Forced Vibration of Continuous Highway Bridges

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This paper presents a correlation of forced vibration theory with dynamic impact tests for three continuous highway bridges and a simple span highway bridge. The experimental impact was determined at the centerline of the single span highway bridge and in the outer and inner spans and at the interior supports for the three types of continuous four-span highway bridges. The effect of the vehicle is assumed to be an oscillating forcing function whose frequency is the frequency of axle repetition and whose force is the oscillating load effect of a constant force traversing a beam. The correlation of the theoretical and experimental impact indicates that the simplifications made in the effect of the vehicles are justified for the bridges tested and the experimental vehicle velocities used. These results show qualitatively that the amount of impact is a function of the ratio of the frequency of axle repetition to the loaded natural frequency of the structure.

•IN THE DESIGN of highway bridges, the static live load is multiplied by a factor to compensate for the dynamic effect of moving vehicles. This factor, commonly referred to as an impact factor, is intended to provide for the dynamic response of the bridge to moving loads and suddenly applied forces. Many investigators have published research that contradicts the current impact formula (1, 4, 18). Some investigators feel not only that the problem of impact deals with the increase in over-all static live load but also that it is an integral part of a dynamic load distribution problem (25).

The current expanded highway program with the large number of required bridge structures emphasizes the need for investigating some of the dynamic behavior problems that have been generally ignored by highway engineers. These problems generally result from the inability of a designer to predict the dynamic response of a bridge structure. Many different investigations have been made of particular portions of the over-all dynamic problem. The results of these varied investigations are inevitably followed by a number of unanswered questions. Ironically, many of the unanswered questions are those of immediate concern in the design of highway bridges, and this emphasizes the need for additional research on the problem of impact.

Nature of the Investigation

This investigation is a study of the dynamic magnification of static load, commonly referred to as impact, resulting from the vibrations produced by a vehicle traversing the length of the bridge. More specifically, the purpose of this investigation is to correlate the response of actual continuous highway bridges under the effects of moving vehicles with vibration theory. The problem is then to determine by means of experimental data, the important parameters affecting bridge vibration and to develop thereby a theoretical correlation of these parameters.

The experimental investigation was designed to determine if the simplifications made in the theoretical impact analysis are justified in the application of this theory to actual structures. In this experimental work the impact was determined at midspan of a single span highway bridge and in the outer and inner spans and at the interior supports for three types of continuous four-span highway bridges. The bridge structures investigated are as follows:

1. A simple span bridge with six post-tensioned prestressed concrete beams 100 ft long constructed to act compositely with a reinforced concrete roadway. The roadway is 30 ft wide with a 3-ft safety curb on both sides.
2. A fully continuous structure, 220 ft long with four aluminum stringers constructed to act compositely with a reinforced concrete roadway. The roadway is 30 ft wide with a 3-ft safety curb on both sides.
3. A fully continuous composite structure 240 ft long and similar to the previous bridge but with four steel wide flange stringers. The reinforced concrete roadway is 28 ft wide with a 3-ft safety curb on both sides.
4. A continuous reinforced concrete roadway 24 ft wide with a 2-ft safety curb on both sides supported by six pretensioned prestressed concrete beams in each of the four spans.

The ends of the simple span beams were encased by a cast-in-place diaphragm at the piers. The continuous roadway slab, constructed to act compositely with the stringers, and the pier diaphragm result in a relatively continuous 198.75-ft bridge.

The types of bridges chosen give a wide range of the various parameters involved in vibration. The aluminum stringer bridge is outstanding in that it allows a comparison of the effect of a lighter material with a smaller elastic modulus in a structure similar in its other aspects to the steel stringer bridge. The continuous pretensioned prestressed concrete bridge resembles the other continuous bridges except that it is only partially continuous in its action.

The mass per unit length is nearly equal for the three continuous bridges. To investigate the effect of a variation in the mass per unit length, the much heavier post-tensioned prestressed concrete bridge was studied. Also these bridges provide a number of variables in their structural qualities which may affect structural damping. The characteristic of damping is theoretically important because it provides an upper limit for the amplitude of forced vibration and might determine the maximum amount of impact for that structure.

Definitions

Impact Factor.—The impact factor used herein, is the ratio of the difference between the dynamic and static effect of a vehicle to the static effect. It is therefore the fractional increase in the static live load; in this case, the vehicle, which is required for the static live load to produce an effect equivalent to that of the dynamically applied live load.

Natural Frequency.—The frequency of a free vibration is called the natural frequency of the elastic system. The elastic system used herein is the bridge structure itself.

Resonance.—When an elastic system is acted on by an external periodic forcing function having the same frequence as a natural frequency of the system, it is in a state of resonance.

Notations

E = modulus of elasticity;
f = natural frequency in cycles per unit of time;
f_L = loaded natural frequency in cycles per unit of time;
$f(x, t)$ = a function of position and time;
g = acceleration due to gravity;
I = moment of inertia;
L = length of span;
M = mass of the load;
m = mass per unit length of span;
N = number of cycles;

n_b	=	damping coefficient;
s	=	spacing of the vehicle axles;
t	=	time;
T	=	a function of time;
v	=	velocity;
W	=	weight of the load;
w	=	frequency of the forcing function;
x	=	horizontal coordinates a distance measured in the direction of the length of the span;
X	=	a function of the horizontal coordinate;
y	=	vertical ordinates deflected displacement due to the static live load;
Y_d	=	vertical ordinates deflected displacement due to the dynamic live load.

THEORETICAL INVESTIGATION

Forced Vibration

The analytical work of Inglis (10), although not touching on this problem, does offer a great deal of insight into an analysis of the effect of multiple axle vehicles traversing a flexural system.

Inglis incorporates the use of a Fourier sine series for the representation of the various bridge loadings. A concentrated load W at section x = a is expressed by a Fourier series in the form

$$\text{Loading} = \frac{2W}{L} \sum_{i=1}^{i=\infty} \sin\frac{i\pi a}{L} \sin\frac{i\pi x}{L} \tag{1}$$

in which L is the span length. The deflection resulting from this load function provides a basis for some simplifications of the load function. The deflection curve must satisfy the relationship

$$EI\frac{d^4y}{dx^4} = \text{Loading}$$

or

$$EI\frac{d^4y}{dx^4} = \frac{2W}{L} \sum_{i=1}^{i=\infty} \sin\frac{i\pi a}{L} \sin\frac{i\pi x}{L} \tag{2}$$

When the load is near the center of a simple span, the static centerline deflection is obtained approximately by using only the first harmonic component of the load series. The resulting static deflection is

$$y = \frac{2WL^3}{\pi^4 EI} = \frac{WL^3}{48.7EI} \tag{3}$$

in which EI is the elastic constant of the beam. Therefore, by using only the first harmonic component of the load, a very close approximation to the exact value of $WL^3/48EI$ is obtained. Thus, only the first harmonic component was used by Inglis for most of his solutions.

The preceding calculation for deflection was made with the assumption that the elastic curve of the beam is free to rotate at the support, which is the elastic curve of a simply supported beam. The exactness of this solution for deflection is a result of the small difference between the simple beam deflection curve for a concentrated load and the deflection resulting from the first component of the harmonic representation of load, a sine curve. Therefore, the type of solution that results in a sine deflection curve is applicable to a simply supported beam, but it requires some justification before it can be applied to a continuous beam. However, to do this it is only necessary to consider the computations for natural frequency by the energy method. It has been shown by Linger and Hulsbos (13) that the assumed sine deflection curve gives a very good approximation in determining the first mode natural frequency. This indicates the closeness of the sine curve to the exact theoretical first mode vibration curve of a continuous beam.

To represent a moving load, the distance that the load travels is taken as vt, where v is the velocity of the load and t is the time required for the load to traverse the distance a. The series representing the moving load of constant magnitude then takes the form

$$\frac{2W}{L} \sum_{i=1}^{i=\infty} \sin \frac{i\pi vt}{L} \sin \frac{i\pi x}{L} \tag{4}$$

Moving Loads of Constant Magnitude.—The oscillations produced in a beam by a single moving load of constant magnitude are found by solving the differential equation of motion (Eq. 5) with the load function given in Eq. 4. The differential equation of motion is

$$EI \frac{d^4y}{dx^4} + m \frac{d^2y}{dt^2} = f(x, t) \tag{5}$$

in which m is the mass per unit length of span, and f(x, t) is load function. The solution of this problem by Inglis (10, p. 27) is shown, using only the primary component of the load function, in Eq. 6. The dynamic deflection due to a single concentrated force is

$$y_d = \frac{2WL^3}{\pi^4 EI} \left[\frac{\sin \frac{\pi x}{L}}{1 - (\frac{v}{2Lf})^2} (\sin 2\pi (\frac{v}{2L}) t - (\frac{v}{2Lf}) \sin 2\pi ft \right] \tag{6}$$

Due to the practical limitations of speed, the term $(\frac{v}{2Lf})^2$ in the denominator is negligible in comparison with unity and can be ignored. Similarly it can be shown that a simplification of this equation for the maximum dynamic deflection at midspan (i.e., vt = L/2 and x = L/2) becomes

$$y_d = y_{st} \left[1-(\frac{v}{2Lf}) \sin 2\pi ft \right] \tag{7}$$

in which y_{st} is the static deflection for the given load function. It is apparent that the right-hand term in the parenthesis is an amplification factor representing the vibratory oscillations resulting from a single force traversing the beam, or that the increased dynamic effect can be represented by a static oscillating load factor of

$$W (\frac{v}{2Lf}) \sin 2\pi ft$$

Therefore, the effect of a series of forces can be represented by a load function whose frequency of application is determined by the repetition of axles and whose magnitude of oscillation is W (v/2Lf).

Repetition of Axles.—The frequency of the impulses representing the passage of axles is given by

$$w = \frac{v}{s} \tag{8}$$

in which s is the spacing of the axles and v is the velocity of the vehicle. The harmonic oscillation which is assumed to represent the dynamic effect of the repetition of axles is then taken as

$$W \left(\frac{v}{2Lf}\right) \sin 2\pi wt \tag{9}$$

The differential equation of motion used in the forced vibration analysis will include the effect of damping:

$$EI \frac{d^4y}{dx^4} + 4\pi n_b m \frac{dy}{dt} + m \frac{d^2y}{dt^2} = f(x, t) \tag{10}$$

in which $4\pi n_b m$ is the damping constant. The harmonic forcing function f (x, t) which represents the dynamic effect of the axle impulses is represented by the first harmonic component of the load series and includes the effect of the mass of the load. The loading function, or forcing function, is

$$f(x, t) = \left(\frac{2}{L}\right) \left[W \frac{v}{2Lf} \sin 2\pi wt - M \frac{d^2\bar{y}}{dt^2} \right] \sin \frac{\pi x}{L} \tag{11}$$

in which $\bar{y}$ is the vertical deflection of the mass. The form of the solution of this partial differential equation may be taken, as shown by Inglis (10), as

$$y = T(t) \sin \frac{\pi x}{L} \tag{12}$$

The solution of Eq. 10 using the forcing function in Eq. 11 is given by

$$y_d = \frac{2WL^3}{\pi^4 EI} \left(\frac{v}{2Lf}\right) \left[\frac{\sin (2\pi wt - \alpha) - e^{-q} \left(\frac{w}{f_L}\right) \sin 2\pi f_L t}{\sqrt{\left(1 - \frac{w^2}{f_L{}^2}\right)^2 + \left(\frac{2n_b w}{f^2}\right)^2}} \right] \sin \frac{\pi x}{L} \tag{13}$$

in which

$q = 2\pi n_b \left(\frac{f_L}{f}\right)^2$;

f = the natural frequency of the bridge; and

f_L= the natural frequency of the bridge with the load on it.

The right-hand term in the numerator of the brackets varies as a function of the damping. Consequently, it dies out as the load passes along the bridge. Therefore, the maximum amplitude of this vibration occurs when the term $\left[\sin(2\pi wt - \alpha) \sin \frac{\pi x}{L} \right]$ is a maximum, and is given by

$$\frac{2WL^3}{\pi^4 EI}\;\frac{\frac{v}{2Lf}}{\sqrt{\left(1-\frac{w^2}{f_L^2}\right)^2+\left(\frac{2n_b w}{f^2}\right)^2}} \tag{14}$$

Because this deflection occurs as a result of the oscillating load factor w(v/2Lf), it is in effect the dynamic variation of the elastic curve about the static deflection position of this curve. Therefore, the impact factor as previously defined, for the maximum amplitude of vibration is the ratio of this amplitude to the static deflection. This impact factor can be written

$$\frac{\frac{v}{2Lf}}{\sqrt{\left(1-\frac{w^2}{f_L^2}\right)^2+\left(\frac{2n_b w}{f^2}\right)^2}} \tag{15}$$

This impact factor closely resembles the amplification factor normally associated with forced vibrations. The ratio (v/2Lf) in the numerator represents the amount of the load effective in the forcing function as the driving force, and is evaluated from the oscillations produced in a beam by a single moving load of constant magnitude. These oscillations, although they result from a single load of constant magnitude, are similar to those of an oscillating driving force. The effect of these oscillations will be increased if a repetition of axles occurs with the moving load of constant magnitude and if these axles are in phase with the oscillating load. The phase difference between these two effects is not considered here since it is possible for the oscillating load effect and the repetitive axle effect to occur together at many different positions in a continuous structure. Instead, these two effects are considered to be in phase, thus giving an upper boundary impact factor for the forced vibration of bridges by the optimum combination of the repetition of axles with the oscillating effect of a smoothly rolling load.

EXPERIMENTAL INVESTIGATION

Test Structures

The bridges tested in this research are part of the Interstate highway system around Des Moines, Iowa. They have all been built within the last seven years and are similar to the type of bridge being built in Iowa's primary and Interstate road system. The approaches to these structures are paved and there is a smooth transition to the bridge roadway. One factor used in selecting the bridges was the uniformity of their actual roadway profile. All of the bridges tested are constructed of longitudinal stringers designed to act integrally with a reinforced concrete roadway slab. However, a variety in this general type of structure was desirable to determine the limitations of the theoretical forced vibration approach presented herein. The variety was obtained by selecting three continuous bridges in which different materials were used to fabricate the longitudinal stringers. The mass per unit length is approximately the same in these bridges. A simple span bridge with a mass per unit length approximately double that of the other structures was also tested.

Simple Span Prestressed Concrete Bridge.—The simple span bridge investigated has six post-tensioned prestressed concrete beams and a span of 100 ft. The stringers are designed and constructed to act compositely with the reinforced concrete roadway slab. The roadway is 30 ft wide with a 3-ft safety curb on both sides (Fig. 1). This structure is one span of a seven-span bridge carrying westbound traffic on Interstate 35 over the Des Moines River north of Des Moines, Iowa. Each span of this bridge is isolated from adjacent spans by a 1-in. expansion joint.

Continuous Aluminum Stringer Bridge.—This structure is a 220-ft continuous four-span bridge with four aluminum stringers which act compositely with a reinforced concrete roadway. This bridge has a 30-ft roadway with a 3-ft safety curb on both sides

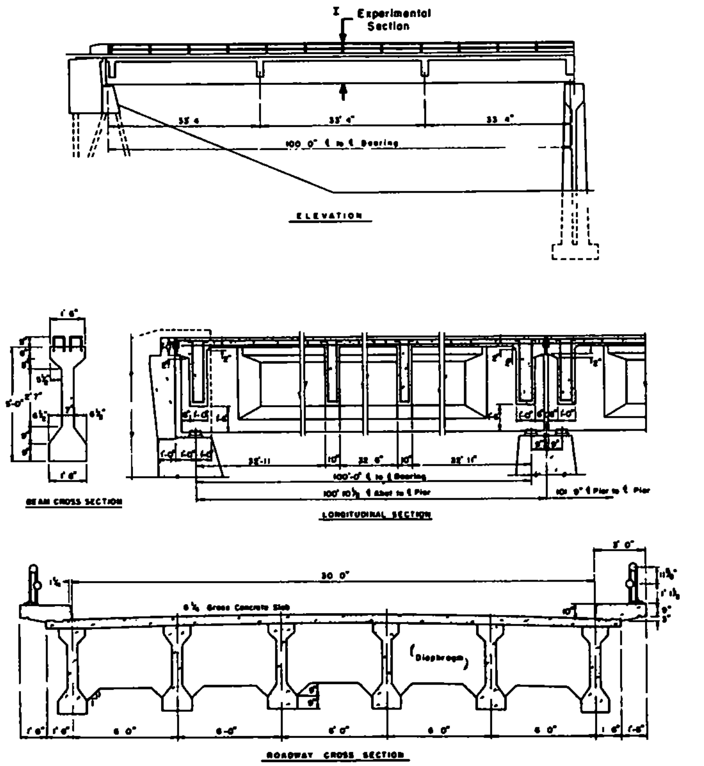

Figure 1. Details of simple span prestressed concrete bridge.

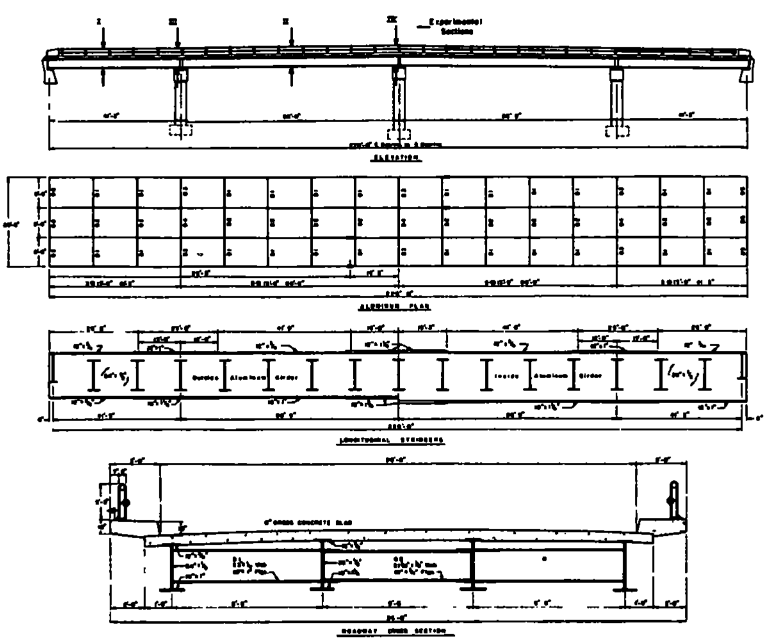

Figure 2. Details of continuous aluminum stringer bridge.

(Fig. 2). It carries traffic on Clive Road over Interstate 35 northwest of Des Moines, Iowa.

Continuous Steel Stringer Bridge.—This 240-ft continuous four-span structure is similar to the previous bridge except for the longitudinal stringers. The four steel stringers act compositely with a reinforced concrete roadway that is 28 ft wide with a 3-ft safety curb on both sides (Fig. 3). This structure carries the traffic on Ashworth Road over Interstate 35 west of Des Moines, Iowa.

Partially Continuous Prestressed Bridge.—This four-span bridge is 198.75 ft long with a 24-ft roadway. The reinforced concrete roadway slab is continuous over the interior supports and has a 2-ft safety curb on both sides. In each of the four spans there are six pretensioned prestressed concrete beams. The ends of the simple span beams are encased by a cast-in-place diaphragm at the piers. These pier diaphragms plus the continuous roadway slab, which acts compositely with the stringers, result in a relatively continuous bridge structure (Fig. 4). This structure carries traffic over Interstate 35 at the Cumming Interchange southwest of Des Moines, Iowa.

Test Vehicles

The vehicle effect has been simplified as much as possible in the theoretical analysis. The only parameters considered to be affected by the vehicles are the forcing function and the loaded frequency of the bridge. The forcing function is a function of the axle spacing and the velocity of the vehicle, and the loaded frequency of the bridge is a function of the ratio of the mass of the vehicle to the mass of the bridge span. The other variables of the loading vehicles, and there are many, were disregarded.

Vehicle A is an International L-190 van-type truck (Fig. 5). This truck, used to check the Iowa State Highway Commission scales, has a wheelbase of 14 ft 8 in. and a

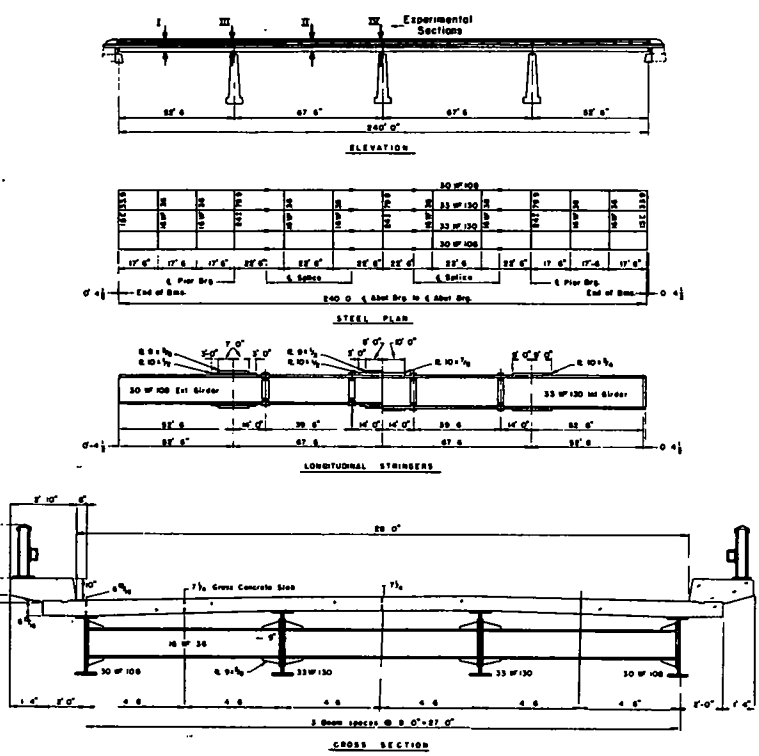

Figure 3. Details of continuous steel WF stringer bridge.

tread of 6 ft. It weighs 40,650 lb with 31,860 lb on the rear tandem axle. The forced vibration resulting from this vehicle at any velocity has two possible frequencies; that is, this vehicle could have the forced vibration frequency determined by the passage of

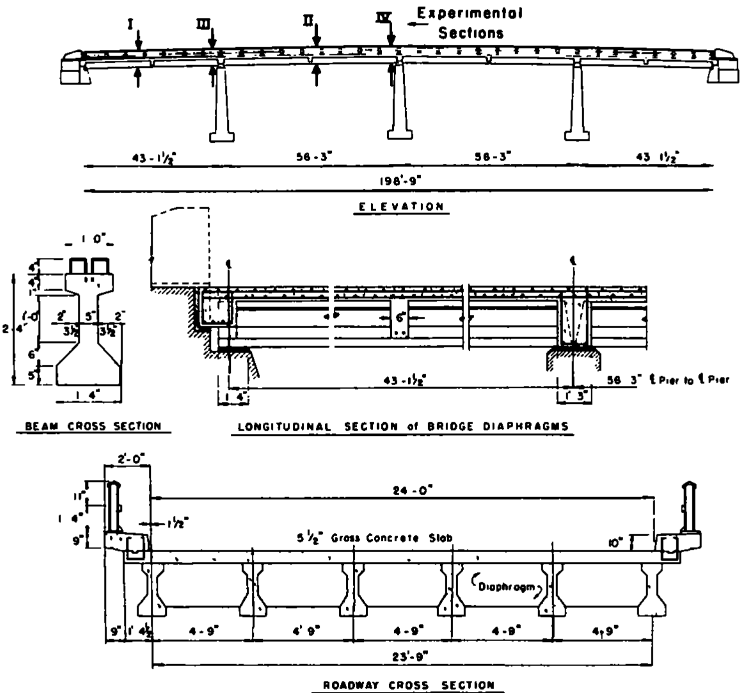

Figure 4. Details of partially continuous prestressed concrete bridge.

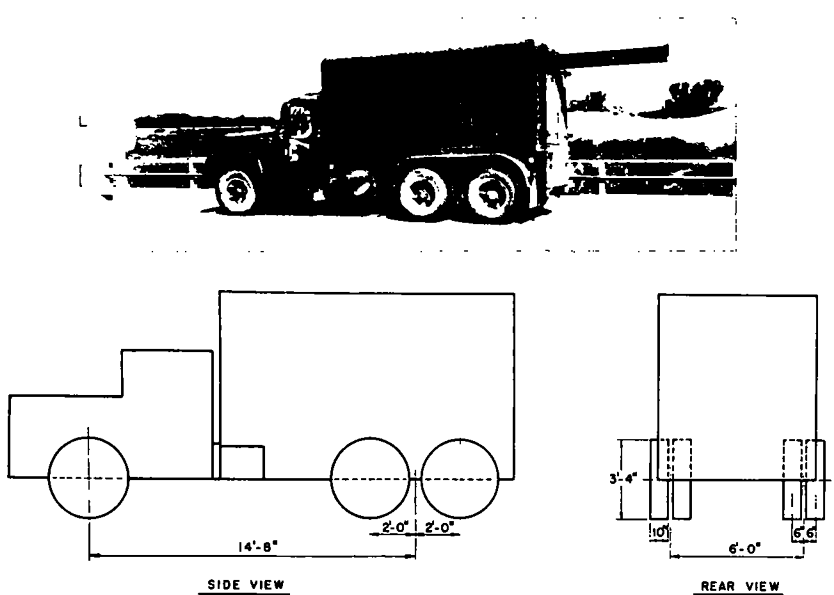

Figure 5. Vehicle A.

the individual axles in the tandem rear axle, in which the forcing frequency is v/4, or it could have a frequency determined by the passage of the front and rear axles, in which the forcing frequency is v/14.67. In the latter the axle spacing has been taken as the distance to the center of the rear tandems.

Vehicle B is a tandem axle, International VF-190 truck tractor pulling a 36-ft Monnon flat bed trailer (Fig. 6). The tractor has a wheelbase of 13 ft 1 in. and a tread of 6 ft. The trailer wheelbase is 23 ft, and the tread of the trailer wheels is 6 ft. The total weight of this vehicle is 73,500 lb, with 32,900 lb on the trailer tandem axle and 31,700 lb on the tractor tandem rear axle. This vehicle has three effective axle spacings, and therefore the forced vibration resulting from this vehicle for any given velocity has three possible frequencies. These three frequencies are v/4 resulting from the individual axle spacings of the tractor and trailer tandem axles, v/13.08 resulting from the tractor wheelbase axle spacing, and v/23 resulting from the trailer wheelbase axle spacing. For the tractor and trailer wheelbase, the axle spacing has been taken as the distance to the center of the tandems.

Instrumentation

To determine the dynamic effect of the vehicles, the static and dynamic bridge moments were computed from the strain measured at the extreme bottom fiber of each stringer. To measure the strains, standard SR-4 strain gages were used. The types of SR-4 gages used were A-1, A-5, and A-9. The resistance to the ground of the SR-4 gages used on the steel and aluminum girders was as follows: the A-1 gages, 1,000,000 to 1,000,000 ohms; the A-5 gages, 500,000 to 1,000,000 ohms. The A-9 gages have approximately a 6-in. gage length and were used to record the strains in the concrete girders.

The strain readings were recorded by a Brush universal amplifier (BL-520) and a Brush direct-writing recorder (BL-274). This equipment produces a continuous record of strain for which the time base can be varied by the speed of the recording paper. The speeds available vary from 1 to to 250 mm per sec. For a check of the time base as determined by the speed of the paper, a 1-sec timer was used to actuate an event marker on the edge of the record. The Brush Universal amplifiers have a number of attenuator settings which vary from 1 μ in. per in. of strain per Attenuator-Line to 1,000 μ in. per in. of strain per Attenuator-Line, and therefore allow a wide choice of amplification of the strain. The power for this Brush recording equipment was obtained from a 10 KW Onan motor generator.

The strains were measured in all the stringers at the centerline of the single span bridge and in the outer and inner spans and at the interior supports for the continuous bridges. This allowed the impact to be evaluated at all the sections of maximum bending

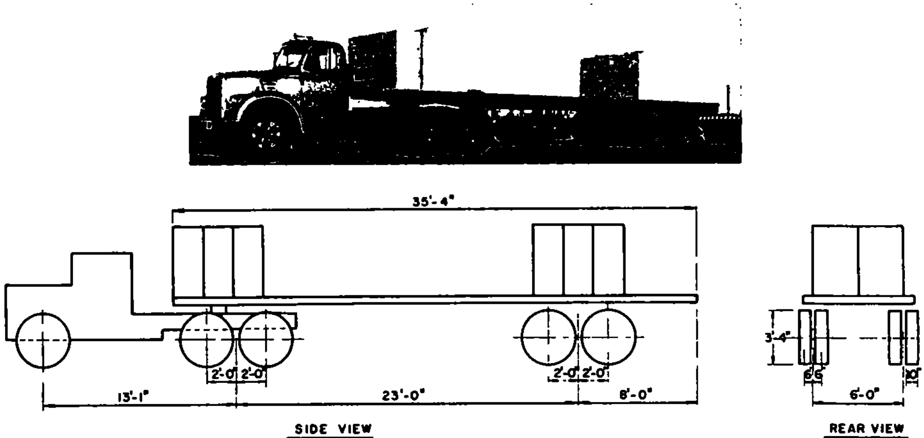

Figure 6. Vehicle B.

moment for the entire length of the bridge structures. Because the continuous bridges are symmetrical about their center interior support it was necessary to instrument only one-half of these bridges with strain gages.

Experimental Sections.—The experimental sections instrumented for the evaluation of the bridge moments are described and shown on the elevation view of each respective bridge plan.

1. Section I is located at a point four-tenths of the outer span from the end support for all the four-span continuous bridges. Section I for the simple span prestressed concrete bridge is at the middle of the span.

2. Section II is located at the middle of the interior span for the continuous steel stringer and aluminum stringer bridges. In the partially continuous four-span prestressed concrete structure, Section II was offset 1 ft 6 in. toward the center interior support to eliminate the effect of a transverse diaphragm at the middle of the interior span.

3. Section III is located at the first interior support of the continuous bridges. To eliminate or reduce any effect the reaction diaphragms might have, Section III was offset from the centerline of the reaction toward the exterior span 1 ft 6 in. and 1 ft 8 in. for the aluminum stringer, the steel stringer, and the prestressed concrete stringer bridges, respectively.

4. Section IV is located at the center interior support of the continuous bridges. This section is offset from the centerline of the reaction a distance equal to the offset of Section III for each respective continuous structure.

All of the bridges were instrumented at each of these section with an SR-4 strain gage at the center of the bottom flange, or the extreme lower fiber of each stringer.

Experimental Neutral Axes.—To obtain the moments required to evaluate the impact, the section moduli or relative section moduli of the stringers were required at the sections where the strains were measured. The effective section of the steel and the aluminum stringers vary considerably depending on their cross-section, due to cover plates or variable flanges, and the proximity of the curbing to the outer stringers. These changes in cross-section result in large changes in the moments of inertia and section moduli from one section to another. The actual section moduli and moments of inertia of the longitudinal stringers were determined experimentally by obtaining the position of the neutral axis of the longitudinal stringers. Because the bridges are symmetrical about their lateral and longitudinal centerlines it was necessary to instrument only one quadrant of each bridge for the determination of the position of the neutral axes of all the experimental sections used to evaluate impact. To obtain the neutral axis five SR-4 strain gages were positioned on each stringer. One gage was located at the center of gravity of the longitudinal stringer, and the other four gages at the extreme fibers and the quarter points of the stringer. The locations of the neutral axes were then used to determine the amount of concrete slab that acts compositely with the stringers. The entire roadway slab thickness was used in these calculations. The moment of inertia was then determined using the necessary amount of slab. A modular ratio of 10 was used for the steel stringer bridge and a ratio of 3.33 was used for the aluminum stringer bridge in these calculations. However, once the position of the neutral axis is known the moment of inertia is independent of the modular ratio used.

In both of the prestressed concrete stringer bridges, the lateral spacing of the stringers is much smaller and the cross-sections of the stringers do not vary along the beams. Moreover, the magnitude of the strains in the web and upper flanges of the prestressed concrete stringers was so small that it made the determination of a neutral axis very uncertain. Therefore, the section moduli of the longitudinal prestressed concrete stringers were assumed to be equal at each section investigated. It was found that in the steel and aluminum stringers in which the experimental neutral axes were determined the actual variation in the section moduli made very little difference in the impact because the impact is a difference in moments or a relative difference in the recorded strains. Thus the assumption made in the prestress concrete bridges will not appreciably affect the results regardless of the exact section moduli.

Experimental Procedure

The impact resulting from the action of the loading vehicles has been derived analytically. To determine experimentally the dynamic effect (the impact), static tests were first performed by the loading vehicle creeping across the bridge with the motor idling. The maximum moment in the bridge cross-section and the longitudinal position of the vehicle were computed. This was used as a base for the evaluation of the results of the dynamic tests. The dynamic tests were then conducted at vehicle speeds beginning at approximately 10 mph and increasing by increments up to the maximum attainable speed. The maximum dynamic moment was obtained in the cross-section for the vehicle in approximately the same longitudinal position as the maximum static moment. The dynamic and static tests were performed along four different lanes on the bridge roadway, two lanes for each direction of travel with one lane corresponding to the highway lane and the other lane at the longitudinal centerline of the bridge. For each assigned lane, the left front tire on the vehicle was guided along a painted stripe indicating the lane on the bridge roadway. During the runs a variation to one side or the other of the painted stripe was never more than 1½ in.

Pneumatic tubes were placed across the bridge roadway at the centerline of one exterior support and at the centerline of the center interior support for the continuous bridges and at the centerline of both exterior supports for the simple span bridge. The signal produced when the vehicle tire passed over this tube activated an event marker on the strain record. Knowing the chart speed and the distance between tubes, the average vehicle velocity was computed. These event markers on the strain record also enabled the longitudinal position of the vehicle to be determined at any time.

The testing of the continuous aluminum and steel stringer bridges was divided into two series for both test vehicles due to the limitation of the number of channels of Brush recording equipment. Section I and III were tested in one series and Sections II and IV in the second series. Both vehicles A and B were used in the dynamic testing of these bridges. The increased number of stringers in both the prestressed concrete bridges necessitated one series of tests for the test vehicle for each experimental section. Only Vehicle A was used in the dynamic testing of these bridges. At each test section the strain was measured at the extreme lower fiber of the stringers. In each series of tests the vehicle made four static runs, one in each lane, and sixteen or twenty dynamic runs, four or five in each lane, depending on the maximum speed obtainable for the particular structure. A continuous strain time record was obtained for each run. Each strain record, therefore, contains a continuous recording of the outer fiber strains for the stringers at the test section, an event marker trace for the longitudinal location of the vehicle and vehicle speed, and a time base with a 1-sec interval.

The test record shown in Figure 7 is a typical dynamic strain record showing the variation of the outer fiber strain as a vehicle moves across the bridge. The static strain time curve has been superimposed on the dynamic strain time curve and is indicated by a dotted line. This record was obtained from a stringer at Section I of the simple span prestressed concrete stringer bridge with Vehicle A traversing the bridge at 38.4 ft per sec.

The maximum static bridge moment is obtained by summing the moments in all the stringers computed from the maximum static strains. Similarly, the total maximum dynamic bridge moment is determined by summing the dynamic moments in all the stringers computed from the maximum dynamic strains. The dynamic effect, or the impact, of the vehicle was then evaluated from the moments as the ratio of the difference of the total dynamic and static bridge moments to the total static bridge moment.

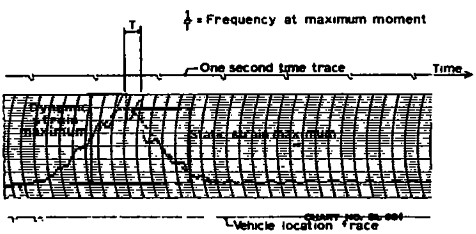

Figure 7. Typical strain record.

The passage of the front axle and each individual axle of the tandems over the pneumatic tubes is clearly shown by the vehicle location trace. In effect, the vehicle is moving from left to right, and this trace indicates the time each individual axle crosses over the centerline of the exterior supports.

The upper event marker, used as the time base, indicates time in 1-sec intervals. This time base was used for determining the vehicle speed and the frequency of bridge vibration. The frequency of vibration of the bridge at maximum moment was determined by using the maximum peak-to-peak period of vibration indicated by the T in Figure 7.

The amplitude of the residual vibration that continues after the vehicle has gone off the bridge was very small in this run. This was the usual case for the concrete stringer bridges; however, the amplitude of residual vibration for the steel and aluminum stringer bridges was usually much larger. The unloaded natural frequency of the bridge and the bridge damping was evaluated from this residual vibration.

RESULTS

Natural Frequencies

The theoretical methods that may be employed to determine the natural frequency of bridges and the effect of live load on the natural frequencies of bridges can be found in many references on vibrations and are discussed by the authors (13).

First Mode of Vibration.—The theoretically computed natural frequencies determined by the authors (5, 13) are compared in Table 1 with the experimentally obtained natural frequencies. The moment of inertia used in the theoretical computations is the moment of inertia of the entire cross-section and includes the sidewalk curb. A modular ratio, the modulus of elasticity of the stringer over the modulus of elasticity of the reinforced

TABLE 1

NATURAL FREQUENCIES

Frequency[1]	Bridge: Aluminum Stringer	Steel Stringer	Continuous Concrete Stringer	Simple Span Concrete Stringer
L_2(ft)	68.75	67.50	56.25	100.0
$\frac{L_1}{L_2}$	0.600	0.777	0.766	-
E_2I_2(psi)	185.1×10^{10}	213.4×10^{10}	197.8×10^{10}	$1,609 \times 10^{10}$
$\frac{E_1I_1}{E_2I_2}$	0.615	1.0	1.0	-
$m_2 \frac{(lb/sec^2)}{(in.^2)}$	0.924	0.889	0.889	1.721
$\frac{m_1}{m_2}$	0.989	1.0	1.0	-
KL_2(rad)	3.400	3.399	3.408	3.1416
$f_{theo.}$ (cps)	3.825	4.34	6.06	3.34
$f_{exper.}$ (cps)	3.97	4.57	7.80	4.26
$\frac{f_{theo.}}{f_{exper.}}$	0.964	0.951	0.780	0.784

[1]Subscripts $_1$ and $_2$ indicate exterior and interior spans, respectively.

concrete roadway slab, of 3.44, 10, and 1.25 was used in the aluminum, steel, and both concrete bridges, respectively. Thus the modulus of elasticity of the reinforced concrete roadway slab and the prestressed concrete stringers were taken as the value used in design for each case; this probably accounts for most of the error in the theoretical frequency determination of the prestressed concrete stringer bridges. It was observed during the experimental testing that the natural frequency of the bridges reduced more than theory indicates it should, when a vehicle first enters the bridge span. However, once the vehicle is on the bridge the reduction in natural frequency, as the vehicle position changes, is similar to the theoretically calculated value; but it is extremely difficult to measure accurately.

Higher Modes of Vibration.—The first mode of vibration was usually found to be prevalent in controlling the response of the bridges to the forcing function of the axles. This was true in most cases and at Sections III and IV where the first mode, or higher odd modes, have the least effect. However, an outstanding exception occurred in the case of the aluminum stringer bridge. In this structure the experimental impact at Section IV, the section at the center interior support, was found to be a function of a higher mode of vibration. The resonance condition in this case is a function of the second mode. This is the first root of the even-mode frequency equation for a four-span symmetrical bridge, and corresponds approximately to the vibration of the beam with both ends fixed. Therefore, it is reasonable to assume that this vibration, when it occurs, will result in the largest dynamic increase in moment at the supports. The second mode frequency computed theoretically agrees closely with the measured frequencies occurring while the vehicle was on the inner spans vibrating the bridge at its second mode. However, this frequency could not be compared with an experimental unloaded natural frequency because this mode of vibration occurred only when the vehicle was on the inner span.

Effect of Vehicle.—The loaded natural frequency is the natural frequency of the bridge which occurs when the vehicle is on the span. This value of loaded natural frequency will control the resonance conditions of the frequency of the vehicle forcing function with the natural frequency of the bridge. This resonance condition has the greatest effect on the amount of impact when the vehicle is near the position of maximum moment.

The reduction in natural frequency due to the mass of the vehicle has been theoretically determined and although it could not be correlated with the experimental reduction, due to the difficulty of measuring it, it is desirable that the effect of the vehicle mass be taken into account.

The different lengths of the spans in the continuous bridges result in a different loaded natural frequency for the load in each span. Therefore, in the correlation of the experimental and theoretical impact, an impact curve is obtained for the loaded frequency as each span is loaded. Moreover, inasmuch as the truck tractor and trailer of Vehicle B were used as separate masses, the reduction in frequency is different for each part of the vehicle. All of the various values of loaded frequency will have an individual impact curve determined by Eq. 15. To reduce the number of these closely spaced curves and to simplify the presentation of the impact data, only two curves are shown for the reduction in natural frequency. These curves are for Vehicle A and the truck tractor of Vehicle B in the outer and inner spans. These two loads have the same effect on the reduction in frequency because their masses are within 0.1 percent of each other. The various theoretical loaded natural frequencies obtained for the vehicles in the outer and inner spans are 98 and 94.9 percent, 97.1 and 95.2 percent and 96.6 and 94.6 percent of the unloaded natural frequencies of the continuous aluminum, steel and concrete stringer bridges, respectively. The loaded natural frequency of the simple span bridge is 95.3 percent of the unloaded natural frequency for Vehicle A at the center of the span.

Forced Vibration

In the determination of impact, the frequency of the forcing function of the vehicle has been taken as the cyclical repetition of the axles. This cyclical repetition is determined by the frequency of passage of the axles across the bridge. To determine the applicability of this concept, it must be shown that the forcing frequency of the axles is predominant in the forced vibration of the bridge, or that the response of the bridge is similar to that of a steady state forced vibration. The frequency of vibration of the structure was determined at the time the vehicle was producing the maximum moment. This value of frequency was obtained by using the one or two cycles of vibration at the maximum amplitudes of vibration. It was found in this experimental work that the natural frequency of the structure was prevalent as the vehicle entered the bridge, and further, that this natural frequency more nearly corresponds to the computed value than to the experimental value of natural frequency obtained after the vehicle had left the bridge. As the vehicle approached the position of maximum moment the frequency became approximately equal to the frequency of the forcing function (Figs. 8 to 12). Because there are two different forcing frequencies available for Vehicle A and three for Vehicle B, there were a number of different frequencies that could be used as the frequency of the forcing function. However, only one axle spacing was predominant in determining the frequency of the forcing function. This is readily shown in Figures 8 to 12 in which the frequency of the bridge vibration caused by the vehicle at the maximum moment point (w = v/s) is shown as a function of the velocity of the vehicle. Variations in this result from the tendency of the bridge vibration to remain near the resonant frequency of the structure at higher speeds where the forcing frequency is impressed by the axle spacing of the vehicle wheelbase.

An exception to the well-defined forcing frequency of the velocity divided by the axle spacing occurred in the continuous prestressed concrete bridge (Fig. 13). This structure was constructed by placing a continuous reinforced concrete roadway over four

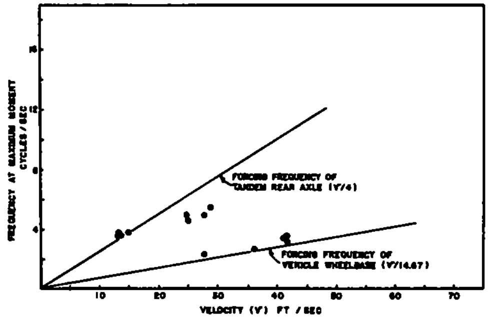

Figure 8. Frequency of forced vibration of simple span concrete bridge for Vehicle A.

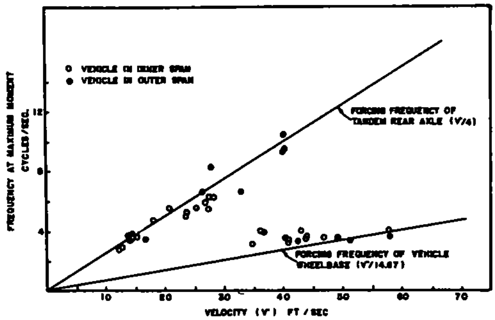

Figure 9. Frequency of forced vibration of aluminum stringer bridge for Vehicle A.

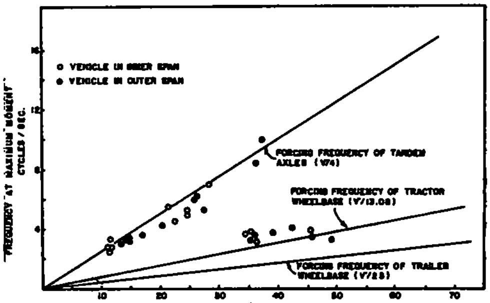

Figure 10. Frequency of forced vibration of aluminum stringer bridge for Vehicle B.

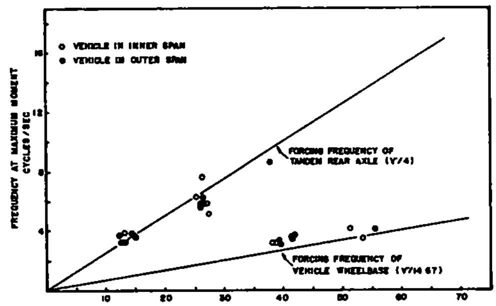

Figure 11. Frequency of forced vibration of steel stringer bridge for Vehicle A.

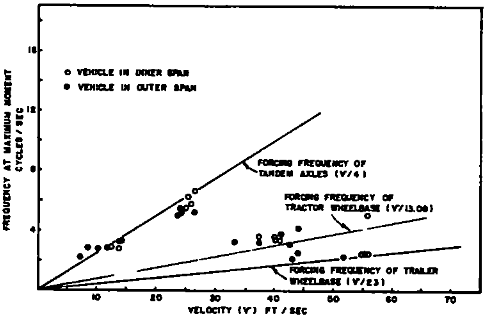

Figure 12. Frequency of forced vibration of steel stringer bridge for Vehicle B.

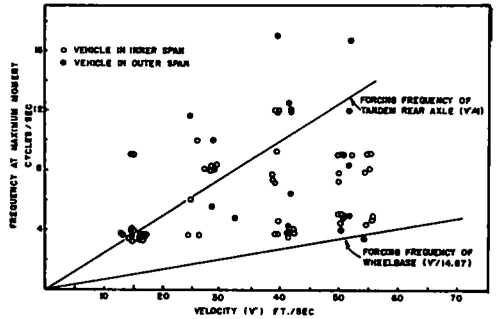

Figure 13. Frequency of forced vibration of continuous concrete stringer bridge for Vehicle A.

spans of simply supported prestressed concrete beams. Unlike the other bridges tested, this bridge does not have a point bearing to allow free rotation at the supports and it is not fully continuous. The interior supports have a 15-in. reinforced concrete diaphragm resting on an 11/32-in. preformed fabric bearing pad. These diaphragms encase the ends of the beams at each interior support and combine with the roadway slab to make the structure partially continuous. The exterior supports have approximately 16 in. of the end of the beam resting on similar 11/32-in. bearing pads. The effect of the large, flat bearing surfaces at the supports heavily damps the vibration of the continuous bridge. These bearings also cause a certain amount of fixity at each support, thus further complicating the vibratory system. Moreover, the pier diaphragms acting with the continuous reinforced concrete roadway slab allow only the negative moments to be transmitted across the piers or interior supports. Positive bending at the piers is eliminated due to the tension in the bottom fibers of the pier diaphragms. These diaphragms are not reinforced to resist tension in that direction. Therefore, it is very difficult to establish a well-defined vibratory system in such an incongruous structure. This is shown in Figure 14 by the random vibration of the structure at the maximum moment which results from the passage of Vehicle A. For this reason, the application of the forced vibration theory presented herein for the determination of the response of this structure to the forcing function of the repetition of axles has little significance.

The impact as determined herein is a function of the amplitude of forced vibration. The derivation of the theoretical impact was made by assuming that the forcing frequency of the axles was predominant in producing the impact. The denominator of the theoretical impact factor is a function of the ratio of the forcing freqency to the loaded natural frequency of the structure and the ratio of the damping factor to the unloaded natural frequency of the structure. The numerator of this impact factor is a function of the ratio velocity to the length of the span. Therefore, because the forcing frequency is the ratio of the velocity of the vehicle to the axle spacing, the magnitude of the theoretical impact will depend on the velocity, axle spacing, length of span, loaded natural frequency, unloaded natural frequency, and the damping factor.

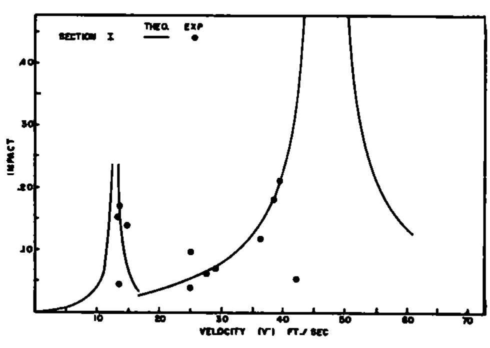

Figure 14. Impact for simple span concrete stringer bridge at Section I for Vehicle A.

The damping factor was obtained experimentally from the decreasing amplitude of the residual vibrations. To determine this experimentally, the amplitude of displacement Y of the strain time curve is measured at time t_0 and at a later time t_N which is N cycles later. The ratio of these amplitudes (Y_0/Y_N) is a constant, for viscous damping, and 1/N times the natural logarithm of this ratio is called the average logarithmic decrement. This quantity therefore does not depend on the way the damping was defined in the original equation of motion and thus is often used as the measure of the damping capacity of a structure. The average logarithmic decrement is then given as

$$\frac{1}{N} \log_e \frac{Y_0}{Y_N}$$

The damping capacity of each bridge is given in terms of the average logarithmic decrement.

Simple Span Prestressed Concrete Bridge.—The correlation of the experimental and theoretical impact for the post-tensioned prestressed concrete bridge is shown in Figure 14. The experimental impact values determined at the centerline of the simple span (Section 1) are shown with the theoretical impact curves obtained by Eq. 15. A loaded natural frequency which is 95.3 percent of the theoretical natural frequency of 3.34 cycles per sec was used in determining the theoretical impact curves. The average logarithmic decrement for this bridge is 0.0916. The resulting amount of damping did not affect the theoretical curves except at resonance. Therefore, for the portion of the impact curves shown in this figure, the effect of the damping is insignificant. Resonance occurs when the ratio of the forcing frequency or the frequency of the repetition of the axles to the loaded natural frequency of the structure is one. This condition occurs two times for Vehicle A. The individual axles of the tandem rear axle unit acting individually cause a resonance at the smaller velocities, and the front axle combined with the tandem rear axle acting as one unit cause resonance at the larger velocities. The impact increases as the ratio of the forcing function to the loaded natural frequency approaches one. The experimental impact values agree with the theoretical impact curves, which, as previously discussed yield an upper limit of impact for the assumptions made in the derivation. The maximum vehicle velocity limited a complete investigation of the wheelbase resonance condition.

Continuous Aluminum Stringer Bridge.—The experimental and theoretical impact for this structure is shown in Figures 15 to 18. The theoretical curves show a good agreement with the experimental impact values. As previously discussed an additional resonance occurred in this structure when the bridge was excited at its second mode of vibration by the individual axles of the tandem rear axle unit. This condition is most prominent at the center interior support. A correlation of the theory presented herein for the upper limit of the wheelbase resonance condition was not obtained due to the limited velocity of the vehicles. Similarly, the resonance condition of the trailer wheelbase could not be investigated. A loaded natural frequency of 98.0 percent and

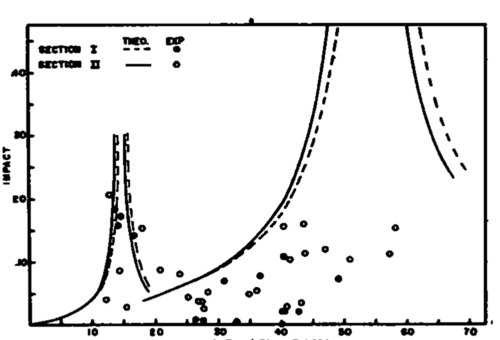

Figure 15. Impact for aluminum stringer bridge at Sections I & II for Vehicle A.

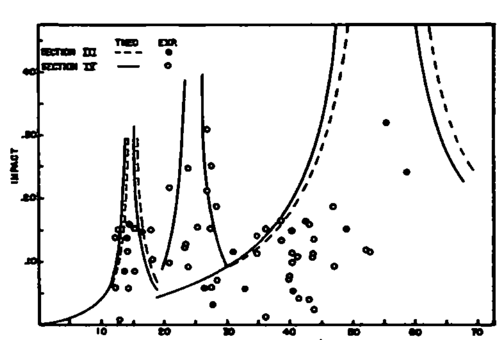

Figure 16. Impact for aluminum stringer bridge at Sections III & IV for Vehicle A.

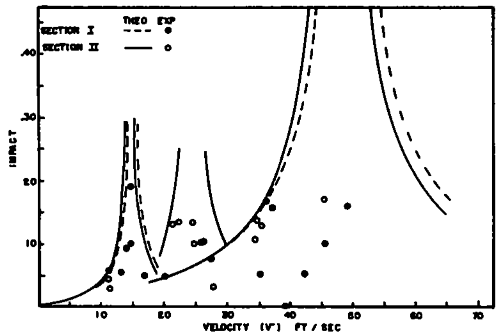

Figure 17. Impact for aluminum stringer bridge at Sections I & II for Vehicle B.

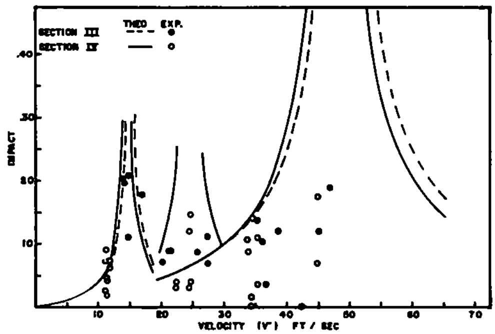

Figure 18. Impact for aluminum stringer bridge at Sections III & IV for Vehicle B.

94.9 percent of the theoretical natural frequency of 3.825 cycles per sec was used for the outer and inner span resonance curves, respectively. The average logarithmic decrement for this bridge is 0.050. The resulting amount of damping did not affect the theoretical curves except at resonance. The maximum values of impact written as a percentage vary from 20.6 to 31.9 percent and 19.1 to 20.8 percent for Vehicles A and B at the positive and negative sections, respectively. Moreover, the resonance condition of the individual axles of the tandem rear axle unit causes an experimental impact almost as large as the resonance condition of the vehicle wheelbase at higher velocities. Therefore, the resonance effect of the repetition of axles is important at the slower speeds.

Continuous Steel Stringer Bridge.—The correlation of the experimental and theoretical impact for this bridge is shown in Figures 19 to 22. More experimental impact values lie outside the theoretical impact envelope in this bridge than in the previous bridges. The greatest discrepancy occurs as the resonance condition is approached from the left side of the figure. That is, the large number of experimental points outside the theoretical envelope at velocities lower than the resonance velocities might result from the loaded natural frequency of the bridge being smaller than the value used to obtain the impact curves. A smaller loaded natural frequency would move the theoretical curves to the left in these figures. However, the theoretical curves shown still qualitatively describe the variations in the experimental impact. There is no indication in this structure of any higher modes of vibration. Moreover, not enough experimental data were obtained for a good evaluation of the resonance condition of the individual axles of the tandem axle unit with the first mode of vibration. Therefore, the experimental impact values for the tandem axles were smaller than those obtained by the resonance condition for the vehicle wheelbase. Also, a large enough velocity was not obtained for the trailer wheelbase to cause a resonance condition. The maximum values of impact, written as a percentage, vary from 44.1 to 26.5 percent and 22.8 to 39.2 percent for Vehicles A and B at the positive and negative sections, respectively. A loaded natural frequency of 97.1 and 95.2 percent of the theoretical unloaded natural frequency of 4.34 cycles per sec was used for the outer and inner span impact curves, respectively. The average logarithmic decrement of this bridge is 0.062. This amount of damping did not affect the theoretical curves except at resonance.

Partially Continuous Concrete Stringer Bridge.—The correlation of the experimental and theoretical impact for this bridge is shown in Figures 23 and 24. A loaded natural frequency of 94.6 percent of the theoretical unloaded natural frequency of 6.06 cycles per sec was used to obtain this curve. The curve for the vehicle on the outer span is not shown inasmuch as it is just to the right of this curve similar to those in the previous figures. This curve includes the effect of damping, which was considerably larger for this structure than for the previous structures. The average logarithmic decrement

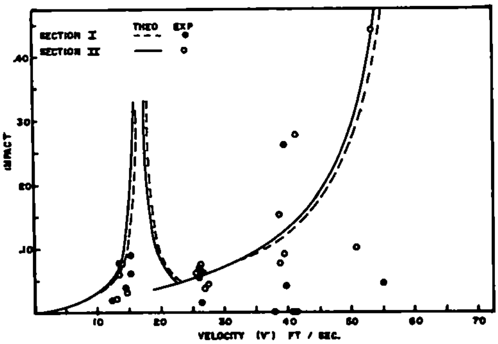

Figure 19. Impact for steel stringer bridge at Sections I & II for Vehicle A.

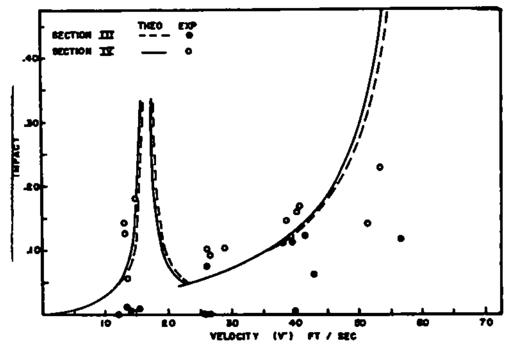

Figure 20. Impact for steel stringer bridge at Sections III & IV for Vehicle A.

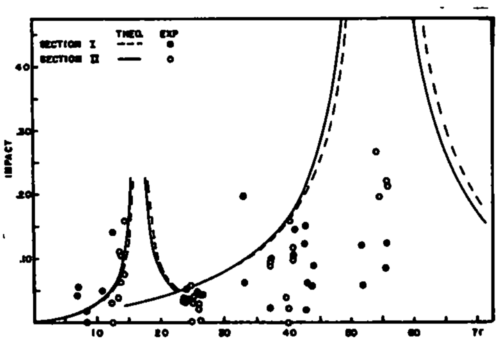

Figure 21. Impact for steel stringer bridge at Sections I & II for Vehicle B.

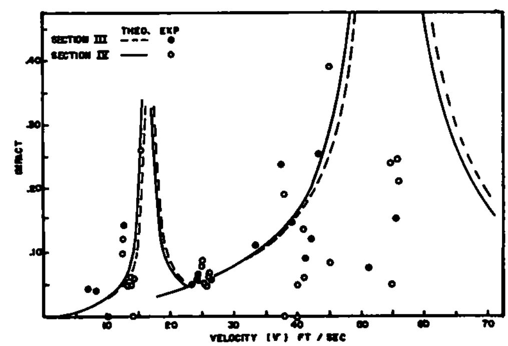

Figure 22. Impact for steel stringer bridge at Sections III & IV for Vehicle B.

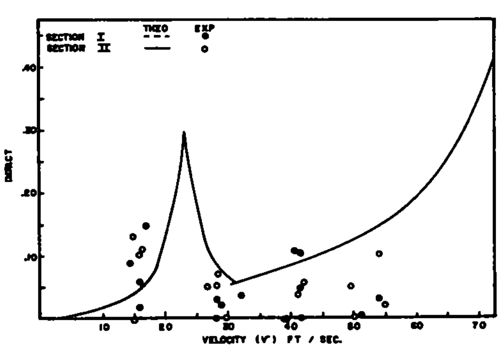

Figure 23. Impact for continuous concrete stringer bridge at Sections I & II for Vehicle A.

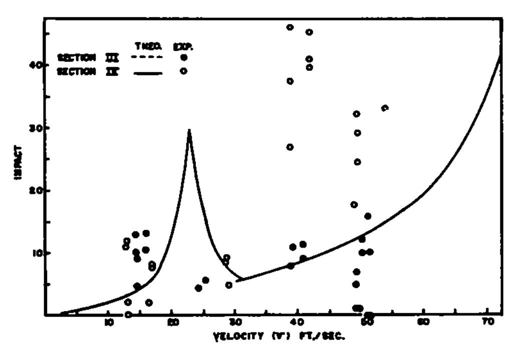

Figure 24. Impact for continuous concrete stringer bridge at Sections III & IV for Vehicle A.

of the residual vibration of this structure is 0.406. This amount of damping results in a reasonable upper limit for the impact curves of 0.298 for the resonance condition caused by the individual axles of the tandem axle unit. Because there are not enough experimental impact data at the velocity corresponding to this resonance condition, this upper limit could not be verified. The experimental impact values show some agreement with the theory at the positive moment sections. However, at the negative moment sections some of the impact values are large at the higher velocities. But, because the total moment in the section is small, the large impact value does not result in an overstress. The large impact results because a very small static live load strain at the center interior support was obtained as the vehicle moved across the bridge with the motor idling. Then, when the load was applied dynamically at larger velocities, the strains became significantly larger. Therefore, the values of impact at Section IV become quite large due to a relatively large increase in small value. This action is the result of the different dynamic and static responses of a bridge that acts as a continuous beam for negative moments at the interior supports and as a simply supported beam for positive moments at these supports. This inconsistent response is, therefore, due to the nonhomogeneous or incongruous structural system. It was shown previously that the experimental unloaded natural frequency of this structure corresponds to the theoretical unloaded natural frequency determined on the basis of a continuous structure. However, the forced vibration frequency of this structure did not correspond at all to the frequency of the forcing function at maximum moment. These inconsistencies in the response of the bridge will not allow the dynamic response of this structure to be analyzed by the forced vibration analysis as presented herein.

CONCLUSIONS

Natural Frequency

The theoretical unloaded natural frequencies of the bridge structures, neglecting damping, agree very well with the experimentally determined unloaded natural frequencies for the aluminum and steel stringer bridges and rather well for the concrete stringer bridges. When the theoretical loaded natural frequency reduction is applied to both frequencies to obtain the theoretical and experimental loaded natural frequencies, it is found that the theoretical loaded natural frequency compares better with the forced vibration resonances than the experimental loaded natural frequency, which was quite difficult to determine accurately.

The reduction in natural frequency due to the addition of the vehicle mass to the vibratory system was analyzed by the authors using an energy method (13). The use of an assumed sinusoidal curve enabled the effect of the mass of the vehicle on the unloaded natural frequency to be taken into account. The accuracy of the sinusoidal curve in determining the unloaded natural frequency also led to the assumption that the deflection curve in the forced vibration analysis of continuous bridges is a sine curve.

Forcing Function

The effect of the vehicle is assumed to be an oscillating forcing function whose frequency is the frequency of axle repetition and whose oscillating force is the oscillating load effect of a constant force. Thus the effect of the vehicle has been simplified as much as possible. The correlation of the theoretical and experimental impact indicates that the simplifications made in the effect of the vehicle are justified for the bridges tested and the experimental velocities used.

Forced Vibration

Impact, as presented herein, is determined by the application of forced vibration theory to the dynamic problem of multiple axle vehicles traversing continuous highway bridges. The results of this study show a good qualitative correlation between the amount of impact and the proximity of the frequency of axle repetition to the loaded natural frequency of the structure, or the resonance condition. The correlation is good for bridge structures that have a well-defined vibratory system. That is, if the structure responds to a forced vibration without changing its natural modes of vibration, as obtained for the free vibrations of the structure, the response of the structure is similar to that of a steady state forced vibration.

RECOMMENDATIONS

The research contained in this study has taken into account, theoretically, the factors in a bridge structure that will affect its response to a forced vibration. The correlation of this theory is obtained by field testing of existing bridge structures. The field testing, as a means of evaluating impact indicates that forced vibration theory is applicable to the problem of bridge impact. However, the factors affecting the response of the bridge structure need to be investigated further. An example of this is the variation in the amount of composite action exhibited between the reinforced concrete slab and the longitudinal stringers along the length of the bridge. This aspect of the bridge structure is not entirely known for continuous bridges, and is basic to the response of the bridge to live loads. Another problem in the response of the bridge is the effect of the live load mass. A theoretical analysis was used in this research, but the irregularities exhibited by the concrete bridges indicate that the effect of the load may be more complicated than the simple effect of the mass of the vehicle. An experimental investigation of this phenomenon could include an additional objective which might be more important than the analysis of the effect of the live load mass. The other problem that could be investigated in a similar manner is the very important problem of an upper limit for the amplitude of forced vibration at resonance. At the onset of this research it was felt that the damping would limit the maximum impact for the resonance condition. It has been shown that this is not true for most of the bridges tested. This problem of the maximum amplitude of forced vibration (or maximum impact) and the lesser problem of the effect of the vehicle mass on the natural frequency of the structure could be investigated by a study of the forced vibration of a structure with a large variable speed oscillator. This oscillator, used in conjunction with a stationary vehicle on the bridge, would provide some answers to the problem of the natural frequency of the bridge and vehicle. In addition, the use of this oscillator with varying amounts of the oscillatory force could be used to investigate the problem of an upper limit for the impact curves. It is felt by the authors that a form of damping, which is evidently not viscous, limits the amplitude of the vibratory motion of the structure, notwithstanding the effect of the springing of the vehicle which has been disregarded. It is possible that this damping becomes a function of amplitude after the amplitude of vibratory motion exceeds a certain value.

The research indicated is based on the theory that before the effect of the various parameters of the vehicle and the roadway surface is integrated into the problem of impact, a thorough knowledge of the vibratory action and response of the structure is desirable.

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