Abstract
Some tolling schemes commonly considered include distance-based tolls as well as derived schemes such as charging a maximum toll (or cap) for the use of the facility or minimum toll, if the distance based toll is less than this value. In order to meet these requirements a new model formulation and algorithm for distance-based toll modelling is developed. It uses the toll cost per link together with minimum and a maximum value of the tolls paid. The model is based on the addition of a set of temporary links to the network, which inherit the tolls and the delays of the original links. The method presented in this paper is general and self-contained. The model may be easily extended for other types of nonlinear toll schemes. A proof is provided for the equivalence of the modified and original network formulations.

Statement of Financial Interest
The method presented in this paper is general and can be implemented by anyone interested to do so. The computational results reported have been done with the Emme software.

Statement of Innovation
This paper presents a new model formulation and algorithm for capped distance-based toll modelling. From a mathematical point of view, it converts a non-additive cost network equilibrium model into an additive cost model.
A network model for capped distance-based tolls

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Abstract

Toll road operators and other toll facility stakeholders require analysis tools to estimate the ridership and projected income for an increasing variety of tolling schemes. Some tolling schemes commonly considered include distance-based tolls as well as derived schemes, such as charging both a minimum and a maximum toll (or cap) for the use of the facility. In addition, different entry ramps may incur different tolls which may be added to a distance-based toll and subject the total toll to a toll cap value. Network equilibrium models that consider such tolls result in non-additive costs on the modeled network. In order to overcome the difficulty associated with non-additive costs a new model formulation and algorithm for distance-based toll modelling with toll caps is developed. It considers the toll cost per link, which may be distance dependent, as well as minimum and a maximum value of the distance based tolls paid. The model uses the addition of a set of temporary links to the network, which inherit the delays and tolls of the original links. The method presented in this paper is general and self-contained. The model may be easily extended for other types of nonlinear toll schemes. A proof is provided for the equivalence of the modified and original network formulations. In order to solve the resulting multi-class network equilibrium model, a multi-threaded bi-conjugate variant of the linear approximation method has been adapted for the particular network model used. The method is illustrated with a small example as well as an instance of capped distance-based toll modelling on a network originating from practice that employs the toll structure considered.
1. Introduction

Distance-based tolls charge the road users proportionally to the distance of their trip, preventing the overuse of short free trips. The impact of such tolls on the user behaviour is considered in several papers as May and Milne (2000), Wen and Tsai (2005), Balwani and Singh (2009), Jou et al. (2012), Meng et al. (2012), Natzel et al. (2011).

In practice, different variants of distance-based tolls are evaluated, such as capped maximum toll, minimum toll and different fees per entry ramp. An example of the toll cost along a path is given in Figure 1. Note that, even though the toll presented in Figure 1 is linear, the method described in this paper extends to all link-wise monotonically increasing functions.

![Toll cost on a path](image)

Figure 1 – An example of the toll cost along a path

In the absence of capping, distance-based tolls are just link additive. These tolls can be modelled by classic traffic equilibrium methods by solving for the user equilibrium (Wardrop (1952)) with a fixed link cost term (see for example Larsson and Patriksson (1998), Hagstrom (1998), Florian (2006)). In the presence of capping, distance-based tolls are not link additive since the cost of a trip does depend on the chosen path. General algorithms for the non-additive traffic equilibrium problem have been considered in several papers, using path enumeration (see for example Bernstein and Gabriel (1997), Lo and Chen (2000), Chen et al. (2010), Qian et al. (2013), Lawphongpanich and Yin (2011)).

The method presented in this paper relies on a network construction and facilitates its solution by any method for computing network equilibrium flows, extending to all link-wise monotonically increasing functions. The method directly uses the toll cost per link instead of the cost of entry/exit ramp pairs. Similar to the ramp-to-ramp method, the algorithm is based on the construction of additional links, which span the toll road, with
the significant difference that the original links are kept as well and are used during the equilibration process. In order to solve the static traffic assignment problem on this augmented network, a multithreaded version of the bi-conjugate linear approximation method (Mitradjieva and Lindberg (2012)) has been adapted for the particular network structure considered.

The method presented in this paper is general and self-contained. The model may be easily extended for other types of nonlinear toll schemes. A proof is provided for the equivalence of the modified and original network formulations.

2. Model formulation

A road network \( G = (N, A) \) consists of nodes \( n \in N \) and directed links \( a \in A \). The traffic demand for origin-destination (O-D) pair \( i \in I \subset N \times N \) is denoted by \( g_i \). These demands use paths \( k \in K_i \), where \( K_i \) is the set of paths used for travel between O-D pair \( i \). The vector of flow on link \( a \) is denoted by \( v_a \). In its simplest form, the cost \( c_a \) on link \( a \) is the sum of the travel time function, denoted by \( s_a(v_a) \), and a toll \( t_a \), that is converted into time units from its money value by a function, which is assumed monotonically increasing. It is assumed that all the link volume-delay functions \( s_a(v_a) \) are monotonically increasing. In order to distinguish between toll and non-toll links, the set of toll links is denoted by \( T \subset A \). The time of a path \( k \in K_i \) is link additive

\[
s_k = \sum_{a \in A} \delta_{ak} s_a(v_a),
\]

where

\[
\delta_{ak} = \begin{cases} 
1 & \text{if } a \in k \\
0 & \text{otherwise} 
\end{cases}.
\]

If \( t_{\text{max}} \) is the toll cap, the toll along a path is

\[
t_k = \min(t_{\text{max}}, \sum_{a \in T} \delta_{ak} t_a).
\]

Hence, the cost of a path is

\[
c_k = \sum_{a \in A} \delta_{ak} s_a(v_a) + \min(t_{\text{max}}, \sum_{a \in T} \delta_{ak} t_a).
\]

The path flows satisfy flow conservation and non-negativity
The link flows are
\[ v_a = \sum_{i \in I} \sum_{k \in K_i} \delta_{ai} h_k . \] (6)

The network equilibrium flows (Wardrop (1952)) satisfy
\[ u_i = \min_{k \in K_i} (c_k), \quad i \in I \] (7)
and
\[ c^*_k - u_i = \begin{cases} 0 & \text{if } h^*_k > 0 \\ \geq 0 & \text{if } h^*_k = 0 \end{cases}, \] (8)

where * denotes the equilibrium values for path travel times and path flows. This is a network equilibrium model with non-additive costs due to the capped tolls.

3. Conversion into a link additive model

In order to formulate the distance-based toll network equilibrium model, it is assumed, without loss of generality, that the toll road forms a contiguous stretch of links. In order to convert non-additive tolls to additive tolls a network construction is used: all the possible paths on the toll highway are enumerated in order to take into account the path dependent cost due to the toll cap. Note that the enumeration needs to be done only for the links of the tolled facility.

The added links, also referred as fictitious or additional, inherit the travel times of the spanned stretch of the road, and the capped toll costs. Unlike the method described in Yang et al. (2004), the original tolled links of the network are not removed since a single tolled link may be used part of a larger path. Since the toll cost on the additional links is known a priori and is fixed during an assignment, we transformed in this way a model with non-additive path costs into a model with additive path costs on an augmented network. A demonstration that the two problems are equivalent is given in the following paragraphs.

The augmented network is indicated by \( \hat{G} = (N, \hat{A}) \) where \( \hat{A} = A \cup D \) is the set of the base network links and \( D \) is the set of the additional links. Additional links are assigned a two-letter index, while links that belong to the base network \( A \) are referred to with only one index. The cost \( c_a \) of a toll link \( a \in T \) can be expressed as:
\[ C_a = S_a \left( \sum_{1 \leq j < a \leq k} v_{jk} + v_a \right) + t_a, \quad j \neq k. \]  

The cost of an added link, which spans the original network links from link \( l \) to \( m \) is:

\[ C_{lm} = \sum_{a=l}^{m} S_a \left( \sum_{1 \leq j < a \leq kn} v_{jk} + v_a \right) + t_{lm}, \quad j \neq k. \]  

Since the toll of an added link can be written as

\[ t_{lm} = \min \left( t_{\text{max}}, \sum_{a=l}^{m} t_a \right), \]

equation (10) becomes

\[ C_{lm} = \sum_{a=l}^{m} S_a \left( \sum_{1 \leq j < a \leq kn} v_{jk} + v_a \right) + \min \left( t_{\text{max}}, \sum_{a=l}^{m} t_a \right), \quad j \neq k. \]  

It is shown that one can derive a classic traffic equilibrium problem with fixed costs added to some links, with non-separable variables since the cost of a link does not depend only on the flow of that link, as indicated in (9). The minimization problem is well defined if and only if the Jacobian matrix of the cost functions is symmetric everywhere (Patriksson (1994)). A proof that this condition is met is presented in the Appendix.

Under the above assumptions, one can show that the equilibrium flows on \( G = (N, A) \) with capped distance-based tolls are equivalent to the equilibrium flows in \( G = (N, A) \) with fixed costs as computed in (11). Assume that equilibrium flows have been computed in \( G = (N, A) \) by solving the associated minimization problem. There may be a non-zero flow through the toll links. In order to traverse a contiguous toll stretch \( l_1 \rightarrow l_2 \rightarrow \ldots \rightarrow l_n \), the flow can be split into two: a part that uses the additional link \( l_{1n} \in D \) and the rest that uses a path containing at least one link from the base network \( l_i \in A \) denoted here by \( l_1 \rightarrow l_{2n} \). At equilibrium, the cost of the two paths must be equal. According to (12), the only difference between the cost of the two paths is due to the toll term, which satisfies

\[ \sum_{l_1}^{l_n} t_j \geq \min \left( t_{\text{max}}, \sum_{l_1}^{l_n} t_j \right). \]

If the equality holds, one can shift the entire flow from path \( l_1 \rightarrow l_{2n} \) to link \( l_{1n} \) without influencing the equilibrium. In the case of strict inequality, the flow on path \( l_1 \rightarrow l_{2n} \) must
be zero according to the user equilibrium condition. Therefore, at equilibrium in \( \widehat{G} = (N, \widehat{A}) \), there are no flows that can parse a contiguous stretch of more than one toll link in the base network, but all this flow will be "absorbed" by the additional links. Since these links are virtual links corresponding to the splitting of trips on the base network based on their toll value, it follows that the two problems are equivalent.

Briefly, the steps of the solution algorithm may be summarized as follows:

- Given \( G = (N, A) \) construct \( \widehat{G} = (N, \widehat{A}) \) by adding the fictitious links in \( D \).
- From toll links in \( T \) compute the toll on fictitious links in \( D \).
- Solve the traffic equilibrium problem in \( G = (N, A) \) with tolls as additional link cost.
- From the flows on fictitious links in \( D \) compute the flows on the toll links in \( T \in G = (N, A) \).

The last step of the algorithm, after computing the equilibrium flows in \( G \), consist in recovering the traffic flow on the toll links by accumulating the flow from the additional links:

\[
v_{acT} = \sum_{1 \leq j \leq a, k \leq n} v_{jk} + v_{acT}, \quad i \neq j.
\]  

(14)

4. Capped distance-based toll study on Sydney M2 highway

The algorithm was implemented\(^1\) by using a multi-threaded bi-conjugate variant of the linear approximation (Mitradjieva and Lindberg (2012)). The solution has been extended for multiple traffic classes as well as for multiple toll facilities, each of them with different cap values.

The method was applied in the evaluation of different the toll cap values over the flow pattern for the eastbound M2 toll highway in Sydney, Australia, which is shown\(^2\) in blue in Figure 2.

\(^1\) Implemented in the Emme 4 software package INRO (2013)

\(^2\) By the courtesy of SKM
This application uses 30 classes of traffic that correspond to different values of time for the private car and truck modes. These 30 classes were derived from continuous value of time distributions.

Figure 2 – The M2 toll highway is shown in blue on the map

Figure 3 – The impact of cap value over link volume on the toll highway
The computational results reported here use 10 different cap values, varying from 0$ to 60$ (0% to 1000% from the prevailing 6$ toll cap, respectively). The 10 corresponding equilibrium assignments were executed using a relative gap of $\sim 10^{-4}$ as a stopping criterion. The monotonically decreasing volumes on four links of the M2 highway are plotted in Figure 3.

The results are easily interpreted to conclude that additional revenues may be realized by increasing the toll cap since certain classes of users with a high value of time are nearly indifferent to the value of the toll cap (it should be mentioned that these analyses are done just for illustrative purposes and do not correspond to any results that were obtained in a study of this toll highway).

5. Conclusions

The method presented in this paper for computing distance-based tolls with cap values has the advantage that it can be solved by any algorithm that computes network equilibrium flows. Other significant contributions in the literature that have addressed network equilibrium flows with non-additive tolls resort to far more complex algorithmic solutions or are restricted to linear dependent distance toll. The original proof presented in this paper requires only link-wise dependency via monotonically increasing functions. The same proof can be applied in order to show that the equilibrium problem solved in Yang et al. (2004) is in fact equivalent to the original equilibrium problem with non-additive path cost for networks with entry–exit based tolls.

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References


Appendix

This appendix contains a formal proof that the terms of the Jacobian matrix of the link cost vector are symmetric.

The terms are already symmetric for links \( a \in A \); they are symmetric for links \( a \in T \) and \( \tilde{a} \in D \) if:

\[
a) \quad \frac{\partial s_{ij}}{\partial v_{kl}} = \frac{\partial s_{kl}}{\partial v_{ij}}, \tag{15}
\]

\[
b) \quad \frac{\partial s_a}{\partial v_{ij}} = \frac{\partial s_{ij}}{\partial v_{a}}, \tag{16}
\]

where \( ij \in D, \ lk \in D, \ 1 \leq i < j \leq n, \ 1 \leq l < k \leq n, \ a \in T \).

Recall from (9) that the partial derivatives are evaluated with respect to the total flow on a toll link \( v_a^{\text{total}} \). The dependency of the total flow \( v_a^{\text{total}} \) on flow \( v_{kl} \) can be described using the relative position of the four indices \( i, j, l, k \) in the integer interval \([1,n]\). If the integer intervals \([i,j]\) and \([k,l]\) are disjoint, there is no dependency, therefore the partial derivatives are zero. If they are not disjoint, there are as many partial derivatives as the number of overlapping indices in the intersection of the two integer intervals. The left term of (15) can be written as:

\[
\frac{\partial s_{ij}}{\partial v_{kl}} = \sum_{i \leq a < j} \frac{\partial s_a}{\partial v_{ij}} + \sum_{i \leq a < j} \frac{\partial s_a}{\partial v_{ji}} = \sum_{a \in [i,j], [k,l]} \frac{\partial s_a}{\partial v_{a}}. \tag{17}
\]

Analogously, the right term of (15) be written as

\[
\frac{\partial v_{kl}}{\partial s_{ij}} = \sum_{k \leq a \leq l} \frac{\partial s_a}{\partial v_{ij}} + \sum_{k \leq a \leq l} \frac{\partial s_a}{\partial v_{ji}} = \sum_{a \in [i,j], [k,l]} \frac{\partial s_a}{\partial v_{a}}, \tag{18}
\]

such that (15) follows.

According to (9) and using the same integer interval argument as above, the left term in (16) is

\[
\begin{cases}
\frac{\partial s_a}{\partial v_{ij}} = \frac{\partial s_a}{\partial v_{a}} = \frac{\partial s_a}{\partial v_{a}} & \text{if } a \in [i,j] \\
\frac{\partial s_a}{\partial v_{ij}} = 0 & \text{if } a \not\in [i,j]
\end{cases} \tag{19}
\]

Using (9) and (10) we get for the right term
\[
\begin{align*}
\frac{\partial s_{ij}}{\partial v_a} &= \frac{\partial}{\partial v_a} \sum_{i=1}^{j} s_i = \frac{\partial s_a}{\partial v_a} \quad \text{if } a \in [i, j], \\
\frac{\partial s_{ij}}{\partial v_a} &= 0 \quad \text{if } a \notin [i, j]
\end{align*}
\]

and (16) follows. \(\square\)