# A Theory of Traffic Flow for Evaluation of Geometric Aspects of Highways

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● IN RECENT YEARS several theories have been proposed wherein the car-following characteristics of a traffic stream have been developed from a variety of considerations such as continuous fluid flow (1), "follow-the-leader" philosophies (2, 3), and other concepts (4). The purpose of these analyses was to obtain a mathematical description of highway capacity as reflected by flow-density relationships. Of interest in this paper is the development of an analytical approach whereby the effects of various aspects of a highway (such as curves, grades, lane width, traffic control devices, type and condition of surfacing) can be evaluated quantitatively under varying traffic density and weather conditions and hence provide rational basis for geometric design procedures.

Unlike previous work, where the pattern of traffic flow (for example, single-lane, congested conditions) is assumed a priori, in the present analysis it is postulated that traffic reacts to a motivating "pressure potential"; that is, the vehicle operator reacts to the prevailing road, traffic, and weather conditions—characterized herein by the pressure potential—in a pattern that determines the behavior of any particular vehicle traversing a given section of roadway. This concept can be expressed quantitatively in a manner analogous to that used in describing one-dimensional viscous flow of a compressible fluid. Solution of the resulting differential equation yields a parameter which is a numerical measure of the salient characteristics of a road. Procedures are then proposed to determine this parameter, using measurable vehicular velocities, which eliminate the need for evaluating the motivating pressure potential.

Application of the new theory in the development of rational procedures for the geometric design of highways is illustrated by suitable examples.

## NOTATION

The following notation is used throughout the paper:

| | | |
|---|---|---|
| $p$ | = | motivating pressure potential; |
| $x$ | = | a position along a roadway; |
| $L$ | = | a particular length of roadway; |
| $N$ | = | number of vehicles; |
| $N_x$ | = | number of vehicles entering a section of roadway at position k; |
| $v$ | = | velocity of vehicle; |
| $w$ | = | effective width of roadway; |
| $\rho$ | = | vehicular density, number of vehicles per unit of effective area, |
| $t$ | = | time; |
| $C_1$, $C_2$ | = | coefficients of proportionality; |
| $k$ | = | coefficient of proportionality; and |
| $a^2$ | = | $(C_1 + C_2)/k$ = a constant. |

## FORMULATION OF PROBLEM

The theoretical development is based on the following assumptions:

The vehicular velocity is proportional to the gradient of the pressure potential,

$$v = -k\frac{\partial p}{\partial x} \tag{1}$$

Making use of Eq. (1), and noting that the quantity $\left[\frac{\partial p}{\partial x}\right]^2$ is of higher order,

$$\frac{\partial^2 p}{\partial x^2} = a^2 \frac{\partial p}{\partial t} \qquad (7)$$

in which, $a^2 = (C_1 + C_2)/k$ = a constant. Eq. 7 is the governing differential equation of the traffic flow process and provides the basis for a quantitative evaluation of salient features of a highway. As the variation in pressure potential with distance and time cannot be measured directly, it is necessary to obviate this requirement if the theory is to be of practical utility. The method adopted to accomplish this end depends on the particular features selected for study, which can best be illustrated by examples.

## APPLICATIONS

Consider a section of highway of length L containing some geometrical aspect to be evaluated quantitatively. This may be a grade or a curve (bridge, traffic control device, etc.) to be studied over a range of traffic volumes. To solve Eq. 7 a suitable set of boundary and initial conditions must be selected. These will depend on the situation being studied; however, for the purpose of illustrating the procedure, the following simplified set of initial and boundary conditions will be assumed:

$$\begin{aligned} p(x, 0) &= p_0 \\ p(L, t) &= p_1 \\ p(0, t) &= p_0 \end{aligned} \qquad (8)$$

For the conditions given by Eqs. 8, the complete solution of Eq. 7 is

$$p(x, t) = p_0 + \frac{\Delta p}{L} x - \frac{2}{\pi} \Delta p \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{-n^2 F_0} \sin \frac{n\pi x}{L} \qquad (9)$$

in which $\Delta p = p_1 - p_0$ and $F_0 = \pi^2 t/a^2 L^2$.

Recalling that $v = -K\, \partial p/\partial x$, Eq. 9 becomes

$$\frac{-v(x, t)}{\frac{k\Delta p}{L}} = 1 - 2 \sum_{n=1}^{\infty} (-1)^{n-1} e^{-n^2 F_0} \cos \frac{n\pi x}{L} \qquad (10)$$

which, noting that $v_{avg} = -k\, \Delta p/L$, can be expressed as

$$\frac{v(x, t)}{v_{avg}} = 1 - 2 \sum_{n=1}^{\infty} (-1)^{n-1} e^{-n^2 F_0} \cos \frac{n\pi x}{L} \qquad (11)$$

If the instantaneous velocity is measured at some point on the roadway, say at $x = L/2$, Eq. 11 gives the relation

$$\frac{v(\frac{L}{2}, t)}{v_{avg}} = 1 - 2 \sum_{m=1}^{\infty} (-1)^{m+1} e^{-4m^2 F_0} \qquad (12)$$

which is seen to be a function of $F_0$ only. Eqs. 11 and 12 can be evaluated once and for all and either tabulated or represented graphically. For example, a plot of Eq. 12 is given in Figure 1 with the parameter $F = \left[F_0\right]^{-1}$. The ratio of the instantaneous velocity at a particular point on a section of roadway to the average of the velocity over the entire section defines a unique F-value that is a numerical rating of the particular geometric aspect being studied. Both of these velocities can be measured directly with present-day devices (5).

The change in vehicular density with respect to the pressure potential is proportional to the density,

$$\frac{\partial \rho}{\partial P} = C_1 \rho \tag{2}$$

from Eq. 2,

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} = C_1 \frac{\partial p}{\partial t} \tag{2a}$$

and

$$\frac{1}{\rho}\frac{\partial \rho}{\partial x} = C_1 \frac{\partial p}{\partial x} \tag{2b}$$

The change in effective width of the roadway with respect to the pressure potential is proportional to the effective width,

$$\frac{\partial w}{\partial p} = C_2 w \tag{3}$$

From Eq. 3,

$$\frac{1}{w}\frac{\partial w}{\partial t} = C_2 \frac{\partial p}{\partial t} \tag{3a}$$

and

$$\frac{1}{w}\frac{\partial w}{\partial x} = C_2 \frac{\partial p}{\partial x} \tag{3b}$$

In the ensuing theoretical development it is assumed that neither access nor egress is available to the section of highway under study. Thus the traffic flow can be considered as a conserved system. It should be noted, however, that the theory can be extended to account for the effects of access and (or) egress provided the number of vehicles entering and (or) leaving the section of roadway is determined.

## THEORY

Considering the traffic flow as a conserved system, the change in the number of vehicles on a length of road, dx, in an interval of time, dt, must equal the difference between the number of vehicles entering the section at position x and the number of vehicles leaving the section at position x + dx (in time, dt). If N is the number of vehicles initially on the length of roadway dx, ($N = \rho w dx$), and $N_x$ is the number of vehicles entering in time, dt, at position x, ($N_x = \rho w v dt$), the preceding statement can be expressed symbolically by

$$\frac{\partial(\rho w)}{\partial t} dtdx = \rho wvdt - \left[\rho wvdt + \frac{\partial(\rho wv)}{\partial x} dxdt\right]$$

or

$$\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho wv)}{\partial x} = 0 \tag{4}$$

Note: If $N_E$ equals the net number of vehicles entering and leaving per unit length of roadway, dx, per unit time, dt, Eq. 4 becomes

$$\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho vw)}{\partial x} = N_E$$

Expanding Eq. 4 and dividing by $\rho w$

$$\frac{\partial v}{\partial x} + \frac{v}{\rho}\frac{\partial \rho}{\partial x} + \frac{v}{w}\frac{\partial w}{\partial x} + \frac{1}{w}\frac{\partial w}{\partial t} + \frac{1}{\rho}\frac{\partial \rho}{\partial t} = 0 \tag{5}$$

which upon introduction of Eqs. 2a, 2b, 3a and 3b, reduces to

$$\frac{\partial v}{\partial x} + (C_1 + C_2)\, v \frac{\partial p}{\partial x} + (C_1 + C_2) \frac{\partial p}{\partial t} = 0 \tag{6}$$

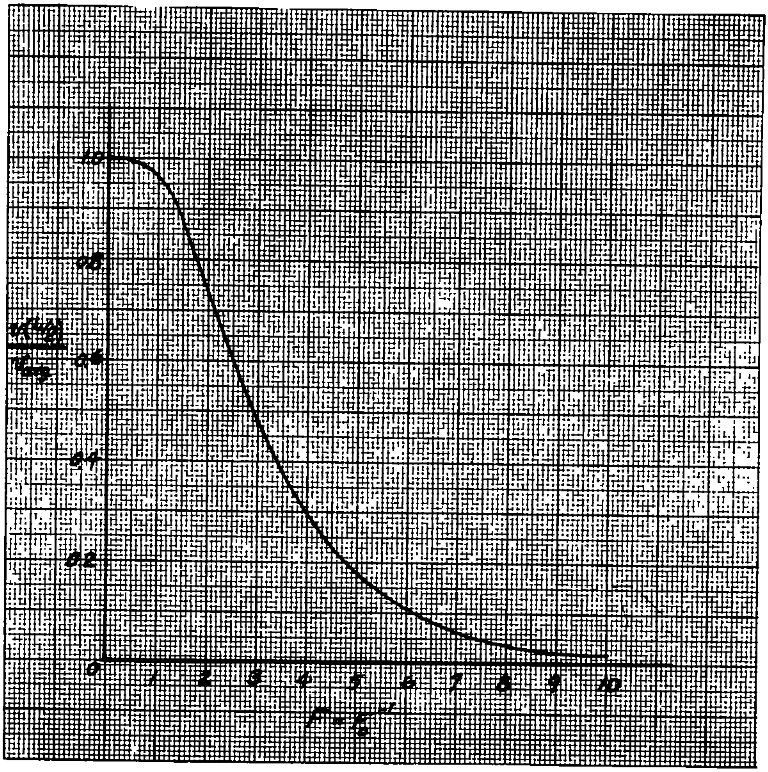


Figure 1. Solution of Eq. 12

By repeating the velocity measurements on the same section of roadway under different ambient conditions (such as minimum and peak traffic densities during the day or night, or different climatic conditions), the resulting changes in the F-value will provide a numerical rating of the particular geometric feature being studied as influenced by the changes in ambient conditions.

By making simultaneous measurements on two sections of the same road, each having a different geometrical feature (such as a grade vs a curve), comparisons of the effects of these two (or any other) features on the resistance to traffic flow can be made on a quantitative basis. A rational answer can them be given to the time-honored question: is a G percent grade more (or less) objectionable than a D degree curve? Repeating the simultaneous measurements at various times will permit making such comparisons under different ambient conditions.

Similarly, velocity measurements before and after traffic control devices are installed, or the road is widened, or an overpass, interchange, acceleration and/or deceleration lane is constructed will permit evaluation of the effectiveness of these devices on a numerical basis. Indeed, the number of potential applications of the new theory appears to be unlimited.

In time, a catalog of F-values can be developed for all types of geometrical and psychological highway appurtenances which will permit future highways to be designed on a more rational basis.

## REFERENCES

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