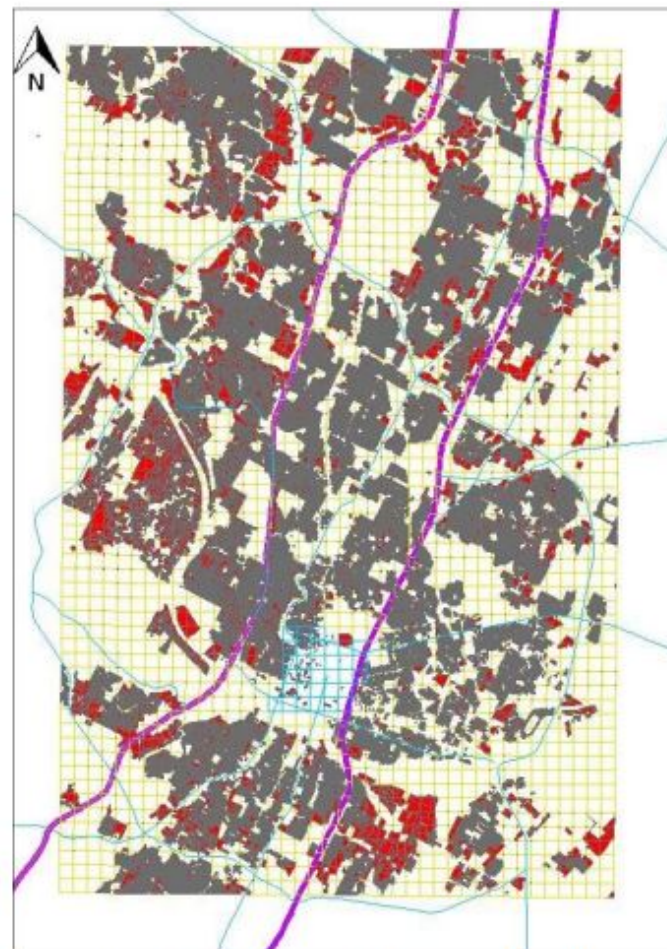


**FIGURE 1b: Commercial land-use distribution**





**FIGURE 1c: Industrial land-use distribution**



**FIGURE 1d: Residential land-use distribution**

# Estimation results of the SSN-MDC model

Variables	Land-use alternatives (base is Undeveloped)					
	Commercial		Industrial		Residential	
	Estimate	t-stat	Estimate	t-stat	Estimate	t-stat
<i>Baseline utility parameters</i>						
Alternative specific constant	-0.388	-1.11	1.273	2.41	-1.812	-1.87
Skew parameter	-0.120	-6.17	-0.187	-5.50	-0.436	-4.45
Distance to MoPac (miles)	-0.072	-4.23	0.165	2.76	-0.068	-4.03
Distance to IH-35 (miles)	-0.120	-4.16	-0.374	-4.99	0.085	3.74
Skew parameter					-0.242	-6.72
Distance to US-183 (miles)	— <sup>+</sup>	—	-0.257	-6.15	—	—
Distance to nearest thoroughfare (miles)	-0.398	-2.31	-1.276	-2.96	0.276	4.15
Skew parameter			0.166	8.12		
Distance to school (miles)	-0.215	-3.78	0.540	3.14	-0.462	-6.81
Distance to hospital (miles)	-0.261	-5.80	0.198	2.84	0.041	1.78
Fraction of grid area under floodplain	-0.018	-8.16	-0.025	-4.76	-0.012	-8.64
Distance to nearest thoroughfare/Distance to floodplain	-0.411	-7.99	-0.396	-3.35	0.107	4.63
Skew parameter	0.284	5.49	0.0	(fixed*)	0.0	(fixed*)
High elevation indicator	-0.272	-5.16	-1.326	-6.09	0.217	3.55
Skew parameter	0.0	(fixed*)				
CBD indicator	—	—	-0.968	-2.73	-0.813	-5.00
<i>Satiation parameters</i>						
	8.750	17.42	3.497	9.63	39.62	12.41
<i>Spatial lag parameters</i>						
	0.297	2.18	0.613	2.05	0.460	3.39

+ A “—” entry in the table indicate that the variable is not statistically significant

\* Fixed because the parameter was not significantly different from zero at not even a 20% level of confidence

**Note: Skew parameters are presented only for those coefficients that are considered random.**

# Estimation results of the SSN-MDC model

		Alternative specific constant			Distance to IH-35	Distance to nearest thoroughfare	Distance to nearest thoroughfare/Distance to floodplain			High elevation indicator
		Commercial	Industrial	Residential	Residential	Industrial	Commercial	Industrial	Residential	Commercial
Alternative specific constant	Commercial	1.00 (fixed)	1.424 (4.12)	0.266 (2.53)	0.00*	0.00*	0.00*	0.00*	0.00*	0.00*
	Industrial		4.175 (4.92)	0.227 (3.11)	0.00*	0.00*	0.00*	0.00*	0.00*	0.00*
	Residential			0.624 (5.02)	0.00*	0.00*	0.00*	0.00*	0.00*	0.00*
Distance to IH-35	Residential				0.278 (4.27)	0.00*	0.00*	0.00*	0.00*	0.00*
Distance to nearest thoroughfare	Industrial					2.172 (3.11)	0.00*	0.00*	0.00*	0.00*
Distance to nearest thoroughfare/Distance to floodplain	Commercial						0.252 (2.89)	0.00*	0.00*	0.00*
	Industrial							0.311 (2.84)	0.00*	0.00*
	Residential								0.186 (4.10)	0.00*
High elevation indicator	Commercial									1.106 (5.18)

\* Fixed because the parameter was not significantly different from zero at not even a 20% level of confidence

# Measures of fit

Summary Statistic	Model			
	SSN-MDC <sup>1</sup>	ASN-MDC <sup>2</sup>	S-MDCP <sup>3</sup>	A-MDCP <sup>4</sup>
Number of observations	2,383			
Composite log-likelihood at convergence of the naïve model	-138,587.10			
Predictive log-likelihood at convergence	-76,239.86	-76,242.97	-76,255.32	-76,276.11
Number of parameters	49	46	43	40
Adjusted composite likelihood ratio test (ADCLRT) between SSN-MDC model and the corresponding model (at any reasonable level of significance)	Not applicable	$[-2*(LL_{SSN-MDC} - LL_{ASN-MDC})]=6.9 >$ Chi-Squared statistics with 3 degrees of freedom	$[-2*(LL_{SSN-MDC} - LL_{S-MDCP})]=30.1 >$ Chi-Squared statistics with 6 degrees of freedom	$[-2*(LL_{SSN-MDC} - LL_{A-MDCP})]=79.2 >$ Chi-Squared statistics with 9 degrees of freedom



# Measures of fit – predicted shares

Summary Statistic			Model							
			SSN-MDC		ASN-MDC		S-MDCP		A-MDCP	
Percentage of grids predicted to invest in...	Actual sample Grids (%)	Average investment (sq mi)	Predicted %	Predicted average Investment	Predicted %	Predicted average Investment	Predicted %	Predicted average Investment	Predicted %	Predicted average Investment
Commercial	54.7	0.0136	58.4	0.0142	59.7	0.0153	60.1	0.0157	62.8	0.0159
Industrial	24.3	0.0134	28.0	0.0148	29.0	0.0161	29.4	0.0162	31.2	0.0169
Residential	82.0	0.0267	78.4	0.0234	76.9	0.0227	76.2	0.0220	75.4	0.0217
Mean absolute percentage error (MAPE)			8.9	9.3	11.6	16.5	12.6	18.3	16.9	21.0
Percentage of grids predicted to invest in...	Actual sample		Predicted percentage	Predicted percentage	Predicted percentage	Predicted percentage	Predicted percentage	Predicted percentage	Predicted percentage	Predicted percentage
	Grids (%)									
Commercial but not Residential	8.2		8.1	8.0	7.9	7.9				
Residential but not Commercial	37.7		36.7	36.0	36.0	35.7				
Both Commercial and Residential	51.7		51.5	51.2	51.0	50.8				
Neither Commercial nor Residential	2.4		4.2	4.8	5.1	5.5				
Mean absolute percentage error (MAPE)			19.8	26.3	30.4	36.3				

**SSN-MDC:** Spatial Skew Normal MDC, **ASN-MDC:** Aspatial Skew Normal MDC, **S-MDCP:** Spatial MDCP, **A-MDCP:** Aspatial MDCP

# Elasticity analysis (example)

- Commercial area is distributed mostly along MoPac and thoroughfares than IH-35 → Should observe higher elasticity value for MoPac and thoroughfares than IH-35.

Scenario	Commercial land-use			
	SSN-MDC	ASN-MDC	S-MDCP	A-MDCP
A 25% increase in distance to MoPac	-11.24 (1.97)	-5.03 (2.00)	-9.87 (1.99)	-4.81 (1.92)
A 25% increase in distance to IH35	-2.12 (2.58)	-4.75 (3.27)	-1.15 (5.47)	-6.80 (4.17)
A 25% increase in distance to nearest thoroughfare	-3.87 (2.66)	-2.67 (4.40)	-3.09 (5.44)	-2.15 (0.38)

- Elasticity effects can be misleading, if spatial interactions or non-normality are neglected.



THE UNIVERSITY OF TEXAS AT AUSTIN  
**CENTER FOR TRANSPORTATION RESEARCH**

# Conclusions

# Conclusions

- ❑ This paper has proposed a **new spatial skew-normal multiple discrete-continuous (or SSN-MDC) model** and an associated estimation method
- ❑ First time (to our knowledge) that a flexible skew-normal distribution for the kernel error term and/or random response coefficients has been used in both spatial- and aspatial-MDC models.
- ❑ Modeling framework can be applied to any MDC context that needs to consider spatial issues and a-not-so-restrictive distribution for unobserved heterogeneity.

# Conclusions

- ❑ Proposed approach is applied to **land-use-change decisions** using the city of Austin's parcel-level land-use data.
- ❑ Results highlight the importance of introducing social dependence effects and non-normal kernel error terms from a policy standpoint.
- ❑ Predicted shares (discrete dimension), predicted investments (continuous dimension), and elasticity effects are not correctly estimated by traditional models (with no spatial correlation and/or non-normal distributions).



# Thank you!

**Prof. Chandra R. Bhat**

*Website:* <http://www.cae.utexas.edu/prof/bhat/home.html>



THE UNIVERSITY OF TEXAS AT AUSTIN  
**CENTER FOR TRANSPORTATION RESEARCH**

# Methodology

# Skew-Normal Distribution

□ Let  $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3, \dots, \eta_L)'$  be a multivariate skew-normal (MVSN) random variable vector of size  $(L \times 1)$  with a location parameter  $\mathbf{0}_L$  and correlation matrix  $\boldsymbol{\Gamma}^*$

□ Then  $\boldsymbol{\eta}$  is obtained through a latent conditioning mechanism as follows:

$$\begin{pmatrix} C_o^* \\ C_1^{*'} \end{pmatrix} \sim MVN_{L+1} \left( \begin{pmatrix} 0 \\ \mathbf{0} \end{pmatrix}, \boldsymbol{\Omega}_+^* \right), \text{ where } \boldsymbol{\Omega}_+^* = \begin{pmatrix} 1 & \boldsymbol{\rho}' \\ \boldsymbol{\rho} & \boldsymbol{\Gamma}^* \end{pmatrix}.$$

where  $\boldsymbol{\rho}$  is a  $(L \times 1)$  vector, whose elements lie between -1 and +1

□ Then  $\boldsymbol{\eta} = C_1^{*'} \mid (C_o^* > 0)$  has the standard skew-normal density function as shown below:

$$\tilde{\phi}_L(\boldsymbol{\eta} = \mathbf{z}; \boldsymbol{\Omega}_+^*) = 2\phi_L(\mathbf{z}; \boldsymbol{\Omega}_+^*) \Phi(\boldsymbol{\alpha}'\mathbf{z}), \text{ where } \boldsymbol{\alpha} = \frac{(\boldsymbol{\Gamma}^*)^{-1}\boldsymbol{\rho}}{(1 - \boldsymbol{\rho}'(\boldsymbol{\Gamma}^*)^{-1}\boldsymbol{\rho})^{1/2}}$$

□ The cumulative distribution function for  $\boldsymbol{\eta}$  may be obtained as:

$$P(\boldsymbol{\eta} < \mathbf{z}) = \tilde{\Phi}_L(\mathbf{z}; \boldsymbol{\Omega}_+^*) = 2\Phi_{L+1}(\mathbf{0}, \mathbf{z}, \boldsymbol{\Omega}_-^*); \quad \boldsymbol{\Omega}_-^* = \begin{pmatrix} 1 & -\boldsymbol{\rho}' \\ -\boldsymbol{\rho} & \boldsymbol{\Gamma}^* \end{pmatrix}.$$



# The aspatial Skew MDC model

$$\max U_q(\mathbf{x}_q) = \left[ \sum_{k=1}^{K-1} \gamma_k \psi_{qk} \ln \left( \frac{x_{qk}}{\gamma_k} + 1 \right) \right] + \psi_{qK} \ln x_{qK} \quad s.t. \sum_{k=1}^K p_{qk} x_{qk} = E_q$$

- $x_{qk}$  : consumption quantity of good  $k$  for individual  $q$
- $p_{qk}$  : unit price of good  $k$
- $E_q$  : total budget of individual  $q$
- $\psi_{qk}$  : baseline (at zero consumption) marginal utility
- $\gamma_k$  : satiation parameter
- $K$  : index of the outside good

# Stochasticity in the model

- Parameterization of the baseline utilities:

$$\psi_{qk} = \exp(\boldsymbol{\beta}'_q \tilde{\mathbf{z}}_{qk})$$

- $\tilde{\mathbf{z}}_{qk}$ : attributes that characterizes good  $k$  and individual  $q$  (including a dummy variable for each alternative except the last one)

$$\boldsymbol{\beta}_q \sim MVSND_D(\mathbf{b}, \boldsymbol{\omega}, \boldsymbol{\Omega}_+^*)$$

$$\boldsymbol{\Omega}_+^* = \begin{pmatrix} 1 & \boldsymbol{\rho}' \\ \boldsymbol{\rho} & \boldsymbol{\Omega}^* \end{pmatrix}$$

# Identification issues

- ❑ Consumer-specific variables can be introduced to only  $K-1$  goods
- ❑ Only the covariance matrix of the error differences is estimable
- ❑ Additional scale normalization needs to be imposed if there is no price variation across goods

# The Spatial Skew MDC model

□ We define

$$\begin{aligned}\bar{\psi}_{qk} &= \bar{\psi}_{qk}^* - \bar{\psi}_{qK}^* = \boldsymbol{\beta}'_q (\tilde{\mathbf{z}}_{qk} - \tilde{\mathbf{z}}_{qK}) \\ &= \boldsymbol{\beta}'_q \mathbf{z}_{qk}, \quad \mathbf{z}_{qk} = \tilde{\mathbf{z}}_{qk} - \tilde{\mathbf{z}}_{qK} \quad \forall k \neq K\end{aligned}$$

$$\text{and} \quad \bar{\psi}_{qK} = \bar{\psi}_{qK}^* - \bar{\psi}_{qK}^* = 0.$$

□ Spatial correlation

$$\bar{\psi}_{qk} = \bar{\psi}_{qk}^* - \bar{\psi}_{qK}^* = \boldsymbol{\beta}'_q \mathbf{z}_{qk} + \delta_k \sum_{q'} w_{qq'} \bar{\psi}_{q'k}, \quad \text{for } k = 1, 2, \dots, K-1$$

$$\bar{\psi}_{qK} = 0 \text{ for } k = K. \quad \bar{\psi}_{qk}^* = \ln(\psi_{qk}) \quad w_{qq} = 0, \quad \sum_{q'} w_{qq'} = 1 \quad \text{and} \quad 0 < \delta_k < 1$$

# Likelihood function form

$$L_{CML}(\boldsymbol{\theta}) = \prod_{q=1}^{Q-1} \prod_{q'=q+1}^Q \det(\mathbf{J}_{qq'}) \times \left( \bar{\boldsymbol{\omega}}_{\tilde{\boldsymbol{\Sigma}}_{qq',C}} \right)^{-1} \left[ 2\varphi \left( \tilde{\boldsymbol{\omega}}^{-1} \tilde{\mathbf{B}}_{qq',C}^*; \tilde{\boldsymbol{\Sigma}}_{qq',C}^* \right) \right] \left[ \Phi(\boldsymbol{\alpha}' \tilde{\mathbf{B}}_{qq',C}^*; \tilde{\boldsymbol{\Sigma}}_{qq',C}^*) \right]$$

- ☐ Pairwise CML
- ☐ Distance band
- ☐ Evaluations using only univariate and bivariate cumulative normal distribution functions (MACML)