On the Kinetic Behavior of Roads

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An analysis is presented of the kinetic behavior of a road supported by a uniform subgrade reaction and subject to various boundary conditions. The static behavior of the road in response to a single centrally located concentrated load force is obtained by direct integration of a differential equation derived to represent the static road model and by a difference equation numerical analysis technique. In addition, the effect of road width on static deflection profile is derived on an approximate basis, and the division of input potential energy between the subgrade and the pavement is determined as a function of road width.

The differential equation of the road model is extended to include time as an independent variable, thus including transient forces in the analysis. The resultant partial differential equation is solved, using difference-differential equations and an electronic differential analyzer, for the dynamic response of the road to a step function of force. The analysis is shown to be valid for any arbitrary road loading force time history.

It is concluded from the results of the study that paved roads can be dynamically analyzed in much the same manner as the vehicles that traverse them have been. The theory presented is uncorroborated by experimental analysis, however, and the necessity of performing experiments to confirm the theory is shown. Suggested paths for continued, more comprehensive, theoretical and experimental analyses are presented.

• THROUGHOUT the history of highway transportation, road design and construction has been based on empirical knowledge of road life as a function of the size, form, and volume of the traffic flow. It has been only comparatively recently that concerted efforts have been made to predict road response to the traffic flow on a quantitative analytical basis. Perhaps the first of the analyses that was generally supported by experimental data was performed by Westergaard (*11*), whose analysis was based on a treatment of the road as a semi-infinite circular plate of finite bending rigidity, uniformly supported on an elastic subgrade. An experimental program that generally confirmed the theoretical analysis technique was performed and reported by Teller and Sutherland (*5*).

Several limitations to the Westergaard analysis immediately present themselves, however. The boundary conditions inherent in the problem due to a finite road width and various types of joints are not included in the analysis of road response to loads spaced remotely from the edge. Also, and perhaps most significantly, the analysis technique does not provide for a solution for the dynamic behavior of the road in response to time-varying road load forces.

Since Westergaard's analysis (*11*),

several analysis techniques have been developed to predict the steady-state frequency response of a road. Samples of this work are reported by Van der Poel and co-workers (*7, 8*); experimental programs that in general confirm the analysis techniques are reported by Nijboer and Jones (*9, 10*). These methods are somewhat limited in that they yield the road steady-state dynamic response to just one form of applied force—a sinusoidal time-varying force. Although it can be argued that the transient response of the road to impulse and step functions of applied force can be computed from the steady-state frequency response in a linear system, the computational task would be enormous for a distributed parameter system, such as a road. Furthermore, experimental

NOMENCLATURE

The symbols used throughout this paper are defined as follows:

Symbol	*Definition*	*Nominal Units*
c	Subgrade damping	lb-sec/cu in.
D	Pavement modulus of rigidity per unit width	in.-lb
E	Young's modulus	psi
F	Applied road load force	lb
h	Pavement thickness	in.
i	Number of stations in difference-equation technique	—
k	Subgrade spring modulus	lb/cu in.
l	Road slab length	in.
M_x	Longitudinal bending moment per unit road width	in.-lb/in.
M_y	Lateral bending moment per unit road width	in.-lb/in.
M_{xy}	Twisting moment per unit road width	in.-lb/in.
m	Pavement mass	lb-sec²/cu in.
N	Pavement width function	—
P	Road surface pressure distribution	psi
q	Intensity of the distributed load	psi
t	Real time	sec.
U_R	Static pavement strain potential energy	in.-lb
U_s	Static subgrade potential energy	in.-lb
U_T	Total static road-subgrade energy	in.-lb
u	Vertical road deflection	in.
u_n	Difference equation road deflection variable	in.
v_x	Longitudinal shear per unit width	lb/in.
v_y	Lateral shear per unit width	lb/in.
w	Road slab width	in.
x	Longitudinal distance from the load center	in.
Δx	Distance between stations in the difference-equation solution technique	in.
y	Lateral distance from the load center	in.
β	Road space frequency parameter	rad/in.
$\delta'(x)$	Dirac delta function	in.$^{-1}$
λ	Road space wavelength	in.
μ	Poisson's ratio	—
ξ	Road vibration damping ratio	—
ρ	Slab length road deflection parameter	—
$\sigma_{x,y}$	Pavement tensile stress	psi
ω_n	Road vibration natural frequency	rad/sec

results show that the natural frequency of road vibrations can be a function of vibration input amplitude, which is not a property of linear systems.

No literature has been found in which an analysis technique was presented to yield the transient response of a road to an arbitrary time-varying road-load force, such as the vertical road load transmitted to the pavement through the tires of a vehicle traversing the road. This paper presents a preliminary treatment of the generalized input force-kinetic road response problem.

PRELIMINARY ROAD MODEL

To predict the over-all behavior of a road in response to a time-varying road-load force, it is necessary to determine the differential forces and moments that exist on any given microscopic element of road, as a function of time, and integrate the results over the entire road surface. The necessity of describing the response of the road in terms of both space and time as independent variables requires the use of partial differential equations in the analysis technique. The following assumptions are made in the analysis:

1. The pavement is supported by a uniformly distributed linear subgrade spring and damping reaction.
2. The pavement material obeys Hooke's law with respect to bending stresses and strains.
3. The load acting on the road is normal to the surface.
4. The road deflections are small relative to pavement thickness.
5. The edges of the pavement are free to move in the plane of the surface.
6. The thickness of the pavement is small in comparison with the dimensions of the surface.

The forces and moments acting on an element of road pavement are diagramed in Figure 1. A derivation of the differential equation of the pavement deflection suface may be found elsewhere (*20*, Chap. 4) and is not repeated here. The resulting equation is

$$D\left[\frac{\partial^4 u}{\partial x^4}+2\frac{\partial^4 u}{\partial x^2 \partial y^2}+\frac{\partial^4 u}{\partial y^4}\right]=q(x, y, t) \quad (1)$$

This equation was originally obtained by Lagrange in 1811, and has since been used extensively in beam, plate, and shall analyses. In the case of the uniformly supported road, the load intensity is given by

$$q(x, y, t) = P(x, y, t) - m\frac{\partial^2 u}{\partial t^2} - c\frac{\partial u}{\partial t} - ku \quad (2)$$

The right side of Eq. 2 represents the applied surface-loading pressure less the surface material inertia, subgrade damping, and subgrade spring reactions. The force due to the static mass of the beam can be considered to be contained in the loading term $P(x,y,t)$. It is not of importance, inasmuch as concern is with pavement deflections relative to the unloaded rest position of the surface.

Past experience has demonstrated that the major difficulty associated with the solution of Eq. 1 is in the application of initial and boundary conditions to the equation to obtain a mathematical solution in closed form. It has been demonstrated that an exact closed-form mathematical solution to most partial differential equations is nonexistent. They must, in general, be solved by approximation techniques such as Fourier or power series expansions, or numerical analysis methods. Eq. 1, when combined with Eq. 2, is no exception. None of the classical closed-form

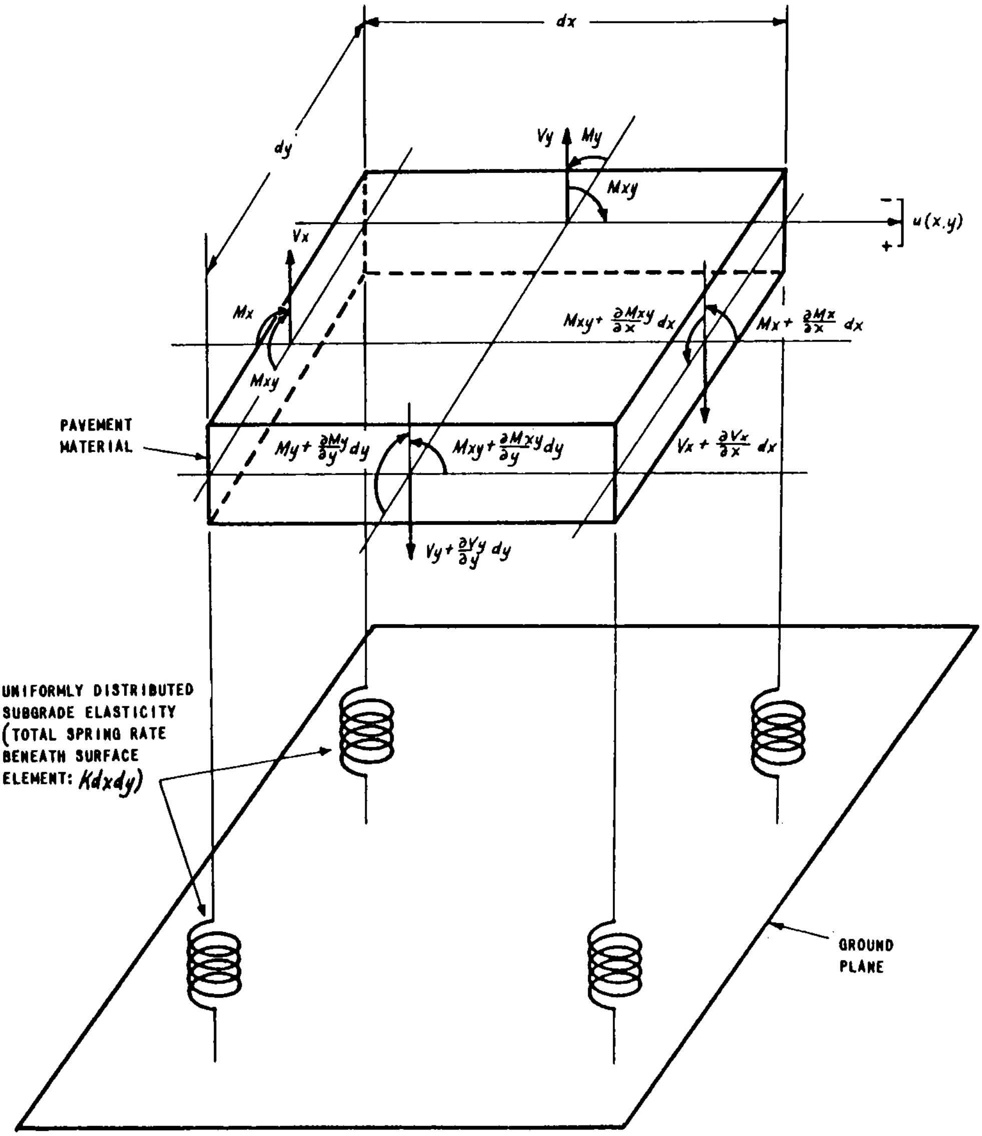

Figure 1. Differential road element.

solution techniques will solve this three-dimensional road equation.

The analysis problem can be simplified considerably by assuming the road width to be small, and solving the resultant narrow-road equation. This is the technique demonstrated in this paper. The unit-width road equation becomes

$$D\frac{\partial^4 u}{\partial x^4}=q(x, t)$$

$$=P(x, t)-m\frac{\partial^2 u}{\partial t^2}-c\frac{\partial u}{\partial t}-ku \qquad (3)$$

Eq. 3 follows from Eq. 1 when road deflections are the same laterally (across the road) at any given longitudinal distance from the point of loading.

It is necessary in the solution of Eq. 3 to specify four boundary conditions in space, and two initial conditions in time. The boundary conditions result from the manner and form of the road construction; the initial conditions result from the prevailing displacement and velocity of each point on the road at the instant that the solution commences. Eq. 3, therefore, represents the equation of the analytical dynamic road model that is considered in this paper.

It should be recognized that the dynamic model represented by Eq. 3 is an "instantaneous model"; that is, it yields the dynamic response of the road to time-varying force inputs, and reaches steady state very rapidly after removal of the transient load force. The coefficients of Eq. 3 can, on the other hand, be time-varying quantities but will vary over a period of minutes, days, months, or years. They vary with the physical condition of the road, ambient temperature, subgrade moisture content, etc. Analysis of the long-term variation in these parameters is beyond the scope of the dynamic road model considered in this paper. Suffice it to say that the kinetic road behavior can be determined for any given road parameters by solution of Eq. 3 at a given instant in time, and that the road parameters can be considered to remain constant during the solution time of the "instantaneous model."

STATIC ANALYSIS TECHNIQUE

Deflections and Bending Moments

Before attempting a solution of Eq. 3 in both the space and time domains, it is advisable to determine the equation of the static road deflection profile so that the steady-state response of the road will be known when the dynamic analysis is performed. The steady-state solution is a special case of the dynamic solution and provides a check on the dynamic solution technique.

Two cases are analyzed—a very long ribbon road with no joints, and a single finite-length slab of pavement. In each case, the load considered is a force uniformly distributed laterally across the road and concentrated in the longitudinal direction to a single point. The force is assumed to act at the origin of the axis system. To solve for the static deflection profile, the static pressure distribution $P(x)$ must be known in terms of the applied load force F. If the width of the road is w, and the load is distributed uniformly across the width as well as longitudinally along the dimension S, the load intensity becomes

$$P(x) = \frac{F}{w\,s}$$

as shown in Figure 2.

Because a longitudinally concentrated load is assumed, the desired loading condition is achieved when dimension S is reduced to zero. The longitudinal load distribution is described by a rectangle of height F/wS and width w, the area of which remains constant at F/w as dimension S is reduced to zero. This is mathematically represented by a space impulse function, called the Dirac delta function, $\delta'(x)$, described more fully elsewhere (*25*). Mathematically

$$\left.\begin{aligned} \delta'(x) &= \infty \quad \text{when } x = 0 \\ \delta'(x) &= 0 \quad \text{when } x \neq 0 \end{aligned}\right\}$$

where

$$\int_{-\infty}^{\infty} \delta'(x)dx = 1$$

Using this notation, and noting that $\delta'(x)$ has units of inverse inches, the pressure distribution can be represented by

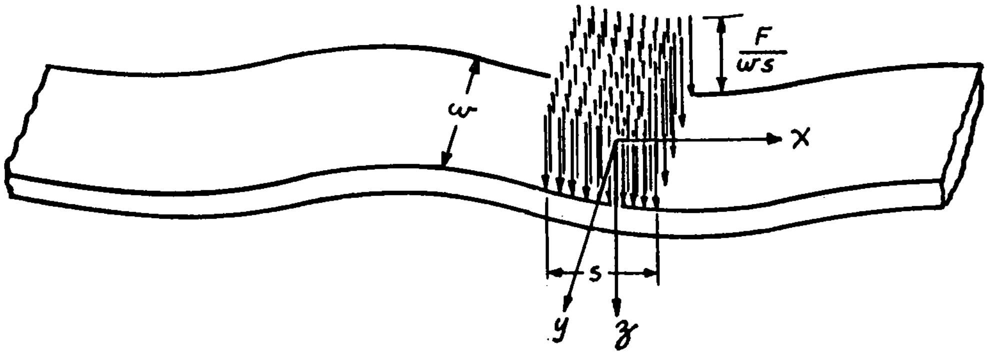

Figure 2. Relation of static pressure distribution $P(x)$ and applied load force F.

$$P(x) = \frac{F}{w}\delta'(x) \quad (4)$$

In the static analysis, all time derivatives can be set equal to zero, and Eq. 3 reduces to the ordinary differential equation

$$D\frac{d^4u}{dx^4} + k\,u = \frac{F}{w}\delta'(x) \quad (5)$$

The solution of Eq. 5 for both infinite and finite length road slabs has been performed elsewhere (*19*, pp. 1-20). The boundary conditions for the infinite road are that the deflection and slope of the deflection curve are zero at $x = \infty$; that is,

$$u(\infty) = 0, \qquad du/dx(\infty) = 0 \quad (6)$$

Also, the slope of the deflection curve is zero beneath the load (because of symmetry in the x-z plane) and the shear at the origin is equal to one-half the magnitude of the load. Expressed mathematically,

$$\frac{du}{dx}(0) = 0, \quad V_x(0) = -D\frac{d^3u}{dx^3} = -\frac{F}{2w} \quad (7)$$

These four boundary conditions can be combined with the general solution of Eq. 3 to give:

$$u(x) = u(0)e^{-\beta x}(\cos\beta\,x + \sin\beta\,x) \quad (8a)$$

$$u(0) = \frac{F\beta}{2k\,w} \quad (8b)$$

$$M(x) = M_x(0)e^{-\beta x}(\cos\beta\,x - \sin\beta\,x) \quad (9a)$$

$$M_x(0) = \frac{F}{4\beta\,w} \quad (9b)$$

in which

$$\beta = \sqrt[4]{\frac{k}{4D}} \quad (10a)$$

and

$$D = \frac{E\,h^3}{12(1-\mu^2)} \quad (10b)$$

The static deflection and bending moment profiles that are represented by Eqs. 8 and 9 are shown as the solid line plots of Figures 6 and 7.

One of the most important parameters in the prediction of the static deflection profile is the factor β, as defined by Eq. 10a. The term β represents the space frequency of the deflection profile wave and is a direct measure of the ratio of the subgrade reaction stiffness to the bending rigidity of the pavement material. The length of the space wave, λ, resulting from a concentrated load on a long uniformly supported pavement section is

$$\lambda = \frac{2\pi}{\beta} \quad (11)$$

The wavelength, λ, decreases as β increases; β can increase by increasing the subgrade spring modulus, k, and/or by decreasing the bending rigidity, D, of the surface material. Thus, for a given subgrade modulus, concrete surfaces generally exhibit longer surface waves than asphaltic surfaces because of the larger bending rigidity of the concrete. This result is borne out by actual test.

An interesting comparison can be made between the narrow road theory and the infinite uniformly supported circular-plate results reported by Westergaard (*11*). Figure 3 shows a comparison of the normalized deflection profiles computed by the two methods and plotted against a common abscissa scale, βx. The agreement indicates that pavement width effects on the longitudinal deflection profile are small, because the Timoshenko analysis is based on unit pavement width, and the Westergaard analysis is based on an infinite pavement width (circular plate radius). In the comparison of bending moment profiles (Fig. 4), it is evident that agreement between the two methods is good, except in the vicinity of the point of load application. The Westergaard analysis (*11*), based on thin plate theory, requires an infinite bending moment in the pavement directly beneath the load. To circumvent this problem, Westergaard used thick plate theory to obtain a local bending moment of finite value, even when the loading area reduced to zero. The local bending moment that results from narrow beam theory is finite for a zero loading area, as indicated in Figure 4, and is of smaller magnitude than that produced by the Westergaard thick plate theory. It is desirable to use the narrow beam theory whenever possible, because of its simplicity when compared with the circular plate theory.

The foregoing analysis deals with the static deflections of a very long pavement section caused by a single concentrated load. Of equal interest are the deflections and moments caused by a concentrated load acting at the center of a slab whose length is shorter than one space wavelength but greater than the width of the road; that is, $w<l<\lambda$. The boundary conditions for this case depend on the method of jointing the pavement. One assumption is that both the shear and the bending moment vanish at the extremes in a given pavement slab. Expressed mathematically

$$V_x(l/2)=V_x(-l/2)=0 \qquad (12)$$

$$M_x(l/2)=M_x(-l/2)=0 \qquad (13)$$

The analysis technique used was that suggested by Timoshenko (*19*, pp. 15–20), and is originally due to Hetényi. The resultant equations are long and cumbersome and are not included in this analysis, except for the following three relations:

$$u(l/2)=u(-l/2)=2\frac{F}{k}\frac{\beta}{w}\left[\frac{\cos h\,\frac{\beta l}{2}\cos\frac{\beta l}{2}}{\sin h\,\beta l+\sin\beta l}\right] \qquad (14)$$

$$u(0)=\frac{F}{2k}\frac{\beta}{w}\left[\frac{\cos h\,\beta l+\cos\beta l+2}{\sin h\,\beta l+\sin\beta l}\right] \qquad (15)$$

$$M_x(0)=\frac{F}{4\beta w}\left[\frac{\cos h\,\beta l-\cos\beta l}{\sin h\,\beta l+\sin\beta l}\right] \qquad (16)$$

Eq. 14 allows computation of the end deflections of the pavement slab. The increase in deflection under the load due to the removal of the subgrade reaction beyond the ends of the slab is given in Eq. 15. Similarly, the decrease in bending moment under the load can be determined from Eq. 16.

The deflection profile of a finite slab length is plotted in Figure 5 for

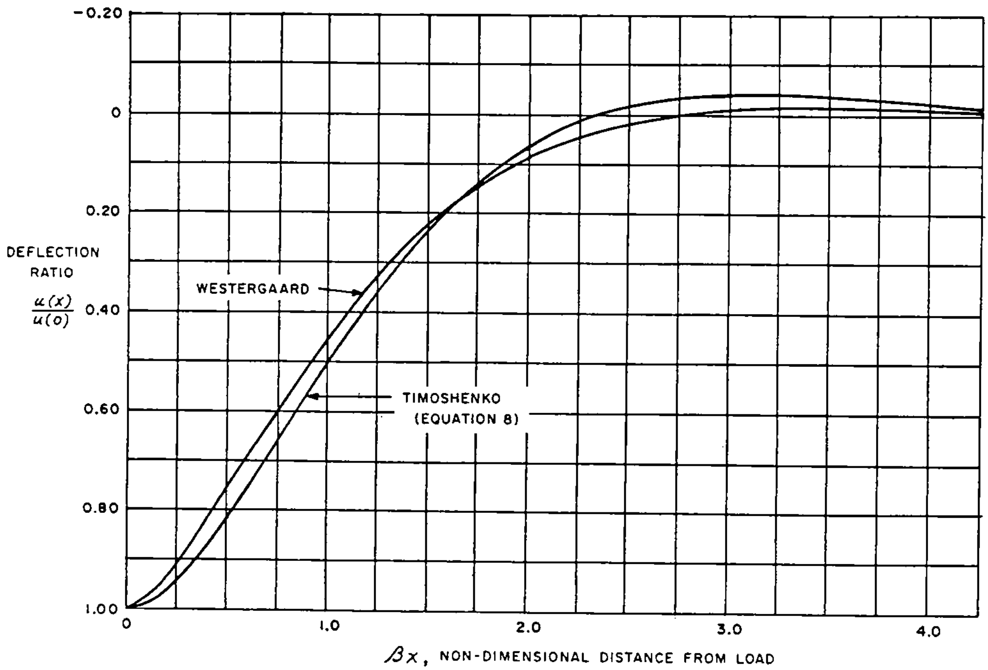

Figure 3. Static comparison of Westergaard circular plate analysis with narrow beam theory; deflection profile.

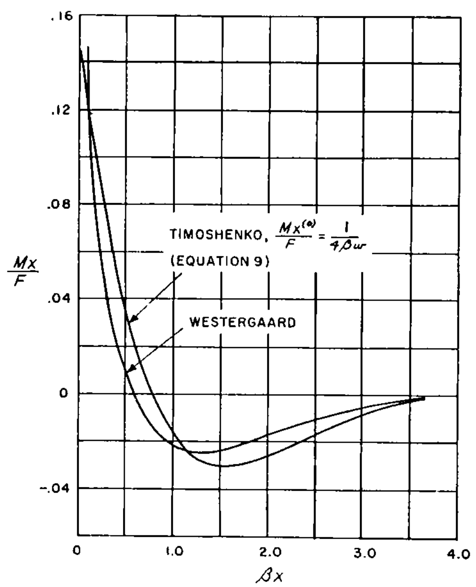

Figure 4. Static comparison of Westergaard circular plate analysis with narrow beam theory; bending moments.

several values of dimensionless slab length βl. In Figure 5, the factor ρ that multiplies the ordinate scale indicates the amount by which the load deflection is increased as the length of the pavement slab is decreased (for a constant load) relative to the deflection beneath the infinite slab length road. It should be noted that a decrease in slab length (with β held constant) results in an increase in deflections and a decrease in bending compared to the long ribbon-pavement response (shown as a dashed line in Fig. 5). Actually, the important parameter is the βl product, which completely determines the static deflection and bending moment response of the road to a single concentrated load. It is not necessary to know β or l individually if all that is desired is the deflection and moment profiles normalized to the values beneath the load. The factors β and l must be individually specified, however, if the actual values of load deflection and pavement bending moment are desired.

TABLE 1

STATIC SOLUTION FOR CENTERLINE DEFLECTIONS OF A LONG CONTINUOUS RIBBON PAVEMENT ROAD

Sta No.	u(x, 0) Equation 8	Analog Computer	Digital Computer
1	0.0118	0.0113	0.0115
2	0.00575	0.00560	0.00562
3	0.00139	0.00152	0.00151
4	−0.000331	−0.000124	−0.000140
5	−0.000548	−0.000420	−0.000420
6	−0.000305	−0.000270	−0.000265
7	−0.0000917	−0.000100	−0.0000989
8	0.0000122	−0.0000180	−0.0000180

Normalized data plotted in Figure 5. Computed for $D=2.56\times10^8$ in.-lb, $k=50$ lb/in.^{3}, $\beta=0.0149$ in.$^{-1}$, $w=114$ in., $h=10$ in., $F=10{,}000$ lb, $u(0, 0)=0.0132$ in. (Eq. 8b).

Figure 5 also demonstrates that the analysis procedure to be used in pavement deflection problems depends on the βl product. The analysis can be separated into three categories (*19*, p. 20), as follows:

(a) Short slabs, $\beta l<0.60$
(b) Medium length slabs, $0.60<\beta l<5$
(c) Long slabs, $\beta l>5$

In case (a), the slab is short enough that pavement bending strength far exceeds subgrade reaction, and the entire slab is pushed into the subgrade as a rigid body. Here, the deflection is constant along the slab and is given by

$$u(x)=\frac{F}{k\,w\,l} \tag{17}$$

and the bending moment is zero. In case (c), the slab is long enough that the end appears to be an infinite distance from the load. The deflections are given by Eq. 8 and the bending moment by Eq. 9. The medium slab-length analysis of case (b) fits neither of the simplifications of cases (a) or (c), and must be analyzed as previously explained.

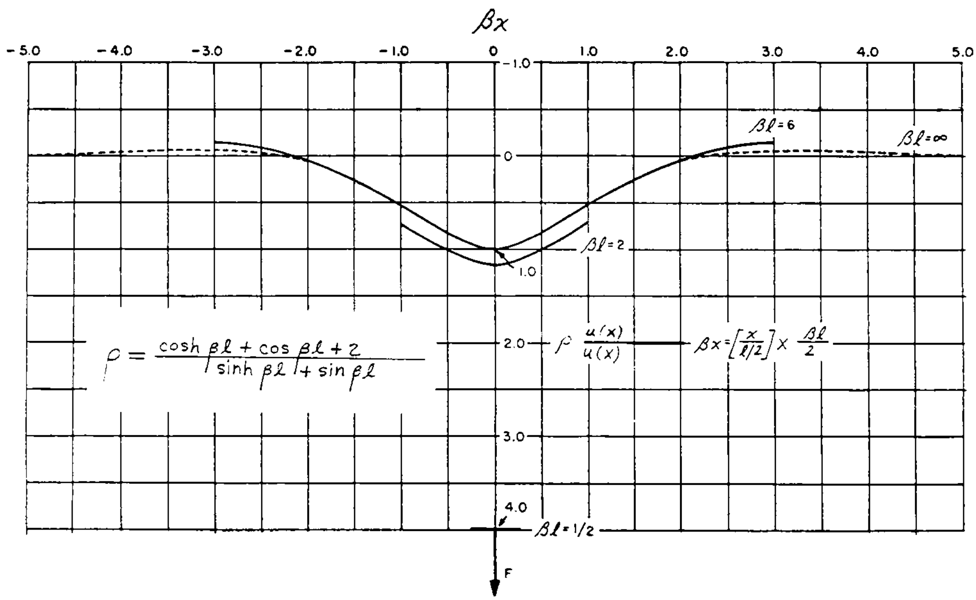

Figure 5. Effect of βl product on static deflection beneath the load, and deflection profile.

It should be noted that all of the foregoing conclusions can be reached without assuming numerical values for the system parameters other than the βl product. A plot of the bending moment profile for $\beta l=2$ (case b) is given in Figure 13. The corresponding deflection profile plot is shown in Figure 12.

Potential Energy Distribution

The static deflection of a road caused by an applied force requires the storage of potential energy in the subgrade and the pavement because of their elastic properties. It is of interest to investigate the distribution of input energy into the pavement surface and subgrade for various boundary conditions. The energy stored in the subgrade can be determined from

$$U_s=\frac{1}{2}k\int_{-w/2}^{w/2}\int_{-l/2}^{l/2}u^2(x, y)\,dx\,dy \quad (18)$$

Similarly, the energy stored in the pavement because of bending is given by

$$U_R=\frac{1}{2}D\int_{-w/2}^{w/2}\int_{-l/2}^{l/2}\left[\left(\frac{\partial^2 u}{\partial x^2}\right)^2+\left(\frac{\partial^2 u}{\partial y^2}\right)^2 +2\mu\left(\frac{\partial^2 u}{\partial x^2}\right)\left(\frac{\partial^2 u}{\partial y^2}\right)\right]dx\,dy \quad (19)$$

Eq. 18 results from summing the energy stored in each differential element of subgrade over the entire area of paved surface. Eq. 19 sums the bending energy in each element of pavement material over the entire pavement surface area. It neglects energy storage in the twisting of the pavement surface, and is derived by Timoshenko (*20*). The potential energy storage due to lateral road contraction can be shown to be negligible when compared with pavement bending strain energy. Consequently, the term

$$2\mu\left(\frac{\partial^2 u}{\partial x^2}\right)\left(\frac{\partial^2 u}{\partial y^2}\right)$$

in Eq. 19 is neglected in the following.

To evaluate the division of potential energy between the subgrade and the pavement using Eqs. 18 and 19 it is necessary to have quantitative knowledge of the variation in deflection due to inclusion of the effects of road width in the analysis. A rigorous analysis of road width effects requires the static solution of the bi-harmonic Eq. 1 with the proper road-edge boundary conditions included. A previous report (*16*), however, indicates that road width effects can be approximated quite satisfactorily by assuming that the width and length solutions can be separated and considered to be independent of each other. Based on this assumption, the static road deflection is given (*16*) by

$$u(x, y)=u(0, 0)e^{-\beta x}[\cos\beta x+\sin\beta x] \times\left[1-\frac{1}{4}(\beta w)^2 N\left\{\frac{1}{6}\left(\frac{2y}{w}\right)^4-\frac{2}{3}\left(\frac{2y}{w}\right)^3+\left(\frac{2y}{w}\right)^2\right\}\right] \qquad \beta w<2 \quad (20a)$$

$$u(x, y)=u(0, 0)\,e^{-\beta(x+y)} \times[\cos\beta(x-y)+\sin\beta(x+y)] \qquad \beta w>5 \quad (20b)$$

in which

$$N=\frac{\cos h\,\beta w-\cos\beta w}{2+\cos h\,\beta w+\cos\beta w} \quad (21)$$

and

$$u(0, 0)=\frac{F\beta}{2k\,w}$$

Combining Eqs. 18, 19, and 20a, the energy division between pavement and subgrade is given for $\beta w<2$ and a very long pavement slab ($\beta l>5$) by

$$U_S=\frac{3}{4}\frac{kw}{\beta}u(0,\ 0)^2$$

$$\times\left[1-\frac{1}{10}(\beta w)^2N+\frac{13}{3{,}240}(\beta w)^4N^2\right] \quad (22)$$

$$U_R=\frac{1}{4}\frac{kw}{\beta}u(0,\ 0)^2$$

$$\times\left[1-\frac{1}{10}(\beta w)^2N+\frac{13}{3{,}240}(\beta w)^4N^2\right] \quad (23)$$

For a very long and very wide pavement (βl and βw both >5), Eqs. 18, 19, and 20b yield:

$$U_S=\frac{9}{8}\frac{k}{\beta^2}u(0,\ 0)^2 \quad (24)$$

$$U_R=\frac{3}{4}\frac{k}{\beta^2}u(0,\ 0)^2 \quad (25)$$

The division of potential energy in the range $2<\beta w<5$ was not investigated because of the lack of a convenient approximation for $u(x,y)$. It is to be expected, however, that the division of potential energy will vary smoothly from narrow to wide road conditions.

The division of potential energy storage between the subgrade and the pavement can be computed from Eqs. 22, 23, 24, and 25 for several values of the space frequency-road width (βw) parameter. The results:

		(Energy %)	
Road Type	βw	Subgrade	Pavt. Strain
Narrow	0	75	25
Medium	2	66	34
Wide	5	60	40

show that the subgrade carries primary responsibility for potential energy storage in a long road, regardless of its width. This leads to the well-known conclusion that subgrade design is at least as important as pavement design. It should be noted that increases in road width result in a smaller percentage of subgrade energy storage, indicating that subgrade design is more critical for narrow roads than for wide ones. Also, the only parameter that affects the potential energy division is βw, where β involves only subgrade modulus and pavement bending rigidity (Eq. 10a). No other parameters are involved in the division of input energy into subgrade and pavement (N is a function of βw also; (see Eq. 21).

It is possible to compute the static load deflection from Eqs. 22, 23, 24 and 25 and the knowledge that the total potential energy supplied to the road by the load is

$$U_T=\tfrac{1}{2}Fu(0,\ 0) \quad (26)$$

The total potential energy is the sum of the subgrade and pavement energies; that is,

$$U_T=U_S+U_R \quad (27)$$

Combining Eqs. 22, 23, 26, and 27, the load deflection for $0\leqslant\beta w\leqslant 2$ is computed to be

$$u(0,\ 0)=\frac{F\beta}{2k\ w}\left[\frac{1}{1-\frac{1}{10}(\beta w)^2+\frac{13}{3{,}240}(\beta w)^4N^2}\right] \quad (28)$$

For the case of a very wide road ($\beta w>5$) Eqs. 24, 25, and 26 combine to give

$$u(0,\ 0)=\frac{4}{15}\frac{F\beta^2}{k} \quad (29)$$

The exact solution to the wide road case was shown (*11*) to be

$$u(0,\ 0)=\frac{1}{4}\frac{F\beta^2}{k}$$

Eq. 29 is within 7 percent of the correct theoretical value, confirming the validity of Eq. 20b for values of $\beta\omega > 5$.

The equations of this section allow approximate prediction of the static deflection and bending moment at any point on a long ($\beta l > 5$), jointless, paved road in response to a centrally located concentrated load. Bending moment values can be converted to tensile stress in the bottom fibers of the pavement by

$$\sigma_{x,y} = \frac{M_{x,y} h}{2I} \qquad (30)$$

where $h/2$ is the distance from the neutral axis to the bottom of the pavement slab. Subscripts x,y of Eq. 30 indicate either x (longitudinal) or y (lateral) stress.

DYNAMIC ANALYSIS TECHNIQUE

Difference-Differential Equations

In direct contrast to the present substantial knowledge of the dynamic behavior of vehicles, little is known about the dynamic behavior of roads. Obviously, the situation is complex, because a road is a distributed parameter system and its response is governed by several independent variables. Vehicles have been successfully analyzed as lumped parameter systems, but roads must be considered to be distributed. As a direct result of the distributed nature of a road, and the necessity of describing its dynamic response in terms of both space and time as independent variables, it becomes necessary to resort to partial differential equations in an analysis of the forces and moments acting on any given small element of road. Integration of these partial differential equations in closed mathematical form is possible only in certain special cases, subject to the initial and boundary conditions of the problem. In general, it is necessary to integrate the equations by resorting to numerical analysis techniques and electronic computing machines.

The particular partial differential equation that must be solved to yield the dynamic road response to applied loading forces is Eq. 3. In keeping with the static analyses of the previous section, the dynamic response of the road to a time-varying, centrally located, concentrated load force is solved for. The dynamic load distribution for this case becomes

$$q(x,t) = \frac{F(t)}{w}\delta'(x) - m\frac{\partial^2 u}{\partial t^2} - c\frac{\partial u}{\partial t} - k\,u$$

Combining this with Eq. 3, gives

$$D\frac{\partial^4 u}{\partial x^4} + m\frac{\partial^2 u}{\partial t^2} + c\frac{\partial u}{\partial t} + k\,u = \frac{F(t)}{w}\delta'(x) \qquad (31)$$

The solution of Eq. 31 for realistic boundary conditions is not possible in closed mathematical form. Previous researchers (*7, 14*) have obtained the frequency response of the road to steady-state sinusoidal excitation forces, using this equation (with no damping), but no solutions have been found in the literature that deal with the response of the road to a generalized time-varying loading force. The most practical way of obtaining the desired dynamic road response is by the application of specialized approximation techniques, of which the difference-differential equation method is the most straightforward.

The difference-differential equation technique is fully described elsewhere (*15, 16, 26*). In Eq. 31, the terms $m(\partial^2 u/\partial t^2)$ and $c(\partial u/\partial t)$ represent the D'Alembert inertial reaction and the subgrade viscous-damping-reaction pressures that occur when a differential pavement element is changing velocity. The term $D(\partial^4 u/\partial x^4)$ is the variation in subgrade pressure due to bending of the pavement, and the term $k\,u$ represents the subgrade pressure at any given point on the road due to the subgrade reaction.

Combining the difference equation approximation of the fourth space derivative (*16*) with the partial differential Eq. 31 results in

$$D\left[\frac{u_{n+2}-4u_{n+1}+6u_n-4u_{n-1}+u_{n-2}}{(\Delta x)^4}\right]$$
$$+m\frac{d^2u_n}{dt^2}+c\frac{du_n}{dt}+k\,u_n=\frac{F(t)}{w\Delta x}\delta'(x_1) \qquad (32)$$

Eq. 32 constitutes a system of ordinary simultaneous differential equations with time as their common independent variable. Before Eq. 32 can be solved, it is necessary to specify four boundary conditions on the space variable, and two initial conditions on the time variable. The reduction of the system to a single independent variable, time, makes it convenient to use analog computing equipment to solve the equations, as long as the required equation coefficient accuracy is not too great (see *16*, Section 5D for a discussion of this point). Analog computers are limited to the use of time as an independent variable and are ideally suited to solving simultaneous linear ordinary differential equations continuously in time.

Continuous Pavement Dynamics

The particular road configuration selected for analysis is a long narrow road with no joints. It has already been demonstrated, in the static case, that values of $\beta w>5$ satisfy this condition. The difference-differential equation technique is normally limited to systems with finite boundaries; that is, the range of the independent variable that has been replaced by the difference equations cannot be infinite, because the number of space increments must be finite. If, however, the static solution values decay to zero beyond a specific finite distance from the load, it is possible to compute the dynamic response of an infinitely long road for the period of time necessary for the road wave to travel to the boundary. No boundary reflections can occur, of course, in the actual system, because the road perturbations can never reach the end of an infinite pavement slab to reflect back toward the load. With damping in the subgrade, an end station can always be found a finite distance from the load where the damping reduces the amplitude of road deflections to very small values, so that no reflected wave can occur. In this case, the solutions obtained in the vicinity of the load application point would be valid both statically and dynamically.

Unfortunately, when subgrade damping is small the large number of computer stations required to allow the pavement deflections to become negligible places the problem beyond the capabilities of most analog computing equipment. Consequently, the present analysis compromises the number of analog computer stations with dynamic solution accuracy for the period of time between the first reflection from the false boundary and the steady-state response. The responses are valid from time $t=0$ until this reflection occurs and after the steady-state is reached, if the pavement slab is considered to be infinite. If the pavement slab is considered to be a long (but finite length) slab that is freely hinged at the boundaries, the dynamic responses obtained herein are representative of the time history of road behavior for all time intervals.

The boundary conditions chosen to approximate the infinite road-length solution were established so that the actual end of the pavement slab occurs when $\beta l=12.7$. At this point, and beyond, the road deflection and slope were forced to be zero. Because the load is centrally located, and its velocity along the road is zero, symmetry of road profile exists around the load in the x-z plane. This fact can be used to advantage to halve the number of computing stations required for a given detail of solution. The following parameters were assumed in the dynamic road analysis for a concrete pavement:

Pavement and subgrade properties

$\mu = 0.15$
$k = 50$ lb/in.^{3}
$E = 3.0 \times 10^6$ psi
$m = 0.00294$ lb $=$ sec^2/in.^{3}
$c = 0.153$ lb $=$ sec./in.^{3}

Pavement slab geometry

$h = 10$ in.
$w = 114$ in.
$l = 850$ in.

Combination road parameters

$D = 2.56 \times 10^8$ in.-lb
$\beta = 0.0149$ rad/in.

$$\omega_n = \sqrt{\frac{k}{m}} = 130 \text{ rad/sec}$$

$$\xi = \frac{c}{2\sqrt{km}} = 0.20$$

$$\lambda = \frac{2\pi}{\beta} = 422 \text{ in.}$$

The value of Poisson's ratio and Young's modulus for concrete, as well as the subgrade stiffness value, were selected as typical values (*11*) for a concrete pavement. The subgrade stiffness value is quite arbitrary, and not at all critical, because the value of β (which controls static deflection profile) depends on the fourth root of the subgrade modulus. Changes in subgrade modulus will affect the road vibrational natural frequency more than the static road profile, because of the road vibration natural frequency dependence on the square root of subgrade modulus.

In the foregoing listing of road parameters, the first two groups are independent parameters; that is, they are primary properties of the system. The third group consists of values that are computed from numbers contained in the first two groups. The natural frequency and damping ratio previously computed are the uncoupled values that result from the simple harmonic motion of a small area plate uniformly supported on the elastic subgrade, with no bending of the plate. In actual practice these values are dependent on the size and stiffness of the loading area, which is an expected result, because the larger or less stiff loading plates bend as they are depressed into the subgrade, thereby reducing the effective subgrade stiffness and the natural frequency. With larger plates, the entire plate may not be in contact with the subgrade at all times, especially when the input force is large. This would also tend to decrease the fundamental natural frequency of pavement vibration. These effects, discussed elsewhere (*7*), result in apparent nonlinear subgrade reaction.

A difference-differential equation analog computer program was written to solve for the dynamic behavior of a concrete paved road with parameter values as just noted. Stations 0 and 1 were defined as adjacent to, and on either side of, the point of load application, and the analysis was performed with nine stations located as shown in Figure 6. For this case, $\Delta x = 50$ in. The required four boundary conditions are

$$\left.\begin{aligned} u_0 &= u_1 \\ u_{-1} &= u_2 \end{aligned}\right\} \text{due to symmetry.}$$

$u_9 = u_{10} = 0$ due to decay of pavement deflections and slope beyond one space wavelength.

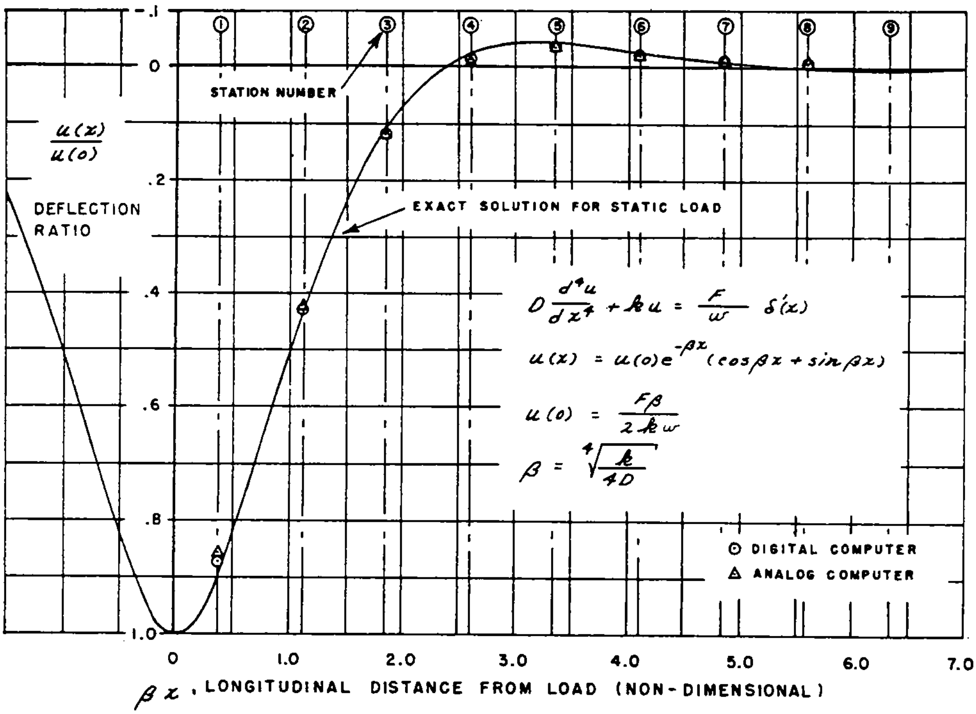

Figure 6. Static longitudinal road deflection profile.

The solid curve in Figure 6 shows the rapid decay of the deflection profile away from the load. The assumption of zero deflection and slope at Station 9 and beyond is verified in this figure. Combining the boundary conditions with Eq. 32 for all eight stations results in a set of eight simultaneous linear ordinary differential equations. These equations were solved on an electronic analog computer for a step-function input of load force. The time history solutions are given in Figures 8, 9, 10, and 11.

The shape of the dynamic deflection and bending moment profiles is completely determined for a given driving force ratio $F(t)/F_0$ and number of stations by the βl product, the natural frequency ω_n, and the damping ratio ξ, where:

$$\frac{m\Delta x^4}{D}=\frac{4}{(i-1)^4}(\beta l)^4 \qquad (33)$$

$$\frac{c\Delta x^4}{D}=\frac{4}{(i-1)^4}(\beta l)^4\left(\frac{2\xi}{\omega_n}\right) \qquad (34)$$

The fact is of interest because it completely defines the road dynamics as a function of certain combinations of parameters and does not require the explicit knowledge of the magnitude of the individual parameters. Thus, it should be possible to predict the dynamic road response to a time-varying road load from relatively easily measured quantities.

The initial values of the pavement deflections and vertical velocities are assumed to be zero in the solution of the equations. These values are completely arbitrary, but must be specified at each station in the computer program.

Figures 6, 7, 8, 9, 10, and 11 show the results given by the analog computer. Figures 6 and 7 give the

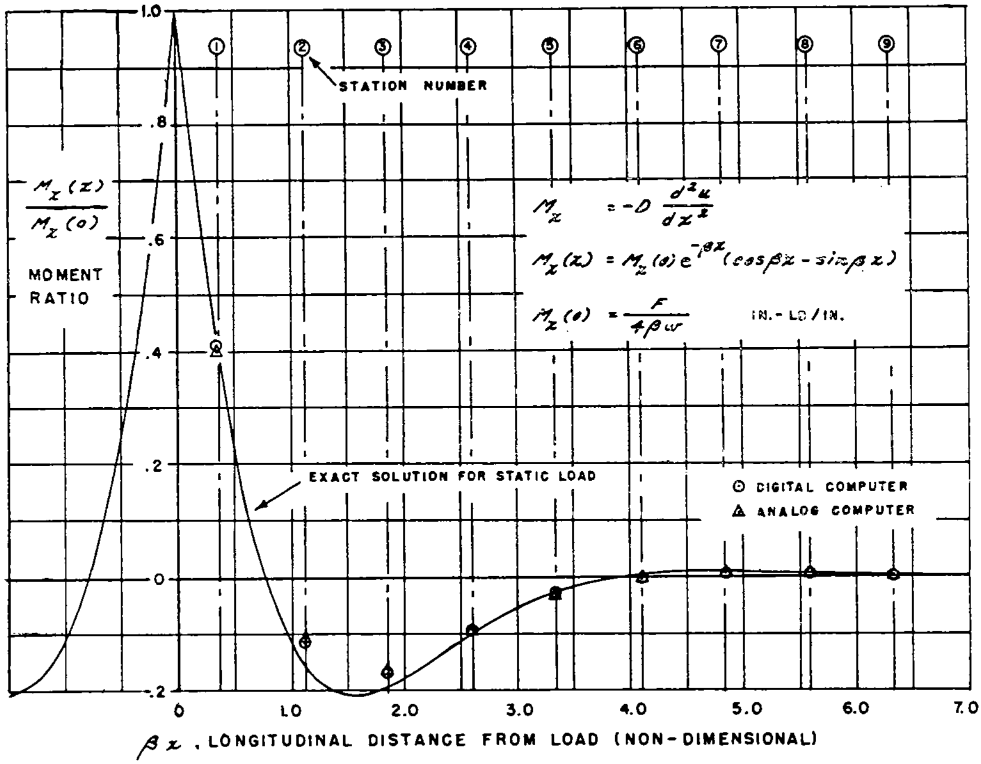

Figure 7. Static longitudinal bending moment.

exact static pavement deflection and bending moment profiles along with the steady-state analog computer solutions. The static profile was also obtained by a simultaneous solution of Eq. 32 with all time-dependent terms equated to zero. This solution, performed on a digital computer, represents the exact difference equation solution of the problem. Any differences that exist between the analog and digital computer results are due to analog computer errors. Differences between the digital computer points and the exact solution curve are due to error in the difference equation technique.

Figures 10 and 11 show the dynamic time-history response of a long pavement slab after the sudden application of a 7,500-lb loading force. It should be noted that the magnitude and time history of the road loading force is completely arbitrary in this analysis. The step function was selected for convenience in the computer mechanization and because it would be the least trouble to duplicate experimentally.

It should be noted in Figures 8, 9, 10, and 11 that the peak deflection and bending moments near the load both exceed the static values by a significant amount and occur somewhat later than the time of load application. The transient is completely decayed in 0.2 sec, but this decay time is dependent on the amount of damping in the subgrade. The smaller the damping, the longer the decay time. The amount of time required for the peak deflection and bending moment to occur is also a function of subgrade damping. The value of sub-

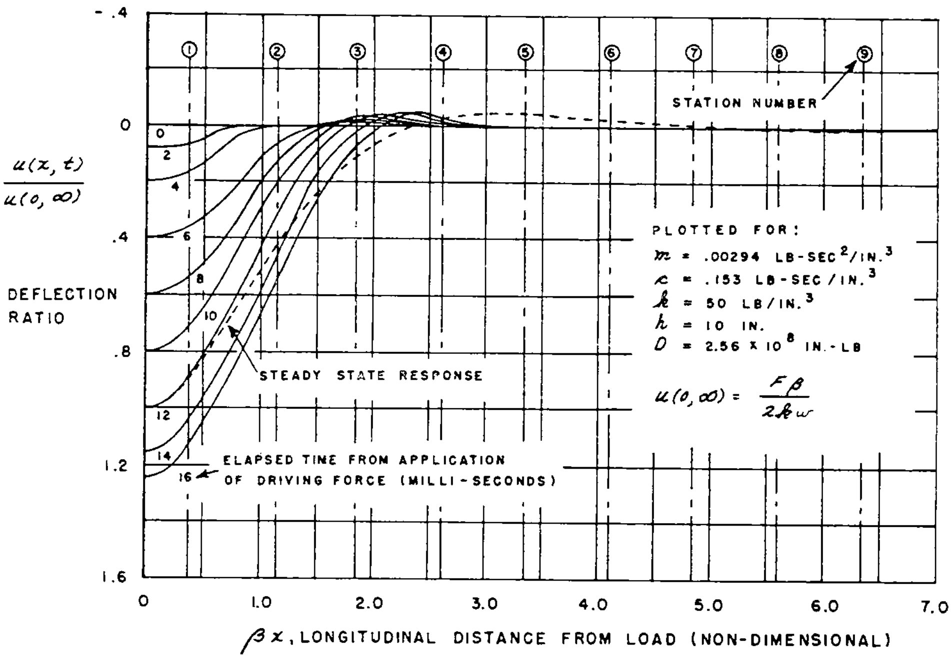

Figure 8. Road deflection dynamic response to a step-function of applied force.

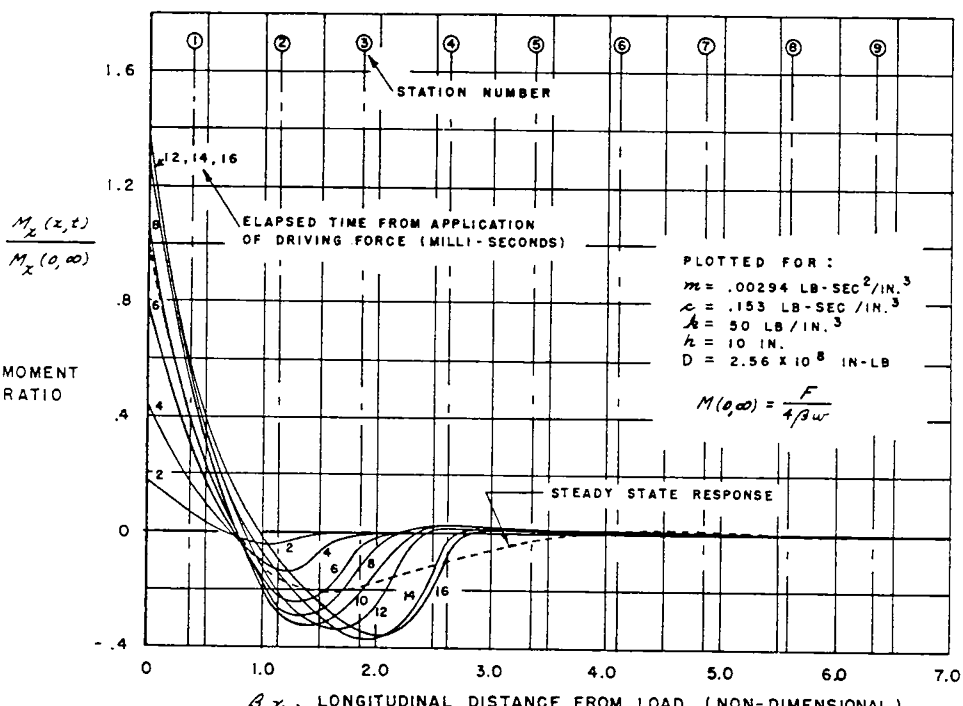

Figure 9. Road bending moment response to a step-function of applied force.

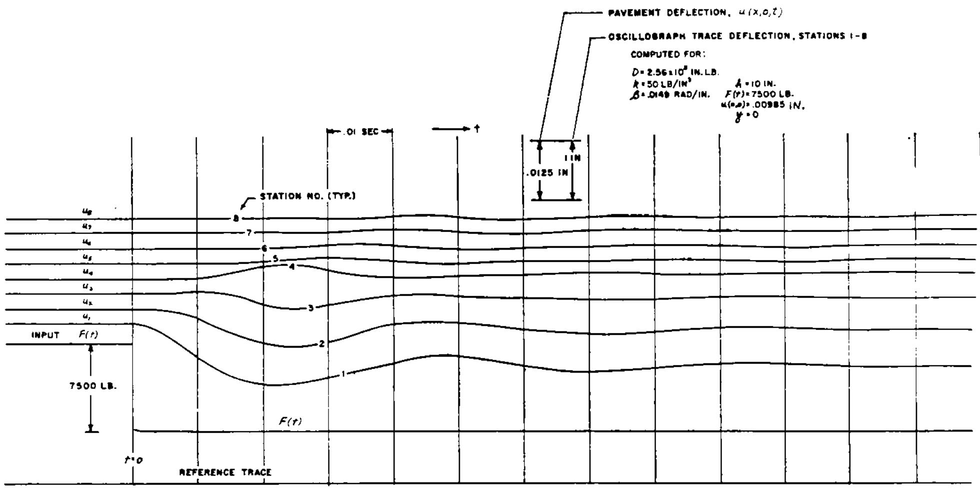

Figure 10. Analog computer solution for centerline pavement displacement step-function response.

grade damping chosen for this solution was selected arbitrarily, and was based on an assumed subgrade damping ratio ($\xi = 0.2$) that was selected as a reasonable value without experimental verification.

Figures 8 and 9 represent the longitudinal pavement deflection and bending moment response to a step function of applied force with time as a parameter. The dynamic response for the first 16 milli-seconds after the application of the road loading force is shown. After about 20 milli-seconds reflections occur at the boundary and the results are not valid for an infinite slab length until the vibrations damp out. The steady-state solution is given by the dashed line in Figures 8 and 9. The buildup

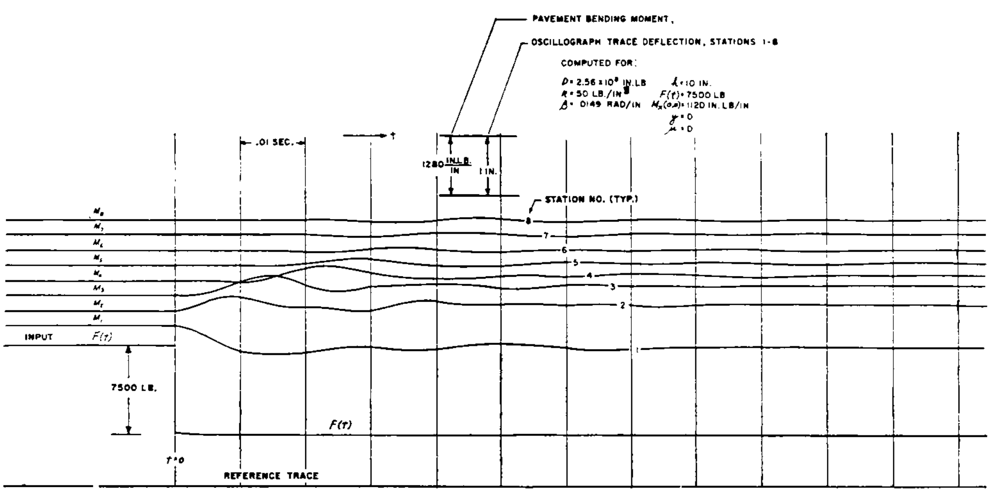

Figure 11. Analog computer solution for longitudinal centerline pavement bending moment step-function response.

of longitudinal wave motion in the solutions should be noted. The phase velocity of these waves is not constant, but is approximately 300 fps and is independent of subgrade damping. The velocity of sound in the pavement is much higher (5,610 fps). Recalling that the load is spatially stationary in this model, the question arises: What effect does a longitudinally moving load have on the dynamic behavior of the road? Because vehicles can traverse the road at speeds that are an appreciable percentage of the phase velocity, it would seem that vehicle speed affects the dynamic road behavior because of pavement wave mechanics. An additional characteristic of a moving load is stress relief caused by not allowing the load to dwell at one point long enough for the road vibration transient to reach its peak value. The present solution technique does not allow the quantitative evaluation of the effects of longitudinal load velocity on dynamic road response, but this subject is currently being pursued at the Cornell Aeronautical Laboratory.

DISCUSSION OF RESULTS

The results of the kinetic road analysis program are discussed in separate sections dealing with statics and dynamics.

Static Road Response

In the static analysis, the road model is assumed to be a rectangular plate of known dimensions. The theory developed results in closed-form mathematical expressions to predict the shape and magnitude of the deflection profile (Fig. 12) and the distribution of internal tensile stresses (Fig. 13) caused by a centrally located concentrated load force acting on a uniformly supported pavement slab with various edge boundary conditions.

It is shown that a single partial differential equation (Eq. 1) can be used to represent the road model, but the solution of the boundary value problem results in different road characteristics that are dependent on the size and form of the loading area(s), the location of the loading area(s) relative to the boundaries of the pavement slab, and the boundary conditions. It is obvious that no single road model solution exists, or can be derived, to represent the road response to all possible combinations of load intensity and boundary conditions. This was also shown to be true in the vehicle model synthesis of Fabian (*4*).

The difference equation technique is a powerful tool in the solution of the road equation. It gives accurate results with less effort than is generally necessary to integrate the road equation mathematically and apply the boundary conditions to eliminate integrating constants. The problem boundary conditions can be applied to the difference equations in an extremely simple and straightforward manner. The major drawback to this method is that the complexity of solution increases rapidly as more detail (that is, more computing stations) is required. Access to a digital computer greatly alleviates this problem, however. It should be noted that the road response will be characteristically smooth, with no discontinuities in the deflection profile, as long as the elastic limit is not exceeded in the pavement material. The difference equation technique is ideally suited to problems of this sort. The method can also be extended to solve the three-dimensional road equation at the expense of added solution complexity.

The shape of the static longitudinal pavement deflection and bending moment profiles can be determined with a knowledge of a single road parameter, the βl product. If βl is numerically less than 0.60, the pavement bending rigidity is so much greater

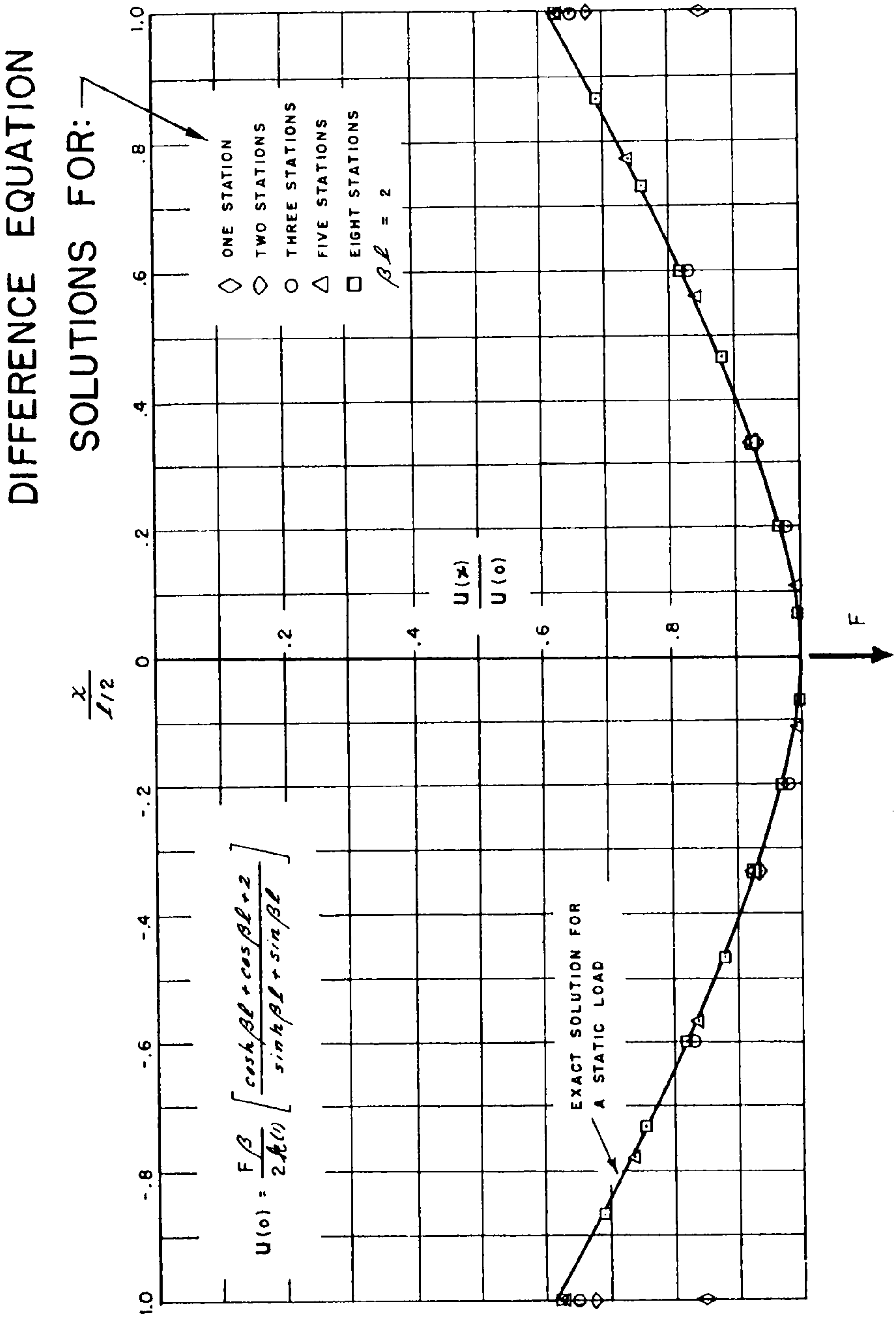

Figure 12. Static longitudinal deflection profile, finite length slab (averaged shear boundary conditions).

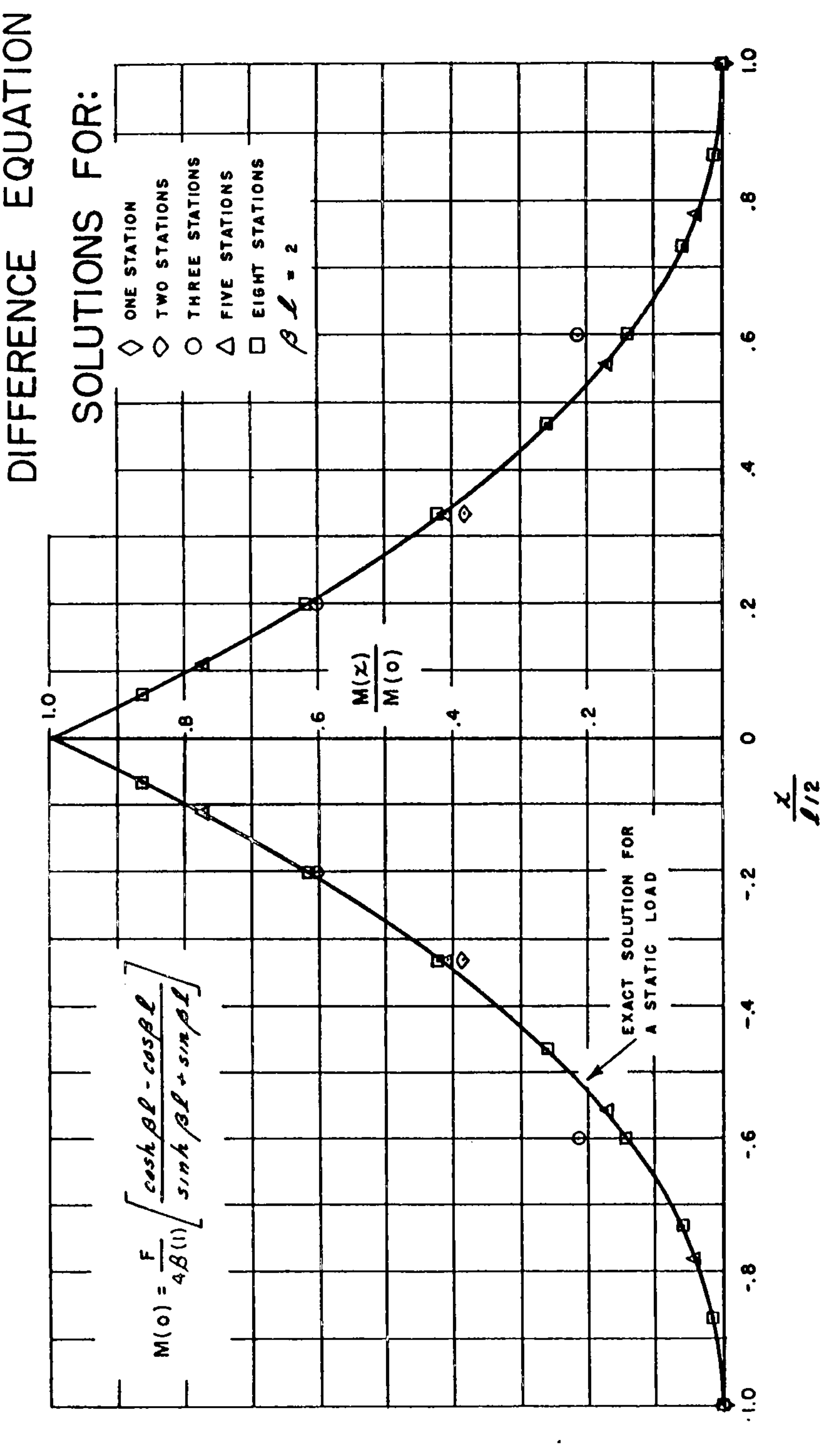

Figure 13. Static longitudinal bending moment, finite length slab (averaged shear boundary conditions).

than the subgrade modulus, and/or the effective lever arm in the generation of bending moment is so small, that the entire loaded pavement slab is impressed into the subgrade with no bending. When βl lies between 0.60 and 5, the deflection profile can be obtained by the relatively complex method of superposition, as explained by Timoshenko (*19*). Values of βl greater than 5 result in a deflection profile the same as the solution of an infinite length pavement slab.

The subgrade is the primary potential energy storage element of the road. When the road is narrow, 75 percent of the potential energy is stored in the subgrade, with the remainder stored in the pavement because of bending strain. As the width increases, the percentage of total potential energy stored in the subgrade decreases, but always remains greater than 50 percent. From these results, it is concluded that subgrade design is especially important for narrow roads, and remains important even for paved surfaces of large expanse in all directions. These conclusions are independent of the type of pavement material.

The theory developed in this report is a pavement theory, and no information regarding the state of stress in the subgrade can be obtained from it.

Dynamic Road Response

All of the conclusions relative to static road behavior are special cases of the dynamic road response; they are the steady-state results to be expected due to application of a single concentrated constant load in the geometrical center of the pavement. If the load is caused to vary as a function of the independent variable time, some additional conclusions are derived relative to the dynamic road response.

The analytical model for a narrow road is represented by Eq. 3, which can be solved in convenient fashion by application of difference-differential equations. The use of automatic computing machinery is almost mandatory with this method, but programing is generally straightforward for either analog or digital computation. The road dynamic response (transient and steady state) can be obtained for any loading time history with the results subject to the initial and boundary conditions of the problem. Again, the partial differential equation and the boundary conditions are necessary to specify a particular road model. The results are valid for a load applied at a fixed location on the road. The model is probably not valid for a load in longitudinal motion, unless the load velocity is much less than the phase velocity of the traveling wave that results from distributed longitudinal shear and bending moment coupling.

The road longitudinal transient response cannot be determined from a knowledge of the βl product alone, as in the steady-state response. In addition to βl, the undamped natural frequency and damping ratio of a section of pavement that is geometrically small enough to vibrate on the subgrade without bending must be known. These three parameters are necessary and are sufficient to allow a dynamic solution that is normalized to the static load deflection, or bending moment, as the case may be. It can be concluded from the foregoing that explicit knowledge of the pavement bending rigidity and subgrade modulus, as well as the over-all pavement slab dimensions, is not required except to determine static load deflection and bending moment values. Furthermore, since βl is assumed to be known for the dynamic road response, it is necessary to specify only any two of the three primary parameters k, D, or l, in addition to ω_n and ξ, to determine completely a given system longitudinal response to any given loading time history.

The coefficients of Eq. 3 will be

constant throughout the short-term dynamic solution for transient response. The resultant road model is an "instantaneous model" that yields the dynamic behavior of the road at a given instant in time. It is recognized that changes do occur in the magnitude of the system parameters on a time scale that varies anywhere from seconds to years. However, the road dynamics are represented in terms of milli-seconds. Consequently, even visco-elastic effects that result in time-varying pavement bending rigidity over a period of seconds or minutes do not affect the road transient response materially during the course of a solution. The instantaneous model can be solved many times, changing coefficients from one time to the next, to determine the net results on the road transient response.

The magnitude of the peak dynamic pavement deflections and bending moments can be materially greater than the static values. To reduce the dynamic stress magnification at resonance, the subgrade damping should be as large as possible. Furthermore, the natural frequency of the road should be as high as possible in the initial design in order to separate vehicle-suspension resonant frequencies from the road resonant frequency. The bending rigidity of the pavement material will affect the frequency of road resonance. As bending stiffness decreases, the effective subgrade stiffness will lessen (that is, the static load deflection will increase with a constant load magnitude), resulting in lower road vibration resonant frequencies. The best designs will incorporate the largest practical values of subgrade modulus damping, and bending rigidity, with the lowest value of pavement mass density.

The road models developed herein can be combined with cumulative damage fatigue criteria to predict the expected pavement life in response to arbitrary time-varying loads. The effect of dynamic magnification factors on tensile stress can shorten pavement life materially from the results to be expected from static (or low frequency) fatigue testing methods of today. The ultimate use of the road model may be in the specification of new design criteria to provide longer lasting pavements.

CONCLUSIONS

The following conclusions result from the kinetic road analysis; all relate to the response of an elastic road structure to a centrally located concentrated load force:

1. A single partial differential equation (Eq. 1) is the basic equation of the elastic road model.
2. Solution of the road equation is subject to initial and boundary conditions that change the form of the solution to fit the particular road characteristics.
3. The dynamic road model equation is not easily solved in closed mathematical form; the difference-differential equation technique is a powerful tool in solving the road equation.
4. An approximate analysis technique has been developed to allow prediction of the static deflection profile and bending stresses from closed form mathematical expressions.
5. The static longitudinal and lateral road-deflection profiles, relative to the load deflection, can be determined from a knowledge of the respective βl and βw products alone, where β is the space frequency parameter, l is the slab length, and w is the slab width. For small values of the product (βl, $\beta w < 0.60$), no bending occurs and the pavement is depressed into the subgrade as a rigid body. For medium values (0.60 to <5), the analysis is performed by linear superposition, and no particular simplifying assumptions are evi-

dent. When the product is large (βl, $\beta w > 5$, the pavement responds as an infinite uniformly supported structure, and the analysis generated for that case is valid. This conclusion also relates to pavement bending moments.

6. The subgrade is the primary potential energy storage element in the road system. For a long pavement slab, the subgrade stores 75 percent of the potential energy; as width increases, the percentage of subgrade energy decreases to around 60 percent for a large road width ($\beta w > 5$). The remaining potential energy is stored as pavement strain energy. The long pavement slab results depend on the βw parameter alone. For shorter length pavement slabs, the subgrade stores an even larger percentage of input potential energy.

7. Subgrade design is an important as pavement design, especially for narrow roads.

8. The dynamic analysis technique is valid for any pavement-subgrade combination that does not allow variation of the road parameters during the course of a solution. This includes almost any road type, including those with visco-elastic pavements. Long-term parameter changes can be handled readily by the technique.

9. Road dynamic response, relative to the deflection beneath the load, is specified completely from a knowledge of βl, βw, natural frequency ω_n and damping ratio ξ. The complete response to a given loading force, (that is, actual magnitudes of deflection) can be determined from a knowledge of these parameters plus w and any two of the three primary parameters: subgrade modulus k, pavement bending rigidity D, or slab length l. This conclusion also relates to bending moments.

10. Road design should maximize natural frequency and damping in order to separate road resonances from vehicle-suspension resonances and to decrease the peak dynamic bending moments and deflections incurred in the pavement by a time-varying load force.

11. Peak dynamic pavement deflections and bending moments can be larger than the static values by a factor that depends on internal damping and can vary over large ranges depending on the form and frequency of the input force. For example, a step function of force applied to a long ribbon pavement with small damping results in peak deflections and bending moments beneath the load that are 40 percent greater than the steady-state values.

12. A traveling wave results from forcing the road at a fixed point. The phase velocity (velocity of propagation) is not constant and is much less than the velocity of sound in the pavement material, because of subgrade effects. A vehicle moving down the highway can attain velocities that are an appreciable percentage of the road phase velocity (around 300 fps for the parameters selected in this report).

13. The methods presented in this paper lend themselves readily to expansion to cover more comprehensive road models and boundary conditions.

ACKNOWLEDGMENT

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