Truss Deflections by Electronic Computation of The Williot-Mohr Diagram

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This paper describes the method and procedure used in a digital computer program to find horizontal and vertical movements of all joints of a truss, given the member stresses and the structure properties. The method presented is a solution by analytic geometry of the well-known graphical method of Williot and Mohr. The discussion of the basic program procedure is accompanied by a simplified flow chart. A sample case is included to illustrate the speed, accuracy, and flexibility of the program.

●THE ADVANTAGES of truss construction are well known, as witnessed by its widespread use in bridge building. A major disadvantage, however, has been that the design calculations are usually very time-consuming, particularly for indeterminate trusses. This disadvantage, however, is becoming less and less significant with the growing use of the digital computer. There have already been several programs developed for stress analysis of determinate and indeterminate trusses. The program described in this paper deals with another important phase—that of deflections. The necessity for truss deflection computations arises both in the design office, as in the determination of secondary stresses, and in the field, as in the erection of continuous trusses by the cantilever method.

Representative of the several procedures for determining truss deflections are the virtual work, the elastic weight, and Williot-Mohr diagram methods. The method of virtual work is perhaps the best and most direct method for computing one component of deflection for one joint, but this process becomes quite lengthy if the true absolute deflections of all joints are needed. The elastic weight method comes a step closer, but even it is limited to only one component for each joint. This method also becomes lengthy if both the horizontal and vertical components are to be computed. The Williot-Mohr diagram yields the resultant deflections of all joints by a single solution. As a graphical method, it too has its disadvantages. Although it is theoretically sound, it is by its very nature limited in accuracy, and for very large structures, it becomes particularly troublesome in matters of scaling and orientation on the paper. But these disadvantages are easily overcome by an algebraic procedure that corresponds to the graphical one. The computations involved in the solution by analytic geometry of the Williot-Mohr diagram are quite simple and are highly repetitive in nature, thus making this method well suited to the electronic computer. The analytic solution is both accurate and fast. It is the purpose of this paper to describe an electronic computer program and the method used therein for the computation of truss deflections.

NOTATION

Specific symbols are defined where they appear in this paper. In general, however, the following rules apply:

1. Superscripts T, W, and M refer, respectively, to the truss diagram, the Williot diagram, and the Mohr correction diagram.
2. Subscripts refer to specific points or vectors. X and Y subscripts are used to denote, respectively, x and y components of vectors.

3. Points on the different diagrams are designated by an upper case letter with a superscript. Thus, for example, A^W is the point on the Williot diagram that corresponds with point (or joint) A^T of the truss diagram.

4. The relative positions or locations of the various points are defined by x and y coordinates. For example, (x_A^W, y_A^W) are the coordinates of point A^W on the Williot diagram.

THE WILLIOT-MOHR METHOD

Though no attempt is made here to develop or prove the Williot-Mohr method for determining truss deflections, the following graphical procedure is described as a basis for the development of the analytic procedure.

The Williot diagram, which is drawn first, gives the movements of all joints of the truss with respect to one of the joints and a member that enters that joint. If the member that was used as a direction reference actually rotates with the deformation of the truss, the Mohr correction diagram must be added to give the true absolute movements of all the joints. The scaling of the Mohr correction diagram depends on the amount that the original reference member actually rotates, and the positioning of the correction diagram depends on the actual movement of the original reference joint. If a truss joint that is actually fixed against translation is used as the reference point for the Williot diagram, then both the Williot diagram deflection and the Mohr diagram correction for that joint would be zero. A second point on the Williot diagram may be located by using one of the members entering the fixed joint as a direction reference. A vector equal to the deformation of that member is drawn from the reference point in the direction in which the opposite joint of that member moves with respect to the fixed joint. The location of the first two points on the Williot diagram is shown in Figure 1. Point F^W on the Williot diagram corresponds to the fixed joint F^T on the truss diagram. The member connecting joints F^T and H^T is used as a direction reference for the Williot diagram. The vector $\overrightarrow{F^W H^W}$ represents the magnitude and direction of the movement of joint H^T with respect to joint F^T, and thereby the position of point H^W on the Williot diagram is established. Note that $\overline{F^W H^W}$ is drawn parallel to $\overline{F^T H^T}$.

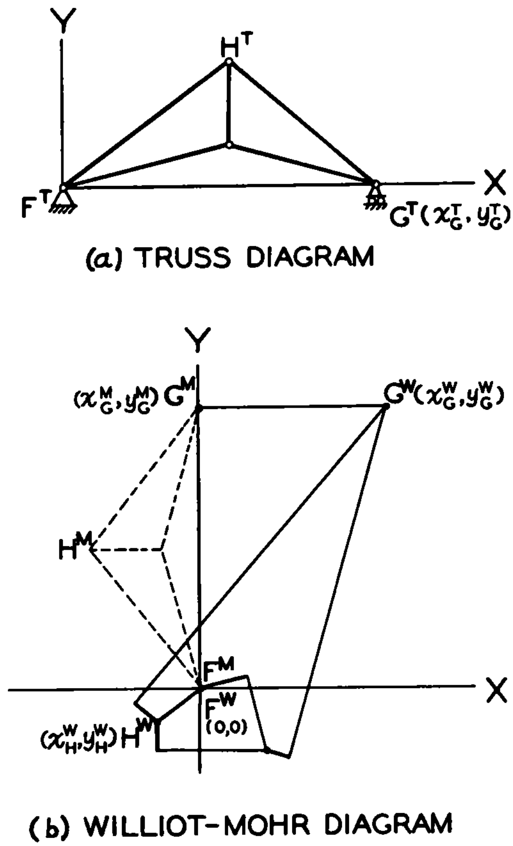


Figure 1. Example truss and Williot-Mohr diagrams.

After having established the first two points, the Williot diagram is completed by proceeding from joint to joint in a series of similar steps. In each step two known Williot diagram points and the deformations of two members are used to establish the position of a third Williot diagram point. One such step is shown in Figure 2, which shows a typical triangular truss panel (Fig. 2a) with joints A^T, B^T, and C^T and the corresponding portion of the Williot diagram (Fig. 2b). Points A^W and B^W are the known Williot diagram points and C^W is the point to be determined. The location of point C^W is determined as follows:

1. The intermediate point W_{AC} is established by the vector $A^W W_{AC}$ which is

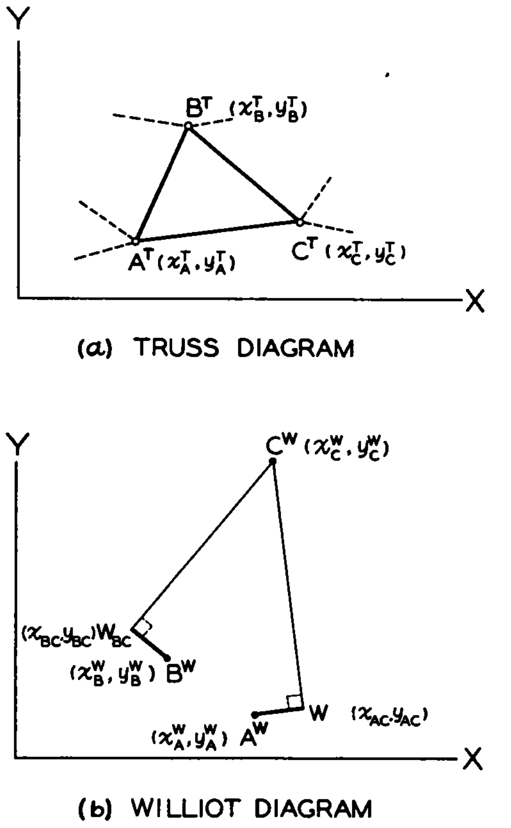


Figure 2. Typical triangular truss panel with corresponding portion of Williot diagram.

proportional in length to the deformation of member AC and is drawn parallel to that member from A^W to W_{AC} in the direction that C moves with respect to A^T. The intermediate point W is established in like manner with respect to member BC.

2. Perpendiculars are erected at points W_{AC} and W_{BC} and are extended until they intersect to locate C^W.

This procedure is repeated until a corresponding Williot diagram point has been established for each joint of the truss.

With the Williot diagram completed, the Mohr correction diagram can be drawn to give the true absolute movements of all the joints. Because the fixed joint of the truss was used as the starting point for the Williot diagram, the correction involves only the rotation of the structure about that joint until the guided joint is brought back into its predetermined path. The Mohr diagram is geometrically similar to the truss configuration, and furthermore, the lines of the Mohr diagram are perpendicular to the corresponding lines of the truss diagram. Referring again to Figure 1, F^M coincides with F^W because F^T is actually fixed in position. Another Mohr diagram point, G^M, is defined by the intersection of a line drawn through G^W parallel to the direction of movement of the guided joint G^T and the line drawn through F^W perpendicular to the line connecting points F^T and G^T. With these two points, F^M and G^M, determined, the rest of the Mohr diagram can be filled in to scale in accordance with the preceding.

THE ANALYTIC PROCEDURE

Input Data

The information needed for the construction of the Williot-Mohr diagram consists of an adequate description of the truss configuration and the deformation (shortening or lengthening) of each member. For the computer program the truss is described by means of two tabulations. The first is a tabulation of joints by number giving a pair of of rectangular coordinates for each joint. The second is a tabulation of members giving the joint numbers of the two joints that each member enters. The coordinates axes are positioned on the truss diagram so that the entire truss lies in the first quadrant, thus eliminating negative coordinates.

In addition to the joint and member tabulations, the numbers of the fixed and guided joints and the direction in which the guided joint moves must be indicated. The direction of movement of the guided joint is expressed as an angle, β', measured from the X-axis. Also, because the output deflections (answers) are to be given in x and y components, the desired directions for these components (such as horizontal and vertical) are indicated by an angle, α, measured from the respective coordinate axes used to describe the truss. For both α and β' the usual sign convention applies; i.e., counterclockwise is positive, clockwise is negative. Note that, regarding algebraic sign, these angles are given with respect to the original coordinate axes used to describe the truss configuration.

The data needed for computing the member deformations include the axial force, cross-sectional area, and length of each member along with the modulus of elasticity of the truss material. The axial forces (stresses), S, and the cross-sectional areas, A, are included in the member table, and the lengths are determined from the joint coordinates. The appropriate conversion factors may be incorporated into the figure for modulus of elasticity in order that all other data may be entered "as found" without regard to units. Based on the data thus provided, the computations for truss deflections can be started.

Preliminary Computations

The guiding principle in the preparation of the format for input data was that these data should be limited as nearly as possible to those quantities that are readily available, leaving conversions and other preliminary computations to the computer program. The first of these preliminary computations to be considered here is that for member deformations. Because the subsequent computations will involve only x and y components, the unit deformation rather than the total deformation is computed for each member at this time. Unit deformation, δ, may be defined as the amount by which a unit length of a member elongates or shortens. In accordance with the definition of modulus of elasticity,

$$\delta = \frac{S}{A\ E} \qquad (1)$$

The algebraic sign, of course, depends on the sign of S, which is considered positive for tension and negative for compression.

Having oriented the original coordinate axes on the basis of convenience for describing the truss configuration, the next step is to transform the coordinates so that (a) the origin coincides with the fixed joint and (b) the X and Y axes lie, respectively, in the directions in which the final deflection components are desired; e.g., horizontal and vertical. The formulas for transformation of coordinates are

$$x = (y'-k)\sin\alpha + (x'-h)\cos\alpha \qquad (2a)$$

$$y = (y'-k)\cos\alpha - (x'-h)\sin\alpha \qquad (2b)$$

in which x and y are the coordinates of a point in the new system, x' and y' are the coordinates of the same point in the old system, h and k are the coordinates of the origin of the new system in terms of the old system, and α is the angle that the new axes makes with the corresponding old axes.

Figure 3a shows a sample truss diagram with angles α and β' labeled and the original coordinate axes used to describe the truss. Figure 3b shows the positioning of the new axes, presuming that in this case horizontal and vertical components of joint deflections are desired. Note that the angle β is found as follows:

$$\beta = \beta' - \alpha \qquad (2c)$$

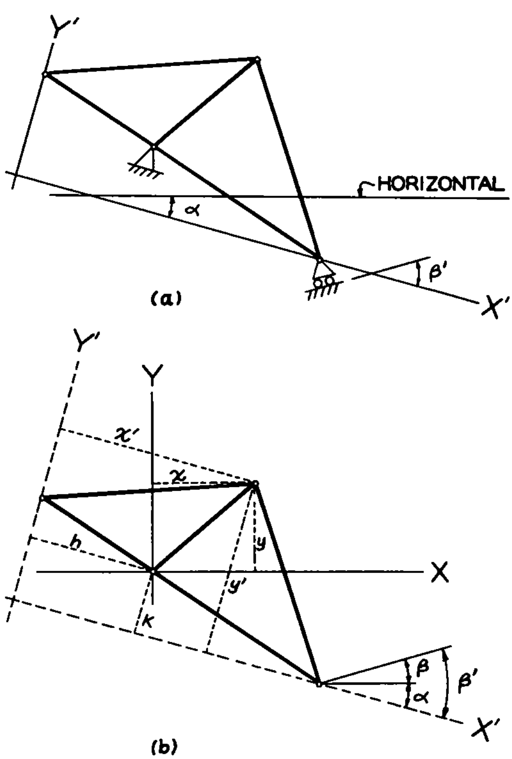


Figure 3. Orientation of coordinate axes.

The Analytic Solution

Once the unit deformations have been computed and the coordinate axes properly oriented as indicated above, the rest of the procedure involves only x and y components. As in the graphical solution, the first two points to be located on the Williot diagram are those corresponding with the fixed joint and the opposite joint of a member that enters the fixed joint. The Williot diagram coordinates (x_F^W , y_F^W)of the point, F^W, which corresponds with the fixed joint, F^T, are both equal to zero (see Fig. 1). If the member, FH, which connects joints F^T and H^T, is used as the direction reference for the Williot diagram, then the Williot diagram coordinates for point H^W are computed by resolving the movement of joint H^T with respect to joint F^T into x and y components. Thus:

$$x_H^W = (x_H^T - x_F^T)\,\delta_{FH}$$

$$y_H^W = (y_H^T - y_F^T)\,\delta_{FH}$$

in which δ_{FH} is the unit deformation for the member FH. But because the truss diagram coordinates for the fixed joint are zero, the equations become

$$x_H^W = x_H^T\,\delta_{FH} \tag{3a}$$

$$y_H^W = y_H^T\,\delta_{FH} \tag{3b}$$

The remaining step-by-step procedure for completing the table of Williot diagram coordinates is again analogous to that of the graphical procedure. However, instead of using the length of a member to compute its deformation, the x and y components of the vector representing the distance from the near end of the member to the far end must be used in order to account for the direction in which the far end moves with respect to the near end. If joints A^T and B^T are at the "near ends" of members AC and BC, respectively, and if the "far ends" of both members enter joint C^T (see Fig. 2a), then the length vectors are computed as follows:

$$L_{XA} = x_C^T - x_A^T \tag{4a}$$

$$L_{YA} = y_C^T - y_A^T \tag{4b}$$

$$L_{XB} = x_C^T - x_B^T \tag{4c}$$

$$L_{YB} = y_C^T - y_B^T \tag{4d}$$

in which L_{XA} is the x component of the vector $\overrightarrow{A^T C^T}$, L_{YA} is the y component of the same vector, etc. For the typical triangular truss panel of Figure 2, the Williot diagram coordinates of points A^W and B^W are known and the coordinates of point C^W are to be computed. The intermediate points W_{AC} and W_{BC} are obtained by adding the respective vectors representing the movement of joint C^T with respect to joint A^T parallel to member AC and the movement of joint C^T with respect to joint B^T parallel to member BC to the points A^W and B^W. Therefore, the coordinates of these intermediate points may be computed by using the following equations:

$$x_{AC} = L_{XA}\delta_{AC} - x_A^W \tag{5a}$$

$$y_{AC} = L_{YA}\delta_{AC} - y_A^W \tag{5b}$$

$$x_{BC} = L_{XB}\delta_{BC} - x_B^W \tag{5c}$$

$$y_{BC} = L_{YB}\delta_{BC} - y_B^W \tag{5d}$$

Finally, the location of point C^W is defined by the intersection of the lines perpendicular to the vectors $A^W W_{AC}$ and $B^W W_{BC}$ and passing through points W_{AC} and W_{BC}, respectively. The general equation of a line given a point(x_1, y_1)on that line and the slope, m, of a line normal to it is

$$y = y_1 - \frac{1}{m}(x - x_1)$$

The slopes of vectors $\overrightarrow{A^W W_{AC}}$ and $\overrightarrow{B^W W_{BC}}$ are equal to $\frac{L_{YA}}{L_{XA}}$ and $\frac{L_{YB}}{L_{XB}}$, respectively. By substituting these slopes and the coordinates of the corresponding points W_{AC} and W_{BC} into the general equation above, the following equations for the lines $\overline{W_{AC}C^W}$ and $\overline{W_{BC}C^W}$ result:

$$y = y_{AC} - \frac{L_{XA}}{L_{YA}}(x - x_{AC})$$

$$y - y_{BC} - \frac{L_{XB}}{L_{YB}}(x - x_{BC})$$

Solving these two equations simultaneously for the coordinates, (x_C^W, y_C^W), of point C^W yields

$$x_C^W = \frac{\frac{x_{AC}L_{XA}}{L_{YA}} - \frac{x_{BC}L_{XB}}{L_{YB}} + y_{AC} - y_{BC}}{\frac{L_{XA}}{L_{YA}} - \frac{L_{XB}}{L_{YB}}} \qquad (6a)$$

$$y_C^W = \frac{\frac{y_{AC}L_{YA}}{L_{XA}} - \frac{y_{BC}L_{YB}}{L_{XB}} + x_{AC} - x_{BC}}{\frac{L_{YA}}{L_{XA}} - \frac{L_{YB}}{L_{XB}}} \qquad (6b)$$

Therefore, Eqs. 4-6 may be used to locate each successive Williot diagram point with respect to the coordinate axes after the two starting points have been located, the first being at the origin and the second by Eqs. 3a and 3b. Special consideration must be given, however, if the result of any one of the Eqs. 4a-4d is zero, as would be the case for a member parallel to either coordinate axis. The following rules apply:

$$\text{If } L_{XA} = 0; \text{ then } y_C = y_{AC} \qquad (7a)$$

$$\text{If } L_{YA} = 0; \text{ then } x_C = x_{AC} \qquad (7b)$$

$$\text{If } L_{XB} = 0; \text{ then } y_C = y_{BC} \qquad (7c)$$

$$\text{If } L_{YB} = 0; \text{ then } x_C = x_{BC} \qquad (7d)$$

Having completed the Williot diagram as indicated above, the next step is to compute the angle, θ, through which the truss must be rotated about its fixed joint to bring the guided joint back into its previously designated path. Because θ is quite small in any practical case, it may be taken as equal to its tangent, thus:

$$\theta - \frac{y_G^W - x_G^W \tan\beta}{x_G^T} \qquad (8)$$

in which x_G^T is the x coordinate of the guided joint (truss diagram) and (x_G^W, y_G^W) are the coordinates of the corresponding Williot diagram point. The coordinates for each point on the Mohr correction diagram (see Fig. 1b) may be computed by the following equations:

$$x_N^M = -\theta\, y_N^T$$

$$y_N^M = \theta\, x_N^T$$

in which x_N^T and y_N^T arethe coordinates of any joint, N^T, of the truss diagram and x_N^M and y_N^M are the coordinates of the corresponding Mohr correction diagram point. The x and y components of the true absolute movement of each truss joint may be obtained by subtracting, respectively, the coordinates of the Mohr diagram points from the coordinates of the corresponding Williot diagram points; thus:

$$D_{XN} = x_N^W + \theta\, y_N^T \qquad (9a)$$

$$D_{YN} = y_N^W - \theta\, x_N^T \qquad (9b)$$

in which D_{XN} and D_{YN} are the x and y components of the deflection of any joint, N^T, respectively.

THE COMPUTER PROGRAM

A computer program for the computation of truss deflections based on the described method was written for the IBM 650 data processing system. This program follows basically the procedure set forth in the simplified flow chart, Figure 4. The program was limited arbitrarily to trusses with up to 99 non-redundant members. It is, of course, also limited to those truss deflection problems that can be solved by the Williot-Mohr method.

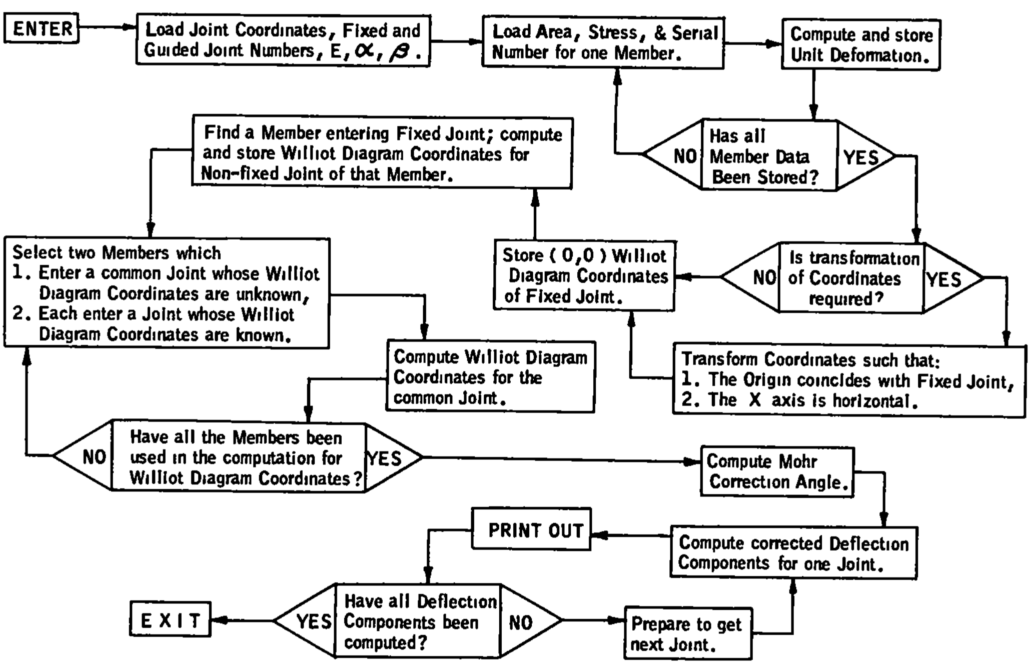


Figure 4. Simplified flow chart for computer program.

Input data include the joint and member tabulations, the modulus of elasticity, and the respective numbers of the fixed and guided joints, as previously indicated. In addition to these, the joint and member counts are required. The output consists of the x and y components of deflection for each joint.

Running time varies from about 4 to 12 sec per joint depending on the order in which the truss members are numbered. If the members are numbered in the order in which they would be used in the construction of the Williot diagram, the searching time for the machine is greatly reduced. Regardless of member numbering, however, the total running time for a 99-member truss would not exceed about 10 min.

SAMPLE CASE

The New Albany-Louisville Bridge on Interstate 64 over the Ohio River was used to illustrate the described truss deflection program. The bridge is a double-decked, tied-arch truss 797.5 ft long from fixed joint to guided joint. Bridge elevations and joint coordinates are shown in Figure 5. Physical properties and dead-load stresses are listed in Table 1. Using for the modulus of elasticity of the truss material 29,000 ksi, the horizontal and vertical components of deflection (Table 2) were computed for each joint in a total of 3 min 5 sec, including read-in and punch-out time, by the IBM 650 digital computer.

TABLE 1

PHYSICAL PROPERTIES AND DEAD LOAD STRESSES[a]

Member Serial Number			Area	Stress
Member No.	From Joint	To Joint	(sq in.)	(kips)
01	01	02	142.00	-710
02	01	03	227.40	-5517
03	02	03	45.20	737
04	02	04	79.10	-474
05	03	04	70.80	-749
06	04	05	45.20	782
07	03	05	202.60	-4821
08	05	06	54.70	-726
09	04	06	96.70	-1095
10	06	07	40.40	711
11	05	07	175.50	-4031
12	07	08	50.00	-572
13	06	08	131.80	-1751
14	08	09	38.20	638
15	07	09	151.80	-3284
16	09	10	39.60	-375
17	08	10	120.40	-2342
18	10	11	36.30	601
19	09	11	130.10	-2620
20	11	12	37.70	-310
21	10	12	130.70	-2838
22	12	13	34.40	563
23	11	13	118.80	-2029
24	13	14	35.20	-182
25	12	14	148.70	-3293
26	14	15	32.60	486
27	13	15	132.20	-1496
28	15	16	33.80	-118
29	14	16	161.00	-3665
30	16	17	32.60	435
31	15	17	101.30	-1056

TABLE 1 (Continued)

PHYSICAL PROPERTIES AND DEAD LOAD STRESSES[a]

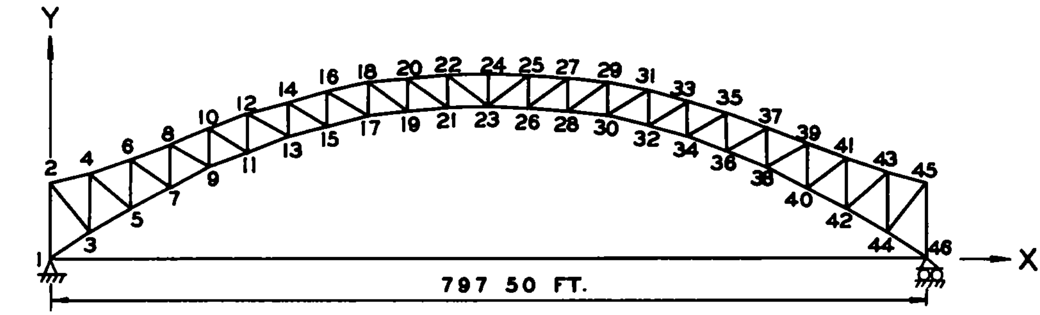
Member Serial Number			Area	Stress
Member No.	From Joint	To Joint	(sq in.)	(kips)
32	17	18	33.80	3
33	16	18	171.80	-3991
34	18	19	34.40	309
35	17	19	101.30	-676
36	19	20	31.10	85
37	18	20	175.50	-4208
38	20	21	39.70	231
39	19	21	83.80	-418
40	21	22	31.10	214
41	20	22	179.50	-4368
42	22	23	42.30	39
43	21	23	75.00	-228
44	23	24	26.70	240
TIE	01	46	132.00	4582

[a]Structure is symmetrical about member 44.

TABLE 2

OUTPUT DEFLECTIONS: HORIZONTAL AND VERTICAL COMPONENTS

Joint No.	Horiz. Comp. (ft)	Vert. Comp. (ft)	Joint No.	Horiz. Comp. (ft)	Vert. Comp. (ft)
01	0.000	0.000	24	0.478	1.779
02	0.245	0.012	25	0.448	1.749
03	0.080	0.185	26	0.475	1.756
04	0.287	0.205	27	0.424	1.672
05	0.161	0.386	28	0.476	1.675
06	0.338	0.407	29	0.409	1.556
07	0.238	0.596	30	0.487	1.556
08	0.401	0.612	31	0.409	1.404
09	0.307	0.809	32	0.508	1.400
10	0.460	0.821	33	0.422	1.227
11	0.365	1.019	34	0.544	1.221
12	0.504	1.030	35	0.452	1.030
13	0.411	1.221	36	0.591	1.019
14	0.533	1.227	37	0.496	0.821
15	0.447	1.400	38	0.649	0.809
16	0.547	1.404	39	0.554	0.612
17	0.469	1.556	40	0.718	0.596
18	0.546	1.556	41	0.618	0.407
19	0.479	1.675	42	0.794	0.386
20	0.532	1.672	43	0.669	0.205
21	0.481	1.756	44	0.876	0.185
22	0.508	1.749	45	0.710	0.012
23	0.478	1.788	46	0.956	0.000



JT.NO.	X	Y	JT.NO.	X	Y	JT.NO.	X	Y	JT.NO	X	Y	JT.NO.	X	Y
02	0.00	70.00	11	181.25	98.35	20	326.25	166.00	29	507.50	161.00	38	652.50	83.31
03	36.25	24.30	12	181.25	134.00	21	362.50	138.84	30	507.50	129.59	39	688.75	106.00
04	36.25	69.33	13	217.50	111.07	22	362.50	169.00	31	543.75	154.00	40	688.75	65.95
05	72.50	46.28	14	217.50	145.00	23	398.75	140.00	32	543.75	121.49	41	725.00	91.33
06	72.50	91.33	15	253.75	121.49	24	398.75	170.00	33	580.00	145.00	42	725.00	46.28
07	108.75	65.95	16	253.75	154.00	25	435.00	169.00	34	580.00	111.07	43	761.25	79.33
08	108.75	106.00	17	290.00	129.59	26	435.00	138.84	35	616.25	134.00	44	761.25	24.30
09	145.00	83.31	18	290.00	161.00	27	471.25	166.00	36	616.25	98.35	45	797.50	70.00
10	145.00	121.00	19	326.25	135.37	28	471.25	135.37	37	652.50	121.00	46	797.50	00.00

NEW ALBANY-LOUISVILLE BRIDGE

Figure 5. Sample case (New Albany-Louisville Bridge).

REMARKS

The method just described for the analysis of truss deflection is nothing new. It is only an analytic version of the commonly used Williot-Mohr diagram. The procedure is also similar in many respects to K. H. Chu's method (1), which was developed for use with a desk calculator. The sign convention is always troublesome in the Williot-Mohr graphical method, but it is taken care of automatically by the computer program. Length of each member is determined from the joint coordinates, a fact that also simplifies the input data.

ACKNOWLEDGMENT

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REFERENCE

1. Chu, K. H., "Truss Deflections by the Coordinate Method." Trans. ASCE, 117: 317-336 (1952).