Some Mathematical Aspects of the Parking Problem

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The comparison between a parking lot and a telephone switchboard is established, and some of the pertinent results for switchboards are applied to parking. Some new data are analyzed from this point of view.

• ONE interesting aspect of the mathematical theories on road traffic that have grown up in the last decade is that they are very largely new. Although considerable effort has been spent in trying to establish meaningful analogies between road traffic and other physical phenomena, none of it has been very successful. On the contrary, students of traffic flow theory have more and more been compelled to introduce new concepts, define new parameters, and obtain new theorems.

Parking is an exception: fortunate if one wishes to employ classical results quickly, unfortunate if one wishes to develop a new theory. The problem of a parking lot is basically the same as the problem of the telephone switchboard. Inasmuch as the latter has been studied intensively for half a century, engineers interested in the design of parking lots would be well advised to consult the procedures of their colleagues in telephony; for example, Erlang (*1*) and Fry (*2*).

This paper mentions several of the ideas on which the statistical theory of telephone traffic is based, quotes some useful results and shows the significance of these results in the design of parking facilities. It also gives some possibilities for improving the operation of parking meters and concludes with a brief mathematical treatment of a new parking problem.

ANALOGY WITH A SWITCHBOARD

A parking lot consists of a number of parking slots, each of which may be either full or empty. Vehicles arrive from time to time and, if a slot is empty, fill it. After a parking time, the vehicles empty the slot and go away. Each vehicle operates independently of the others; when all slots are filled, vehicles may wait and form a queue (system with queueing) or go away (system with loss). In the terminology of automatic telephony, the vehicles are equivalent to calls, the slots to lines, and parking time to holding time. In both fields, systems with loss are more common, but systems with queueing sometimes occur.

A complete statistical description of such a system consists of a distribution of interarrival times (that is, times between successive arrivals) and a distribution of parking times. In case of a system with queueing, it would also be necessary to specify the queue discipline; usually "first come, first served." Following traditional notation, the mean interarrival time is denoted by $1/\lambda$, the mean parking time by $1/\mu$ and the number of slots by N. The quantity $\rho=\lambda/N\mu$ is called the relative traffic intensity and measures how well the lot is able to deal with the demand. When $\rho<1$, the situation is said to be stable.

The simplest assumption is that the demand for parking occurs at random and that departures are also at random. In a later section, the few data available support this conjecture. If arrivals to the parking lot occur at random with mean rate λ, then the distribution of interarrival times is known to be the negative exponential:

$$\lambda e^{-\lambda x},\ 0<x<\infty \tag{1}$$

which can also be written in cumulative form:

$$\text{Prob (time between consecutive arrivals}>x)=e^{-\lambda x} \tag{2}$$

or in "counting" form:

$$\text{Prob (exactly } n \text{ arrivals in unit time)} = \frac{e^{-\lambda}\lambda^n}{n!},\ n=0,1,2,\ldots. \tag{3}$$

the Poisson distribution.

Three similar expressions with λ replaced by μ, characterize random parking times and random departures from the lot. Of course, the parameters λ and μ will vary considerably from hour to hour during a typical day, so that the parking process is only temporarily homogeneous. It can be shown (*3*, p. 377) that under the assumptions of random arrivals and random departures, the probability that exactly n of the slots are occupied (assuming temporary equilibrium) is also given by the Poisson distribution:

$$\text{Prob (} n \text{ slots occupied)} = \frac{e^{-\lambda/\mu}(\lambda/\mu)^n}{n!},\ n=0,1,2,\ldots. \tag{4}$$

assuming an infinitely large lot ($N\rightarrow\infty$). This result will be approximately valid whenever the value of ρ is small enough to insure that the possibility of all slots being filled is very unlikely.

In case the traffic intensity is so large that the lot may be quite possibly overflowing, the Poisson distribution (Eq. 4) needs only to be truncated; that is, stopped at the value N. However, in order that the probabilities add to unity, it is also necessary to divide by their sum, so that

$$\text{Prob (} n \text{ slots occupied)} = \frac{\dfrac{(\lambda/\mu)^n e^{-\lambda/\mu}}{n!}}{\displaystyle\sum_{j=0}^{N}\frac{(\lambda/\mu)^j e^{-\lambda/\mu}}{j!}},\ n=0,1,2,\ldots, N$$

$$= \frac{\dfrac{(N\rho)^n}{n!}}{1+N\rho+(N\rho)^2/2!+\ldots+(N\rho)^N/N!} \tag{5}$$

This result is also proved by Feller (*3*). A particular case ($n=N$) of Eq. 5 gives the probability that the lot is full, and hence is the probability of a loss to the system in a system with loss, or an increment to the queue in a system with queueing. It is called "Erlang's loss formula," and can be written

$$L_N(\rho) = \frac{\dfrac{(N\rho)^N}{N!}}{1+N\rho+\dfrac{(N\rho)^2}{2!}+\ldots+\dfrac{(N\rho)^N}{N!}} \tag{6}$$

Eq. 6 has been used very extensively in the design of telephone switchboards. If it is decided in advance how probable the loss of a call is to be; *i.e.*, the choosing of L, and if there is a value for the demand ρ, then the correct number of lines N can easily be determined from existing tables of the Poisson distribution.

Mori (*4*) and Kometani and Kato (*5*) have analyzed the operation of several parking lots in Japan, with a view to testing the validity of the hypotheses of random arrivals and departures on which Eq. 6 is based. Using appropriate statistical procedures, they confirm the propriety of these assumptions for the parking lots studied.

However, in Japanese parking lots, the customers catered to are persons making short business stops, rather than workers and shoppers. This fact helps explain how the random arrival and random departure model obtains.

SANTA MONICA STUDY

A study was carried out in Santa Monica, Calif., to obtain extremely detailed data on parking. The gross fluctuations in parking are already well known, but some interesting conclusions might be forthcoming from a short period of precise observation. Therefore, a parking lot of 150 slots was observed for 10½ hr during a typical weekday (Wednesday, June 15, 1960) and every significant movement recorded. The results (*6*) of this investigation can be briefly summarized. The principal document consists of a master list of each car arriving and departing, and each person feeding a meter, together with the slot number, the time of day, the amount of money deposited, the time showing on the meter, the number of passengers, and the total lot occupancy at that time. It is then easy to deduce the amount of time used by each car, this time being divided into three categories: "green time," *i.e.*, time paid for; "red time," *i.e.*, time illegally parked; and "blue time," *i.e.*, time inherited from the previous occupant of the slot.

This particular parking lot was also used by certain cars holding parking permits, sold for a monthly fee, which exempts them from feeding the parking meters. There were two categories of parkers, of which one (permit holders) always used green time.

The first problem in the analysis of the master list is to see if the random switchboard model previously described is valid. In doing so, it was necessary to deal separately with the two types of parkers, as well as to make allowance for varying values of λ and μ. It turned out that it was sufficient to divide the period under study into three parts: growth, stability, and decay, as shown in Figure 1. If cash customers and permit holders are handled separately, and only one of the three time periods referred to, then the assumption of random arrivals and departures seems fairly well justified. Figures 2 and 3 show these results for departures, and the fit with the Poisson distribution for arrivals is given in Table 1.

TABLE 1
OBSERVED AND THEORETICAL NUMBER OF DEPARTURES IN TEN-MINUTE PERIODS
(10:30 to 6:00)

Time Period	No. of Departures	
	Observed	Theoretical
0	0	0
1	2	0
2	2	1
3	3	3
4	3	5
5	7	7
6	7	7
7	5	7
8	6	5
9	2	4
10	4	2
11 or more	4	4

VALUE OF PARKING

During the period of observation, a total of $37.20 was deposited into

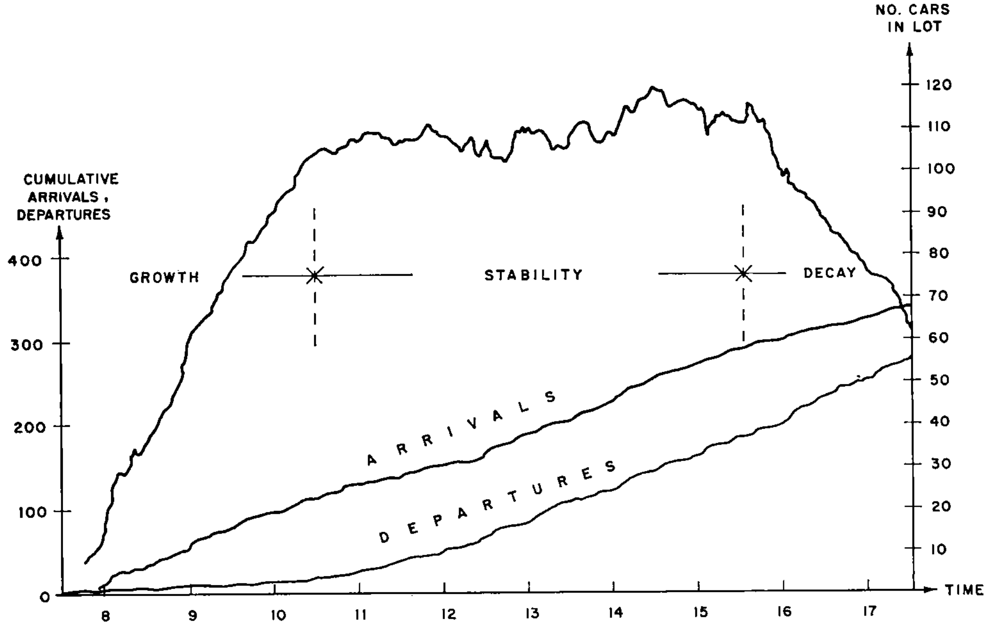

Figure 1. Number of cars in lot, cumulative arrivals and departures.

the meters, and 34,622 min. of meter time were used by non-permit holders. This represents about $0.07 per hr paid for parking, neglecting the slight effect of cars parked before the experiment began or remaining after it finished. Figure 4 shows arrivals classified by amount deposited. Permit holders consumed 8,737 min of parking. It is difficult to calculate their financial contribution. If 21 working days per month during which the permit is used are assumed, the daily cost is about $0.30. (A permit sells for $6.00.) Inasmuch as 33 permit holders were seen, the daily value would be $9.90. However, the monthly income from the 65 permits current was known to be exactly $390.00. It appears that many permit holders did not use the lot during the period under investigation; possibly they store their cars in this lot at night.

One of the most interesting practical consequences of this particular study is the exact evaluation of blue time and red time, and the consequent improvement in municipal revenue if these could by some means be reduced or eliminated. The income from all parking meters in Santa Monica for the year preceding the study is given in Table 2, and the lot being studied is part of the first category, "Downtown and Pier."

The income from this parking lot consists of not only meter receipts, but also permit sales and "bail forfeiture" for parking tickets issued.

TABLE 2

NUMBER AND REVENUE OF VARIOUS GROUPS OF METERS

Area	No. of Meters	Revenue [1] ($)
Downtown and pier	1,077	87,448.70
Wilshire	137	8,962.35
Ocean Park	79	6,317.79
Douglas Aircraft	159	5,373.02
Ocean Avenue	358	16,066.00
Beach	204	21,295.53
Total	2,014	145,463.39

[1] For year ending June 30, 1959.

During the period of observation, the police issued two parking tickets, which one assumes produced \$2.00 in revenue at the current price of \$1.00 a ticket (subsequently raised to \$2.00 a ticket). This is far less than possible; Figure 5 shows the number of cars illegally parked during the entire day.

Therefore, the income from this day's parking can be considered to be \$37.20 in cash, \$9.90 from permit

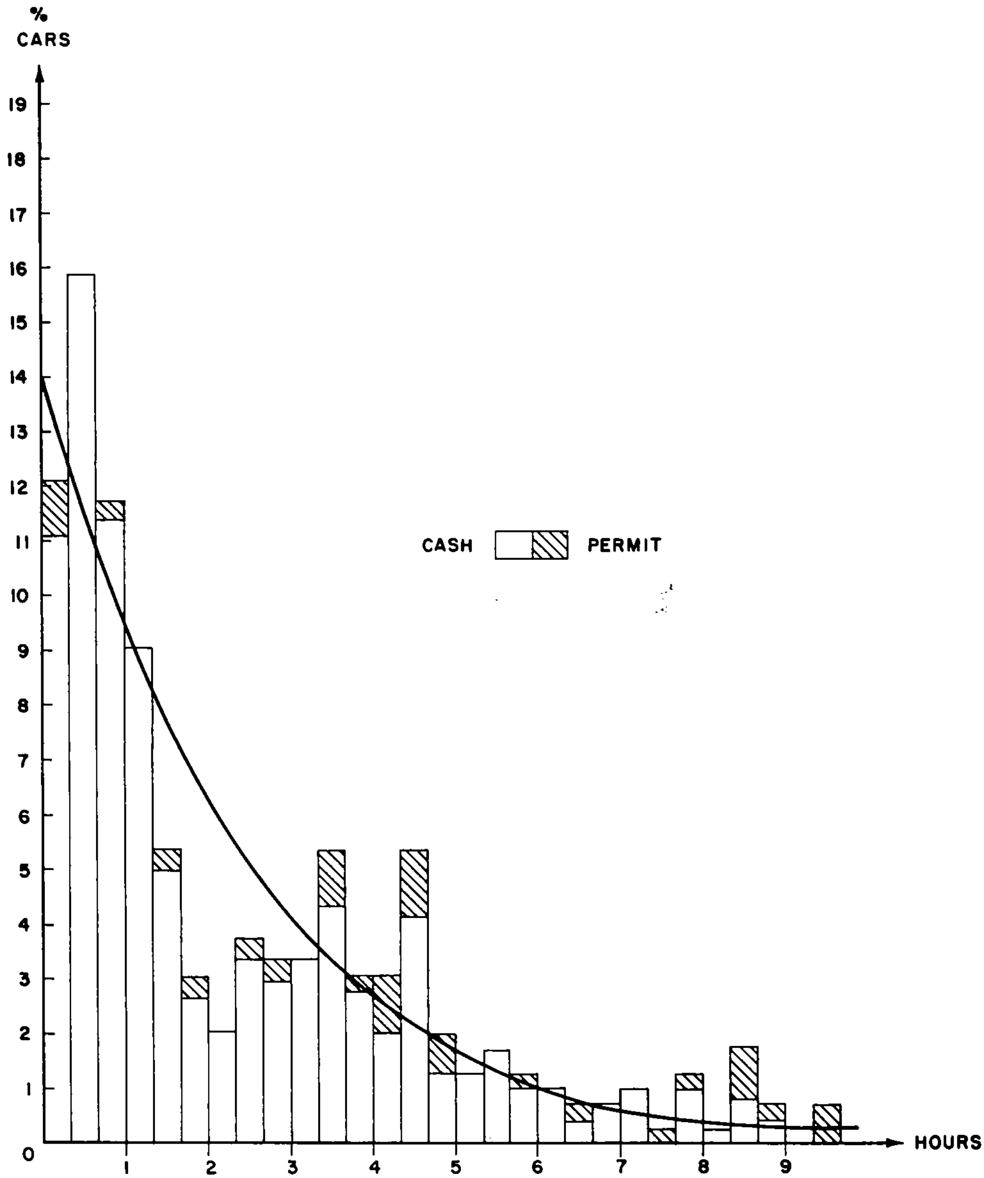

Figure 2. Percentage of cars parking various lengths of time; all departing cars included.

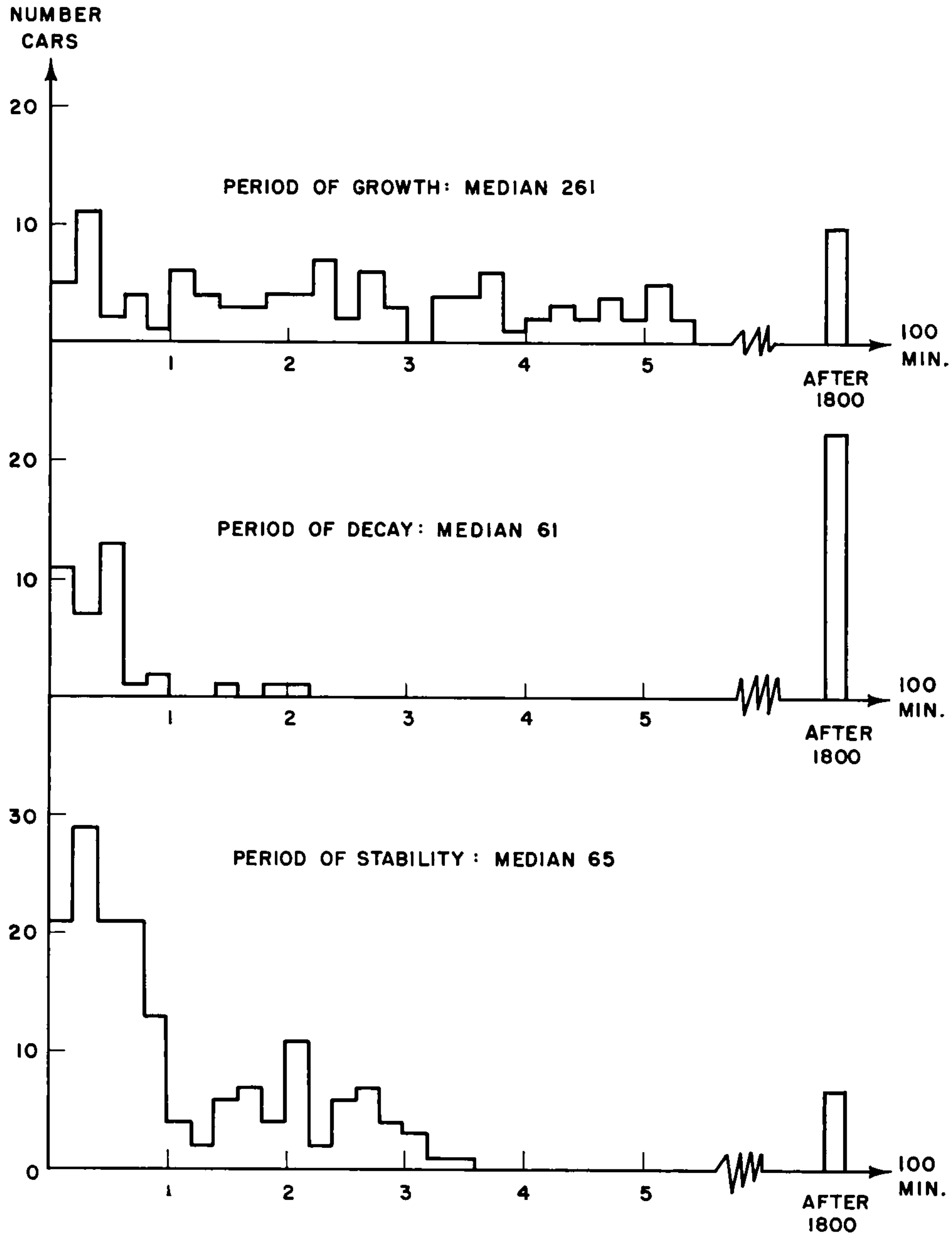

Figure 3. Number of cars parking various lengths of time; all cars arriving in various periods.

holders, and \$2.00 from bail forfeiture—a total of \$49.10.

The maximum possible for the operation of 150 meters over a 10½ hr period is \$78.75 (at \$0.05 per hr), of which \$29.65 was lost. The loss consists of red time and blue time parking, and of times when slots were empty. The loss proportions shown by the study are \$20.10 from empty slots, \$4.05 from red time, and \$4.50 from blue time—a total of \$29.65.

These figures are very likely not quite typical, because the presence of investigators on the lot was observed in many cases to make arrivals change their mind and put money in the meters. In some instances they would feed the meters when their subsequent behavior showed this to be absolutely unnecessary, and in one case even offered to pay us a fine for some small amount of red time used.

Thus, nearly one-third of the

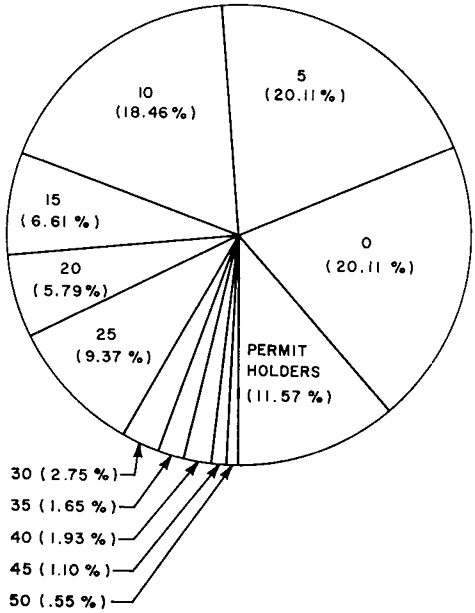

Figure 4. Arrivals classified by amount deposited.

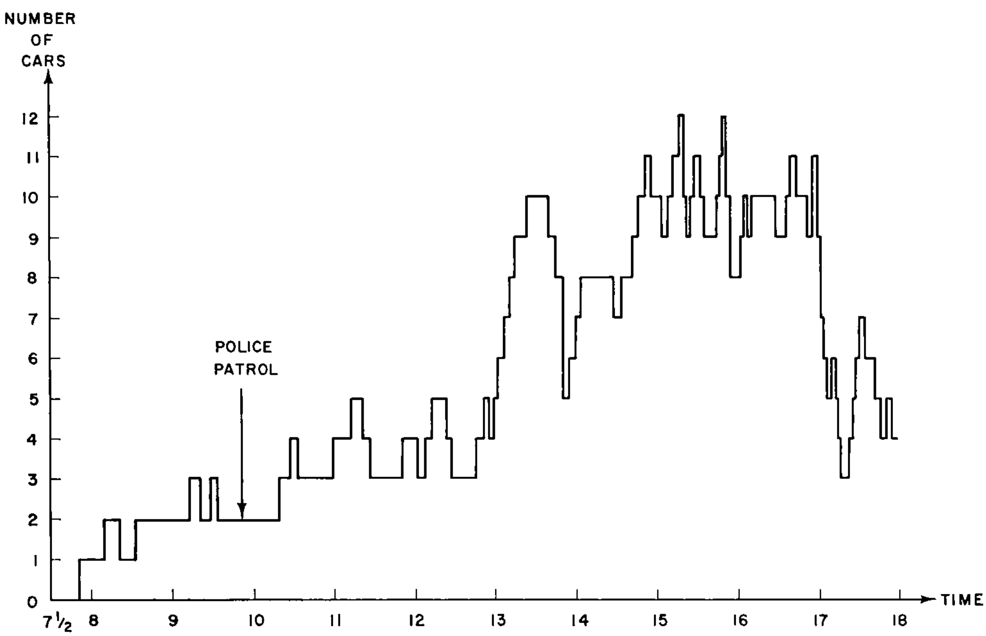

Figure 5. Cars illegally parked.

amount lost during the day could have been recovered. If this experience had been typical (and from the remarks, can hardly be much less) then the increase in revenue to Santa Monica from the elimination of red and blue time would be $33,436.36 for the year.

It seems, therefore, that some serious effort should be made to induce persons using the parking facilities to pay for their time. The question of red time is probably one of enforcement, in which parking revenue is balanced against enforcement costs in a sensible way.

Is it possible to zero each meter before a new car is parked? The suggestions in this direction usually involve electric and mechanical systems, magnets, road tapes, photoelectric cells, or similar devices. It seems that these have two principal disadvantages: cost and reliability. Power cables must be laid to each meter, and the whole area booby-trapped with devices to tell whether a car is coming in or going out or shuffling back and forth.

There are two other possible sources of power for meter zeroing; namely, the arriving driver and the departing driver. It might be possible to invent parking meters that would have to be zeroed before use or after use, and yet which could not be meddled with by passing busybodies; for example, one for arriving driver, and one for departing driver.

Arriving Driver

A driver arrives and finds enough green time already showing on the meter. How can he be persuaded to drop in a coin (assuming that the instrument could be constructed so that when a coin is dropped, the flag first falls to zero, and then records the correct amount)? When the coin is dropped, the meter feeds out a small gummed piece of paper with the meter number printed on it. The driver pastes this on his windshield and is liable to a ticket without this, even if the meter shows green. Because the meter number is printed on the paper, this driver cannot go all over town using up green time.

Departing Driver

It would be desirable to have the departing driver throw a switch to deprive his successor of green time. He must be offered some incentive, and guard against anyone else doing so. The incentive will be money, and the guard a key. Each parking meter could be provided with a key such as are now used in luggage checking lockers. He must deposit $0.50; then the key can be removed. When he returns, he puts the key back and receives his change from parking—at least a dime. As he puts back the key and collects his change, the same force zeroes the meter.

OFF-STREET PARKING

The basic formulation of a parking lot was shown to correspond closely to that of a telephone switchboard. This does not mean, however, that there are no individual variations on that premise. A mathematical model illustrating one sensible difference between the two systems can be sketched. The difference referred to is that the choice of a telephone line is done mechanically according to some predetermined principles, while the choice of a parking slot is made by the individual driver to suit his own purposes.

Assuming that a driver wishes to park as near as possible to some particular objective and is offered parking lots $P_1, P_2, \ldots$ in increasing order of distance (or inconvenience) from his objective, if he finds a slot in P_n he can park at once. If the lot P_n is full, he may wait and watch P_n, and will be able to take supposedly the first available slot. Which lot should he watch? In order for this to be a sensible question, some restrictions must be placed on the problem. Clearly the lower numbered lots are more desirable in that they are nearer his objective. Assume that the remote lots are more desirable in that there is a greater chance of finding a parking place, that is, if he wishes to attend a movie on zeroth street, shall he search first street, which is congested, or second street, which is less so, or third street . . . .?

Suppose the number of slots in P_n is N_n and the traffic intensity in P_n is $\rho_n=\lambda_n/\mu_n$. Then the probability of P_n being full is given by Eq. 6 with parameters N_n and ρ_n, abbreviated

$$L_n=L_{N_n}(\rho_n) \tag{7}$$

Consequently, there is zero delay with probability $1-L_n$ in the n^{th} parking lot. Otherwise let the delay be t, with density function $f\ (t)$ and cumulative density $F\ (t)$. Then

$$\begin{aligned} F(x)&=\text{Prob}(t\leqq x)=1-\text{Prob}(t>x)\\ &=1-\text{Prob}(\min[x_1,\ldots,x_{N_n}]>x) \end{aligned} \tag{8}$$

in which $x_1, \ldots, x_{N_n}$ are the N_n random variables of parking time for the N_n slots in the n^{th} lot.

Each of these variables is assumed to have density function,

$$\mu_n e^{-\mu_n x}$$

so that Eq. 8 can be written

$$\begin{aligned} F(x)&=1-\text{Prob}[x_1>x,\ldots,x_{N_n}>x]\\ &=1-\left[e^{-\mu_n x}\right]^{N_n}\\ &=1-e^{-\mu_n N_n x} \end{aligned} \tag{9}$$

Differentiating Eq. 9, the continuous density for delay in the n^{th} lot becomes

$$f(x)=\mu_n N_n e^{-\mu_n N_n x} \tag{10}$$

Therefore, the whole probability distribution for delay in the n^{th} lot can be written in terms of the Dirac delta function for the discrete component and Eq. 10 for the continuous component.

$$f_n(x)=[1-L_n]\,\delta(x)+L_n\mu_n N_n e^{-\mu_n N_n x} \tag{11}$$

Multiplying Eq. 11 by x and integrating, the average delay in the lot P_n becomes

$$D_n=L_n\mu_n N_n\int_0^\infty x e^{-\mu_n N_n x}\,dx=L_n/\mu_n N_n \tag{12}$$

There are several applications that might be made of Eq. 12. For example, if explicit expressions for the various traffic intensities were given, the value of D_n could be compared with the inconvenience of parking in P_n, and some optimum obtained. To illustrate this for a simple but plausible set of assumptions, assume all the lots to be the same size, so that the subscript on N can be dropped. Next, the parking times are assumed independent of the lot parked in, but the demand is inversely proportional to the nearness of the lot; namely, the other parkers are also going to the same movie. This would permit dropping the subscript on μ, and to write $\rho_n=(C/n)$. Therefore,

$$L_n=\frac{(C/n)^N/N!}{\sum_{j=0}^{N}\frac{(C/n)^j}{j!}} \tag{13}$$

and with the constants C and N specified, L_n could be computed as a function of n. If the disadvantage of the n^{th} parking place consisted of a simple linear constraint (*i.e.*, if the streets in this town are equally spaced and if it takes time T to walk a block), then the values of Eq. 13 would only have to be compared with nT, and an optimum difference obtained.

ON-STREET PARKING

In this section a new model is presented to analyze some features of on-street, or curb parking. Figure 6 shows how a typical downtown area can be broken up into zones. Each square represents a city block and the broken lines outline the zones. The property of each of these zones is that no more than one block need be walked to pass from one zone to the next. Here, to prevent any ambiguity, the city block, referred to before, will henceforth be called a square, and the full distance from corner to corner along one side of the square will be called a block. All zone outlines are contingent on the center of attraction, or destination designated by an X in the diagram, and each destination will determine a different zone pattern. (Because it is unlikely that all persons parking in a downtown area will have the same destination, it is therefore unlikely that all persons who happen to know the following procedures will concentrate their parking efforts in a single specific zone, thus upsetting the results of the procedure.) Then, the maximum distance to be walked from zone n to the destination is n blocks.

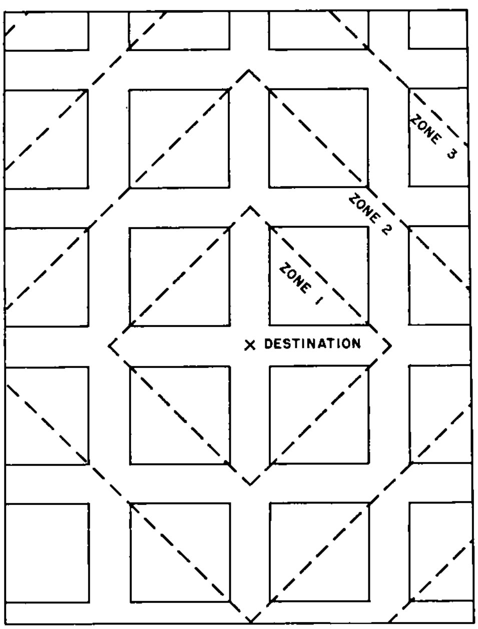

Figure 6. Rectangular street network divided into zones.

This problem is an optimal choice of zone in which to try to park. The following approximations will be made to simplify the presentation of this method. The streets shall be considered to have very small width. The squares are regular and of uniform dimensions, and each is surrounded by the same number of parking slots, the slots running from corner to corner with no omissions.

Let N be the number of parking slots along one side of any square. Then the number of parking slots surrounding each square is $4N$. From the diagram, in zone 1, there are $8N$ slots; in zone 2, $24N$ slots; in zone 3, $40N$ slots; and in zone n, there are $(2n-1)\,8N$ slots. If the number of slots in zone n is defined as N_n, then,

$$N_n = (2n-1)8N \qquad (13)$$

Let the length of one block be A, and the length of each parking slot be a. Then $A = Na$. Now, the average distance of zone n from the destination is $(2n-1)A/2$, or $(2n-1)$ $Na/2$ blocks. This, of course, considers only travel along the perimeter of a square to be allowed. If v is the walking speed, the average time it takes to walk from zone n to the destination is

$$t_n' = (2n-1)\,Na/2v \qquad (14)$$

If now it could be estimated how long, on the average, it would take to find a parking slot in any particular zone, then the zone could be picked for which the total average time involved in reaching the destination would be a minimum.

The problem is now to find out how long it will take to find an empty parking slot in any zone, or equivalently, how many parking slots in zone n can be expected to be passed before there is an empty one? Let this number be $\bar{k}_n$ (the bar indicating that it is a mean value) so that if the search speed is V, the average time it will take to find a parking slot is

$$t_n = \bar{k}_n\, a/V \tag{15}$$

and the total average time involved in reaching the destination by parking in zone n is the sum of Eqs. 14 and 15,

$$T_n = t_n + t_n' = \frac{\bar{k}_n a}{V} + \frac{(2n-1)Na}{2v} \tag{16}$$

Now, for given n, what is the expected $\bar{k}_n$? Assuming equilibrium conditions, the probability that j slots are filled in lot n is given by Eq. 5. It seems reasonable that ρ is constant for each zone. Certainly $1/\mu$, the mean parking time, can be considered constant; but λ, the mean arrival rate, will vary because the zones are of different sizes. From the earlier discussion on the variety of destinations, no area will be more in demand than any other. Then, λ should be approximately linearly proportional to the size of the zone, or synonymously to the number of slots in each zone, so that $\lambda_n = CN_n$, where C is a constant. Then,

$$\rho = \lambda_n/N_n\mu = CN_n/N_n\mu = C/\mu = \text{constant} \tag{17}$$

If the j-filled slots in zone n are independently distributed, then the probability that the first k_n slots searched are all filled is, according to Fry (*2*),

$$\frac{\binom{j}{k_n}}{\binom{N_n}{k_n}}, \quad k_n = 0, 1, 2, \ldots, j \tag{18}$$

and the probability of having to test more than k_n is found by

$$p(>k_n) = \sum_{j=k_n}^{N_n} \frac{\binom{j}{k_n}}{\binom{N_n}{k_n}}$$

$$\text{Prob } (j \text{ slots occupied}) \tag{19a}$$

or

$$p(>k_n) = \sum_{j=k_n}^{N_n} \frac{j!(N_n-k_n)!}{N_n!(j-k_n)!} \frac{(N_n\rho)^j/j!}{\sum_{i=0}^{N_n} \frac{(N_n\rho)^i}{i!}}$$

$$= \frac{(N_n-k_n)!}{N_n!} \frac{\sum_{j=k_n}^{N_n} \frac{(N_n\rho)^j}{(j-k_n)!}}{\sum_{i=0}^{N_n} \frac{(N_n\rho)^i}{i!}} \tag{19b}$$

Letting $i = j - k_n$,

$$p(>k_n) = \frac{(N_n\rho)^{k_n}(N_n-k_n)!}{N_n!} \frac{\sum_{i=0}^{N_n-k_n} \frac{(N_n\rho)^i}{i!}}{\sum_{i=0}^{N_n} \frac{(N_n\rho)^i}{i!}} \tag{19c}$$

Now, the probability of having to search exactly k_n slots before finding an empty one is

$$p(k_n)=p(>k_n-1)-p(>k_n) \tag{20}$$

and the average number of slots expected to be searched in zone n is

$$\bar{k}_n=\sum_{k_n=0}^{N_n} k_n p(k_n)$$

$$=p(1)+2p(2)+3p(3)+\ldots$$
$$=p(>0)-p(>1)+2p(>1)-$$
$$2\,p(>2)+\ldots-(N_n-1)p(>N_n-1)+$$
$$N_n p(>N_n-1)-N_n p(>N_n) \tag{21}$$

This model does not allow for the search to involve more than N_n slots, so that if N_n slots are searched unsuccessfully, then the driver drops the search in the most favorable zone, moves to the next most favorable, and initiates a new search. Hence, the term $N_n p(>N_n)$ in Eq. 21 is zero, and the equation becomes

$$\bar{k}_n=\sum_{k_n=0}^{N_n-1} p(>k_n) \tag{22}$$

Then, the average time it will take to find a parking place, if the time of entering zone n is $t=0$, is

$$t_n=\frac{\bar{k}_n a}{V} = \frac{a}{V}\sum_{k_n=0}^{N_n-1} p(>k_n) \tag{23}$$

and the total average time involved in reaching the destination is, from Eq. 16,

$$T_n=\frac{a}{V}\sum_{k_n=0}^{N_n-1}\frac{(N_n\rho)^{k_n}\,(N_n-k_n)\,!}{N_n!}\;\frac{\displaystyle\sum_{i=0}^{N_n-k_n}\frac{(N_n\rho)^i}{i!}}{\displaystyle\sum_{i=0}^{N_n}\frac{(N_n\rho)^i}{i!}}+\frac{(2n-1)Na}{2v} \tag{24}$$

Since t_n' is an increasing function, and t_n a decreasing one, it can be expected that there is some zone n for which T_n is a minimum. That would be the zone in which to seek a parking slot in order to reach one's destination in the least time.

Each square has a zone boundary cutting it diagonally, so that if it were elected to circle a square, rather than concentrate all one's efforts in a single zone, it would be best to choose the two zones for which the times were less than the times involved in parking in any other zone.

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