A network model for capped distance-based tolls

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Capped distance-based tolls

- Distance-based tolls charge the road users proportionally to the distance of their trip, or more general, to the usage of the road.
The classical network equilibrium formulation which adds a fixed cost to each link is no longer valid since the tolls are non-additive due to the capping.

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\[ t_{a \rightarrow d} = \min \left( t_{a \rightarrow d}^{\text{max}}, t_a + t_b + t_c + t_d \right) \]
The network equilibrium problem is formulated on the augmented network $A \cup D$, partitioned into three disjoint sets:

- Links in the base network without tolls $A - T$

  $$c_a = s_a(v_a), \quad a \in A - T$$

- Links in the base network with tolls $T$

  $$c_a = s_a\left(\sum_{1 \leq j \leq a \leq k \leq n} v_{jk} + v_a\right) + t_a, \quad j \neq k, \quad a \in T$$

- Additional links $D$

  $$c_{lm} = \sum_{a=l}^{m} s_a\left(\sum_{1 \leq j \leq a \leq k \leq n} v_{jk} + v_a\right) + t_{lm}, \quad j \neq k, \quad lm \in D$$
Model formulation

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  - Links in the base network without tolls $A - T$
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    \[ c_a = s_a \left( \sum_{1 \leq j \leq a \leq k \leq n} v_{jk} + v_a \right) + t_a, \quad j \neq k, \quad a \in T \]
  - Additional links $D$
    \[ c_{\bar{lm}} = \sum_{a=l}^{m} s_a \left( \sum_{1 \leq j \leq a \leq k \leq n} v_{jk} + v_a \right) + t_{\bar{lm}}, \quad j \neq k, \quad \bar{lm} \in D \]
Model formulation

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- Additional links $D$
  \[ c_{\tilde{m}} = \sum_{a=l}^{m} s_a\left(\sum_{1 \leq j \leq a \leq k \leq n} v_{jk} + v_a\right) + t_{\tilde{m}}, \ j \neq k, \ \tilde{m} \in D \]
Mathematical properties

- With the cost vector $C(v)$ defined for each link type, the equivalent optimization problem for the user equilibrium is

$$\min \int_0^v C(x)^T \, dx$$

- We showed that this is possible iff the Jacobian matrix of the cost vector $C(v)$ is symmetric everywhere

$$\frac{\partial s_{\hat{ij}}}{\partial v_{kl}} = \frac{\partial s_{\hat{kl}}}{\partial v_{\hat{ij}}}, \quad \hat{ij} \in D, \quad \hat{kl} \in D$$

$$\frac{\partial s_a}{\partial v_{\hat{ij}}} = \frac{\partial s_{\hat{ij}}}{\partial v_a}, \quad a \in T, \quad \hat{ij} \in D$$
Algorithm description

- The algorithm can be summarized as follows:
  - Augment the original network with additional links
  - From the toll on base network links compute the toll and define the cost functions on the additional links
  - Solve the traffic user equilibrium problem on the augmented network with tolls as fixed link cost
  - Accumulate the flows on the additional links and assign it to the toll links

- Any method for obtaining UE flows can be used
- A parallelized bi-conjugate variant* of the linear approximation method has been implemented in Emme 4.1

*As described in Mitradjieva, M. and Lindberg, P.O. (2013)
A numerical example

- Mountain route with no alternative

cost 22
A numerical example

- Mountain route with no alternative
- Demand 600 trips from 1 to 2
A numerical example

- Mountain route with no alternative
- Demand 1200 trips from 1 to 2
A numerical example

- Mountain route with no alternative
- Demand 1800 trips from 1 to 2
A numerical example

- Mountain route with no alternative
- Demand 2400 trips from 1 to 2
A numerical example

Mountain route with toll road alternative

cost 22

cost: travel time 5 + toll 20
A numerical example

- Mountain route with toll road alternative
- Demand 2400 trips from 1 to 2
A numerical example

- Mountain route with toll road alternative
- Demand 3000 trips from 1 to 2
A numerical example

- Mountain route with toll road alternative
- Demand 3600 trips from 1 to 2
A numerical example

- Mountain route with toll road alternative
- Demand 4200 trips from 1 to 2
A numerical example

- Mountain route with toll road alternative
- Demand 4800 trips from 1 to 2
A numerical example

- Mountain route with toll road alternative cap $18
- Demand 4800 trips from 1 to 2
A numerical example

- Mountain route with toll road alternative cap $16
- Demand 4800 trips from 1 to 2
A numerical example

- Mountain route with toll road alternative cap $15
- Demand 4800 trips from 1 to 2
A numerical example

- Mountain route with toll road alternative cap $14
- Demand 4800 trips from 1 to 2
The method has been applied in Sydney, Australia

- Land area – 4,700 km²
- Population – 4.7 million
- Employment – 2.3 million
- Zones – 2,123
- Nodes – 14,900
- Links – 41,350

Key features:
- different cap values
- ~20km travelled
- congested road
- significant % of trucks
Sydney highway toll study

- 30 classes of traffic corresponding to distributed values of time for the private car and truck modes
- Extraction of cordon matrices, economic benefits, etc
- 4 time periods (AM, IP, PM, NT)
Sydney M7 highway toll study

- Uncapped toll
  - max flow 7000 vph
  - average tolled distance 12.0 km

- Toll cap AUD 7.20
  - max flow 7300 vph
  - average tolled distance 12.5 km
Sydney M7 highway toll study

- Uncapped toll
  - max flow 7000 vph
  - average tolled distance 12.0 km

- Toll cap AUD 3.60
  - max flow 8200 vph
  - average tolled distance 14.4 km
different cap values were used, varying from 0$ to 15$

**Toll link volume vs. cap**

- **link A**
- **link B**
- **link C**
- **link D**

**cap value in $**

**volume on link**

Sydney M2 highway toll study
Conclusions

- This method has the advantage that it can be solved by any algorithm that computes user equilibrium flows.
- Previous contributions addressing UE flows with non-additive tolls resort to path enumeration or are restricted to uncapped linear dependent distance toll.
- This method is easy to apply and does not require post processing that uses approximations.
- This method has also been extended to treat ramp-to-ramp tolls.

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The Evolution of Transport Planning