Tables of Stresses in Three-Layer Elastic Systems

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The tables of values of stresses arising in a three-layer elastic system loaded over a circular area, published by Acum and Fox, have been extended to cover a wider range of parameters. Values of the vertical stresses at both interfaces are given, and are accurate to within 1 percent. The calculations were performed on the Ferranti Mark I* computer at Koninklijke/Shell Laboratorium, Amsterdam.

The work forms part of a program for the development of a method for the design of flexible roads.

•IN GENERAL, road design involves the determination of the thicknesses and elasticities of the various layers in the road structure to insure that the stresses and strains developed are within permissible limits. The present analysis assumes that the design is based on a knowledge of the stresses at the interfaces between the layers on the axis of symmetry below a circular loaded area.

The first analysis to take into account the different elastic properties of the various layers was given by Burmister (1) in 1943. In a series of subsequent papers (2) he derived expressions for the stresses and displacement in two- and three-layer systems. However, he did not publish any stress values and only a small number of values of the surface deflection at the centre of the loaded area for a two-layer system. Fox (3) produced extensive tables of stresses in two-layer systems in 1948, and in 1951 Acum and Fox (4) produced tables for the normal and radial stresses in three-layer systems at the intersections of the axis of symmetry with the interfaces. The tables in this paper form an extension, to a much wider range of the parameters, of those of Acum and Fox. (Stress and displacement values for one given system were published by Schiffman (5) in 1957).

The road system considered in this paper consists of three layers, as in Figure 1. The stresses that have been computed are listed in Table 1, where the numerical subscripts refer to the layer.

Throughout the tables it is assumed that Poisson's ratio is equal to 0.5 and that there is full friction at the interfaces. For convenience, the stresses are printed on the assumption that compression stress is positive.

Values of these stresses are given in Appendix A for all combinations of the following parameter values:

$E_1/E_2 = k_1 = 0.2, 2.0, 20.0, 200.0$
$E_2/E_3 = k_2 = 0.2, 2.0, 20.0, 200.0$
$a/h_2 = a_1 = 0.1, 0.2, 0.4, 0.8, 1.6, 3.2$
$h_1/h_2 = H = 0.125, 0.25, 0.5, 1.0, 2.0, 4.0, 8.0$

Each stress value is expressed as a fraction of the applied load. Stresses for all values of a_1 appear in a block and the blocks are arranged in groups of four in ascending order of k_2; these groups are arranged in order of k_1 and finally in sets in order of H.

The calculations were performed on the Ferranti Mark 1* computer at Koninklijke/Shell Laboratorium, Amsterdam. The machine program was written to calculate 36 stress values at the same time; viz., 6 stresses for each of 6 values of a_1. The infinite range of integration (see Appendix B) was truncated at such a point that the contribution from the omitted portion was negligible. The finite part of the range was then split into a number of equal intervals and the integrals were then evaluated over each interval

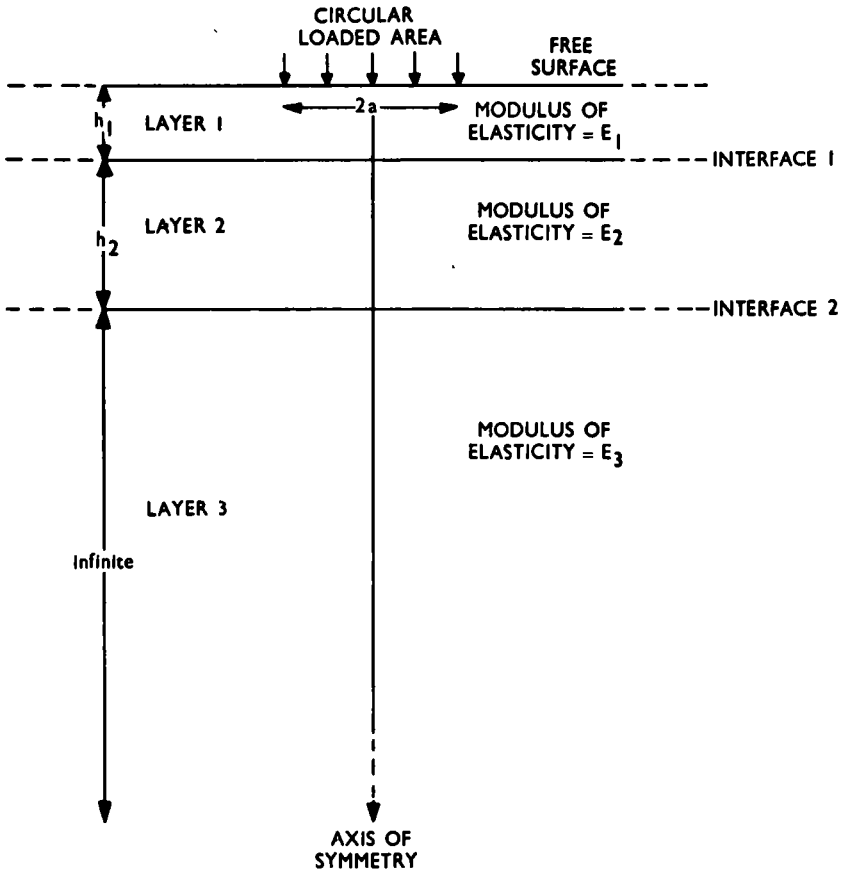

Figure 1. Structure of layered system.

TABLE 1

NOTATION FOR STRESSES GIVEN IN APPENDIX A

Stress	First Interface	Second Interface
Vertical	σ_{z_1}	σ_{z_2}
Vertical and radial (upper layer)	$\sigma_{z_1} - \sigma_{r_1}$	$\sigma_{z_2} - \sigma_{r_2}$
Vertical and radial (lower layer)	$\sigma_{z_1} - \sigma_{r_2}$	$\sigma_{z_2} - \sigma_{r_3}$

separately using a Gauss quadrature formula. Some difficulty was encountered near the origin due to the fact that the integrand had an essential singularity on the negative real axis, this was overcome by subdividing the first interval.

The following checks (which are perfect apart from a constant factor) were built into the computer program.

$$\sigma_{z_1} - \sigma_{r_1} = k_1 (\sigma_{z_1} - \sigma_{r_2}) \quad \text{(first interface)}$$

$$\sigma_{z_2} - \sigma_{r_2} = k_2 (\sigma_{z_2} - \sigma_{r_3}) \quad \text{(second interface)}$$

In most cases the two sides of these equations agreed to five significant digits but where this was not so the disagreement was still less than 1 percent. All numbers in the tables, however, are correct only to within one digit in the last decimal place printed.

As a further check some results were obtained for comparison with those of Acum and Fox. These are presented in Appendix C with the corresponding values from Acum and Fox's paper in brackets.

A typographical error was found in the published solution of Acum and Fox and was corrected before the tables were calculated.

A brief mathematical description of the problem is given in Appendix B, together with some notes on the assumptions involved in setting up the problem and the correct expressions for the stresses.

ACKNOWLEDGMENTS

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Appendix A

$H = 0.125$

$k_1 = 0.2$

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_r$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						$k_2 = 0.2$
0.1	0.66045	0.12438	0.62188	0.01557	0.00332	0.01659
0.2	0.90249	0.13546	0.67728	0.06027	0.01278	0.06391
0.4	0.95295	0.10428	0.52141	0.21282	0.04430	0.22150
0.8	0.99520	0.09011	0.45053	0.56395	0.10975	0.54877
1.6	1.00064	0.08777	0.43884	0.86253	0.13755	0.68777
3.2	0.99970	0.04129	0.20643	0.94143	0.10147	0.50736
						$k_2 = 2.0$
0.1	0.66048	0.12285	0.61424	0.00892	0.01693	0.00846
0.2	0.90157	0.12916	0.64582	0.03480	0.06558	0.03279
0.4	0.95120	0.08115	0.40576	0.12656	0.23257	0.11629
0.8	0.99235	0.01823	0.09113	0.37307	0.62863	0.31432
1.6	0.99918	-0.04136	-0.20680	0.74038	0.98754	0.49377
3.2	1.00032	-0.03804	-0.19075	0.97137	0.82102	0.41051
						$k_2 = 20.0$
0.1	0.66235	0.12032	0.60161	0.00256	0.03667	0.00183
0.2	0.90415	0.11787	0.58933	0.01011	0.14336	0.00717
0.4	0.95135	0.03474	0.17370	0.03838	0.52691	0.02635
0.8	0.98778	-0.14872	-0.74358	0.13049	1.61727	0.03086
1.6	0.99407	-0.50533	-2.52650	0.36442	3.58944	0.17947
3.2	0.99821	-0.80990	-4.05023	0.76669	5.15409	0.25770
						$k_2 = 200.0$
0.1	0.66266	0.11720	0.58599	0.00057	0.05413	0.00027
0.2	0.90370	0.10495	0.52477	0.00226	0.21314	0.00107
0.4	0.94719	-0.01709	-0.08543	0.00881	0.30400	0.00402
0.8	0.99105	-0.34427	-1.72134	0.03259	2.67934	0.01340
1.6	0.99146	-1.21129	-6.05643	0.11034	7.35978	0.03680
3.2	0.99332	-2.89282	-14.46408	0.32659	16.22830	0.08114

H = 0.125

k_1 = 2.0

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_2}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						k_2 = 0.2
0.1	0.43055	0.71614	0.35807	0.01682	0.00350	0.01750
0.2	0.78688	1.01561	0.50780	0.06511	0.01348	0.06741
0.4	0.98760	0.83924	0.41962	0.23005	0.04669	0.23346
0.8	1.01028	0.63961	0.31981	0.60886	0.11484	0.57418
1.6	1.00647	0.65723	0.32862	0.90959	0.13726	0.68630
3.2	0.99822	0.38165	0.19093	0.94322	0.09467	0.47335
						k_2 = 2.0
0.1	0.42950	0.70622	0.35303	0.00896	0.01716	0.00858
0.2	0.78424	0.97956	0.48989	0.03493	0.06647	0.03324
0.4	0.98044	0.70970	0.35483	0.12667	0.23531	0.11766
0.8	0.99434	0.22319	0.11164	0.36932	0.63003	0.31501
1.6	0.99364	− 0.19982	− 0.09995	0.72113	0.97707	0.48853
3.2	0.99922	− 0.28916	− 0.14461	0.96148	0.84030	0.42015
						k_2 = 20.0
0.1	0.43022	0.69332	0.34662	0.00228	0.03467	0.00173
0.2	0.73414	0.92096	0.46048	0.00899	0.13541	0.00677
0.4	0.97493	0.46583	0.23297	0.03392	0.49523	0.02476
0.8	0.97806	− 0.66535	− 0.33270	0.11350	1.49612	0.07481
1.6	0.96921	− 2.82859	− 1.41430	0.31263	3.28512	0.16426
3.2	0.98591	− 5.27906	− 2.63954	0.68433	5.05952	0.25298
						k_2 = 200.0
0.1	0.42925	0.67488	0.33744	0.00046	0.04843	0.00024
0.2	0.78267	0.85397	0.42698	0.00183	0.19043	0.00095
0.4	0.97369	0.21165	0.10582	0.00711	0.71221	0.00356
0.8	0.97295	− 1.65954	− 0.82977	0.02597	2.32652	0.01163
1.6	0.95546	− 6.47707	− 3.23855	0.08700	6.26638	0.03133
3.2	0.96377	−16.67376	− 8.33691	0.26292	14.25621	0.07128

H = 0.125

k_1 = 20.0

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						k_2 = 0.2
0.1	0.14648	1.80805	0.09040	0.01645	0.00322	0.01611
0.2	0.39260	3.75440	0.18772	0.06407	0.01249	0.06244
0.4	0.80302	5.11847	0.25592	0.23135	0.04421	0.22105
0.8	1.06594	3.33600	0.16930	0.64741	0.11468	0.57342
1.6	1.02942	1.81603	0.09080	1.00911	0.13687	0.68436
3.2	0.99817	1.75101	0.08756	0.97317	0.07578	0.37890
						k_2 = 2.0
0.1	0.14529	1.81178	0.09059	0.00810	0.01542	0.00771
0.2	0.38799	3.76886	0.18844	0.03170	0.06003	0.03002
0.4	0.78651	5.16717	0.25836	0.11650	0.21640	0.10820
0.8	1.02218	3.43631	0.17182	0.34941	0.60493	0.30247
1.6	0.99060	1.15211	0.05761	0.69014	0.97146	0.48573
3.2	0.99393	- 0.06894	- 0.00345	0.93487	0.88358	0.44179
						k_2 = 20.0
0.1	0.14447	1.80664	0.09033	0.00182	0.02985	0.00149
0.2	0.38469	3.74573	0.18729	0.00716	0.11697	0.00585
0.4	0.77394	5.05489	0.25274	0.02710	0.43263	0.02163
0.8	0.98610	2.92533	0.14627	0.09061	1.33736	0.06687
1.6	0.93712	- 1.27093	- 0.06355	0.24528	2.99215	0.14961
3.2	0.96330	- 7.35384	- 0.36761	0.55490	5.06489	0.25324
						k_2 = 200.0
0.1	0.14422	1.78941	0.08947	0.00033	0.04010	0.00020
0.2	0.38388	3.68097	0.18405	0.00131	0.15781	0.00079
0.4	0.77131	4.80711	0.24036	0.00505	0.59391	0.00297
0.8	0.97701	1.90825	0.09541	0.01830	1.95709	0.00979
1.6	0.91645	- 5.28803	- 0.26440	0.06007	5.25110	0.02626
3.2	0.92662	-21.52546	- 1.07627	0.18395	12.45058	0.06225

H = 0.125

k_1 = 200.0

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						k_2 = 0.2
0.1	0.03694	2.87564	0.01438	0.01137	0.00201	0.01005
0.2	0.12327	7.44285	0.03721	0.04473	0.00788	0.03940
0.4	0.36329	15.41021	0.07705	0.16785	0.02913	0.14566
0.8	0.82050	19.70261	0.09851	0.53144	0.08714	0.43568
1.6	1.12440	7.02380	0.03512	1.03707	0.13705	0.68524
3.2	0.99506	2.35459	0.01177	1.00400	0.06594	0.32971
						k_2 = 2.0
0.1	0.03481	3.02259	0.01511	0.00549	0.00969	0.00485
0.2	0.11491	8.02452	0.04012	0.02167	0.03812	0.01906
0.4	0.33218	17.64175	0.08821	0.08229	0.14286	0.07143
0.8	0.72695	27.27701	0.13639	0.27307	0.45208	0.22604
1.6	1.00203	23.38638	0.11693	0.63916	0.90361	0.45430
3.2	1.00828	11.87014	0.05935	0.92560	0.91469	0.45735
						k_2 = 20.0
0.1	0.03336	3.17763	0.01589	0.00123	0.01980	0.00099
0.2	0.10928	8.66097	0.04330	0.00509	0.07827	0.00391
0.4	0.31094	20.12259	0.10061	0.01972	0.29837	0.01494
0.8	0.65934	36.29943	0.18150	0.07045	1.01694	0.05085
1.6	0.87931	49.40857	0.24704	0.20963	2.64313	0.13216
3.2	0.93309	57.84369	0.28923	0.49938	4.89895	0.24495
						k_2 = 200.0
0.1	0.03307	3.26987	0.01635	0.00025	0.02809	0.00014
0.2	0.10810	9.02669	0.01513	0.00098	0.11136	0.00056
0.4	0.30639	21.56482	0.10782	0.00386	0.43035	0.00215
0.8	0.64383	41.39878	0.20949	0.01455	1.53070	0.00765
1.6	0.84110	69.63157	0.34816	0.05011	4.56707	0.02284
3.2	0.86807	120.95981	0.60481	0.15719	11.42045	0.05710

H = 0.25

k_1 = 0.2

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						k_2 = 0.2
0.1	0.27115	0.05598	0.27990	0.01259	0.00274	0.01370
0.2	0.65109	0.12628	0.63138	0.04892	0.01060	0.05302
0.4	0.90404	0.14219	0.71096	0.17538	0.03744	0.18722
0.8	0.95559	0.12300	0.61499	0.48699	0.09839	0.49196
1.6	0.99703	0.10534	0.52669	0.81249	0.13917	0.69586
3.2	0.99927	0.05063	0.25317	0.92951	0.11114	0.55569
						k_2 = 2.0
0.1	0.27103	0.05477	0.27385	0.00739	0.01409	0.00704
0.2	0.66010	0.12136	0.60631	0.02893	0.05484	0.02742
0.4	0.90120	0.12390	0.61949	0.10664	0.19780	0.09890
0.8	0.94923	0.06482	0.32410	0.32617	0.56039	0.28019
1.6	0.99029	− 0.00519	− 0.02594	0.69047	0.96216	0.48108
3.2	1.00000	− 0.02216	− 0.11080	0.95608	0.87221	0.43610
						k_2 = 20.0
0.1	0.26945	0.05192	0.25960	0.00222	0.03116	0.00156
0.2	0.66161	0.11209	0.56045	0.00877	0.12227	0.00611
0.4	0.90102	0.08622	0.43111	0.03354	0.45504	0.02275
0.8	0.94012	− 0.07351	− 0.36756	0.11653	1.44285	0.07214
1.6	0.97277	− 0.40234	− 2.01169	0.33692	3.37001	0.16850
3.2	0.99075	− 0.71901	− 3.59542	0.73532	5.10060	0.25503
						k_2 = 200.0
0.1	0.27072	0.04956	0.24778	0.00051	0.04704	0.00024
0.2	0.65909	0.10066	0.50330	0.00202	0.18557	0.00093
0.4	0.89724	0.04243	0.21242	0.00791	0.70524	0.00353
0.8	0.93596	− 0.24071	− 1.20357	0.02961	2.40585	0.01203
1.6	0.96370	− 1.00743	− 5.03714	0.10193	6.82481	0.03412
3.2	0.97335	− 2.54264	−12.71320	0.30707	15.45931	0.07730

$H = 0.25$

$k_1 = 2.0$

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						$k_2 = 0.2$
0.1	0.15577	0.28658	0.14329	0.01348	0.00277	0.01384
0.2	0.43310	0.72176	0.36088	0.05259	0.01075	0.05377
0.4	0.79551	1.03476	0.51738	0.19094	0.03842	0.19211
0.8	1.00871	0.88833	0.44416	0.54570	0.10337	0.51687
1.6	1.02425	0.66438	0.33219	0.90563	0.14102	0.70510
3.2	0.99617	0.41539	0.20773	0.93918	0.09804	0.49020
						$k_2 = 2.0$
0.1	0.15524	0.28362	0.14181	0.00710	0.01353	0.00677
0.2	0.42809	0.70225	0.35112	0.02783	0.05278	0.02639
0.4	0.77939	0.96634	0.48317	0.10306	0.19178	0.09589
0.8	0.96703	0.66885	0.33442	0.31771	0.55211	0.27605
1.6	0.98156	0.17331	0.08665	0.66753	0.95080	0.47540
3.2	0.99840	− 0.05691	− 0.02846	0.93798	0.89390	0.44695
						$k_2 = 20.0$
0.1	0.15436	0.27580	0.13790	0.00179	0.02728	0.00136
0.2	0.42462	0.67115	0.33557	0.00706	0.10710	0.00536
0.4	0.76647	0.84462	0.42231	0.02697	0.39919	0.01996
0.8	0.92757	0.21951	0.10976	0.09285	1.26565	0.06328
1.6	0.91393	− 1.22411	− 0.61205	0.26454	2.94860	0.14743
3.2	0.95243	− 3.04320	− 1.52160	0.60754	4.89878	0.24494
						$k_2 = 200.0$
0.1	0.15414	0.26776	0.13388	0.00036	0.03814	0.00019
0.2	0.42365	0.63873	0.31937	0.00143	0.15040	0.00075
0.4	0.76296	0.71620	0.35810	0.00557	0.57046	0.00285
0.8	0.91600	− 0.28250	− 0.14125	0.02064	1.92636	0.00963
1.6	0.88406	− 3.09856	− 1.54928	0.07014	5.35936	0.02680
3.2	0.89712	− 9.18214	− 4.59107	0.21692	12.64318	0.06322

H = 0.25

k_1 = 20.0

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						k_2 = 0.2
0.1	0.04596	0.61450	0.03072	0.01107	0.00202	0.01011
0.2	0.15126	1.76675	0.08834	0.04357	0.00793	0.03964
0.4	0.41030	3.59650	0.17983	0.16337	0.02931	0.14653
0.8	0.85464	4.58845	0.22942	0.51644	0.08771	0.43854
1.6	1.12013	2.31165	0.11558	1.01061	0.14039	0.70194
3.2	0.99676	1.24415	0.06221	0.99168	0.07587	0.37934
						k_2 = 2.0
0.1	0.04381	0.63215	0.03162	0.00530	0.00962	0.00481
0.2	0.14282	1.83766	0.09188	0.02091	0.03781	0.01891
0.4	0.37882	3.86779	0.19339	0.07933	0.14159	0.07079
0.8	0.75904	5.50796	0.27540	0.26278	0.44710	0.22355
1.6	0.98743	4.24281	0.21213	0.61673	0.90115	0.45058
3.2	1.00064	1.97494	0.09876	0.91258	0.93254	0.46627
						k_2 = 20.0
0.1	0.04236	0.65003	0.03250	0.00123	0.01930	0.00096
0.2	0.13708	1.90693	0.09535	0.00488	0.07623	0.00381
0.4	0.35716	4.13976	0.20699	0.01388	0.29072	0.01454
0.8	0.68947	6.48948	0.32447	0.06741	0.98565	0.04928
1.6	0.85490	6.95639	0.34782	0.20115	2.55231	0.12762
3.2	0.90325	6.05854	0.30293	0.48647	4.76234	0.23812
						k_2 = 200.0
0.1	0.04204	0.65732	0.03287	0.00024	0.02711	0.00014
0.2	0.13584	1.93764	0.09638	0.00095	0.10741	0.00054
0.4	0.35237	4.26004	0.21300	0.00372	0.41459	0.00207
0.8	0.67286	6.94871	0.34743	0.01399	1.46947	0.00735
1.6	0.81223	8.55770	0.42789	0.04830	4.36521	0.02183
3.2	0.82390	10.63614	0.53181	0.15278	10.93570	0.05468

H = 0.25

k_1 = 200.0

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						k_2 = 0.2
0.1	0.011139	0.866114	0.000433	0.005389	0.000090	0.004451
0.2	0.041180	2.771354	0.001357	0.022334	0.000357	0.017784
0.4	0.141196	6.883021	0.003415	0.090224	0.001365	0.068224
0.8	0.426o3	13.119664	0.006598	0.331785	0.004624	0.231118
1.6	0.94520	13.779134	0.006896	0.883371	0.010591	0.529555
3.2	1.10738	2.772901	0.001365	1.110259	0.008008	0.443037
						k_2 = 2.0
0.1	0.00909	0.896553	0.000483	0.002259	0.000407	0.000203
0.2	0.032669	3.110763	0.001554	0.011027	0.001611	0.000806
0.4	0.110584	8.337852	0.004189	0.044000	0.006221	0.003110
0.8	0.330477	18.895534	0.009478	0.114513	0.021860	0.110930
1.6	0.666786	31.118909	0.015595	0.342940	0.058553	0.029277
3.2	0.898447	23.898500	0.114493	0.884545	0.089191	0.044595
						k_2 = 20.0
0.1	0.000776	1.083738	0.000544	0.000065	0.000361	0.000043
0.2	0.002741	3.859448	0.001797	0.000257	0.003421	0.000171
0.4	0.008631	10.330923	0.005155	0.001014	0.013365	0.000663
0.8	0.023137	26.441442	0.013207	0.003844	0.049135	0.002457
1.6	0.046835	57.146409	0.028732	0.013143	1.553333	0.007692
3.2	0.071083	99.229034	0.049645	0.037342	3.660964	0.018048
						k_2 = 200.0
0.1	0.000744	1.119099	0.000596	0.000014	0.001311	0.000007
0.2	0.002616	4.000968	0.002005	0.000056	0.005223	0.000026
0.4	0.008141	11.996405	0.005982	0.000224	0.020551	0.000103
0.8	0.021293	32.997364	0.016487	0.000871	0.077534	0.000388
1.6	0.040876	82.777997	0.041390	0.003234	2.639622	0.001320
3.2	0.056613	189.137439	0.094687	0.011049	7.602287	0.003801

H = 0.5

k_1 = 0.2

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						k_2 = 0.2
0.1	0.07943	0.01705	0.08527	0.00914	0.00206	0.01030
0.2	0.27189	0.05724	0.28621	0.03577	0.00804	0.04020
0.4	0.66375	0.13089	0.65444	0.13135	0.02924	0.14622
0.8	0.91143	0.15514	0.77571	0.38994	0.08369	0.41843
1.6	0.96334	0.13250	0.66248	0.72106	0.13729	0.68647
3.2	0.99310	0.06976	0.34879	0.89599	0.12674	0.63371
						k_2 = 2.0
0.1	0.07906	0.01617	0.08085	0.00557	0.01074	0.00537
0.2	0.27046	0.05375	0.26877	0.02190	0.04206	0.02103
0.4	0.65347	0.11770	0.58848	0.08222	0.15534	0.07767
0.8	0.89579	0.11252	0.56258	0.26429	0.47045	0.23523
1.6	0.94217	0.04897	0.24486	0.60357	0.90072	0.45036
3.2	0.99189	0.01380	0.06900	0.91215	0.94385	0.47192
						k_2 = 20.0
0.1	0.07862	0.01439	0.07196	0.00175	0.02415	0.00121
0.2	0.26873	0.04669	0.23345	0.00692	0.09519	0.00476
0.4	0.65188	0.09018	0.45089	0.02676	0.36003	0.01800
0.8	0.87401	0.01269	0.06347	0.09552	1.19151	0.05958
1.6	0.39563 −	0.24336 −	1.21680	0.28721	2.95409	0.14770
3.2	0.95392 −	0.53220 −	2.66100	0.66445	4.86789	0.24339
						k_2 = 200.0
0.1	0.07820	0.01243	0.06213	0.00041	0.03632	0.00018
0.2	0.26803	0.03912	0.19558	0.00163	0.14576	0.00073
0.4	0.64904	0.06006	0.30029	0.00643	0.56051	0.00280
0.8	0.86406 −	0.10447 −	0.52234	0.02436	1.06771	0.00984
1.6	0.36677 −	0.67154 −	3.35768	0.08540	5.77669	0.02388
3.2	0.89703 −	1.86126 −	9.30628	0.26467	13.63423	0.06817

H = 0.5

k_1 = 2.0

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						k_2 = 0.2
0.1	0.04496	0.08398	0.04199	0.00903	0.00181	0.00906
0.2	0.15978	0.28904	0.14452	0.03551	0.00711	0.03554
0.4	0.44523	0.72313	0.36156	0.13314	0.02634	0.13172
0.8	0.83298	1.03603	0.51802	0.42199	0.07992	0.39962
1.6	1.05462	0.83475	0.41737	0.85529	0.13973	0.69863
3.2	0.99967	0.45119	0.22560	0.94506	0.10667	0.53336
						k_2 = 2.0
0.1	0.04330	0.08250	0.04125	0.00465	0.00878	0.00439
0.2	0.15325	0.28318	0.14159	0.01836	0.03454	0.01727
0.4	0.42077	0.70119	0.35060	0.06974	0.12951	0.06477
0.8	0.75633	0.96681	0.48341	0.23256	0.41187	0.20594
1.6	0.93447	0.70726	0.35363	0.56298	0.85930	0.42965
3.2	0.98801	0.33878	0.16939	0.88655	0.96353	0.48176
						k_2 = 20.0
0.1	0.04193	0.08044	0.04022	0.00117	0.01778	0.00089
0.2	0.14808	0.27574	0.13787	0.00464	0.07027	0.00351
0.4	0.40086	0.67174	0.33587	0.01799	0.26817	0.01341
0.8	0.69098	0.86191	0.43095	0.06476	0.91168	0.04558
1.6	0.79338	0.39588	0.19794	0.19803	2.38377	0.11919
3.2	0.85940	− 0.41078	− 0.20539	0.49238	4.47022	0.22351
						k_2 = 200.0
0.1	0.04160	0.07864	0.03932	0.00024	0.02515	0.00013
0.2	0.14676	0.26853	0.13426	0.00095	0.09968	0.00050
0.4	0.39570	0.64303	0.32152	0.00374	0.38497	0.00192
0.8	0.67257	0.74947	0.37474	0.01416	1.36766	0.00684
1.6	0.74106	− 0.02761	− 0.01381	0.04972	4.08937	0.02045
3.2	0.75176	− 1.88545	− 0.94273	0.15960	10.25631	0.05128

H = 0.5

k_1 = 20.0

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						k_2 = 0.2
0.1	0.01351	0.16526	0.00826	0.00596	0.00098	0.00488
0.2	0.05079	0.58918	0.02946	0.02361	0.00386	0.01929
0.4	0.16972	1.66749	0.08337	0.09110	0.01474	0.07369
0.8	0.47191	3.23121	0.16156	0.31904	0.04967	0.24834
1.6	0.97452	3.54853	0.17743	0.82609	0.11279	0.56395
3.2	1.09911	1.27334	0.06367	1.08304	0.09527	0.47637
						k_2 = 2.0
0.1	0.01122	0.17997	0.00900	0.00259	0.00440	0.00220
0.2	0.04172	0.64779	0.03239	0.01023	0.01744	0.00872
0.4	0.13480	1.89817	0.09491	0.03993	0.06722	0.03361
0.8	0.35175	4.09592	0.20480	0.14419	0.23476	0.11738
1.6	0.70221	6.22002	0.31100	0.42106	0.62046	0.31023
3.2	0.97420	5.41828	0.27091	0.82256	0.93831	0.46916
						k_2 = 20.0
0.1	0.00990	0.19872	0.00994	0.00063	0.00911	0.00046
0.2	0.03648	0.72264	0.03613	0.00251	0.03520	0.00181
0.4	0.11448	2.19520	0.10976	0.00988	0.14116	0.00706
0.8	0.27934	5.24726	0.26236	0.03731	0.51585	0.02579
1.6	0.50790	10.30212	0.51511	0.12654	1.59341	0.07967
3.2	0.70903	16.38520	0.81926	0.35807	3.69109	0.18455
						k_2 = 200.0
0.1	0.00960	0.21440	0.01072	0.00013	0.01355	0.00007
0.2	0.03526	0.78493	0.03925	0.00054	0.05395	0.00027
0.4	0.10970	2.44430	0.12221	0.00214	0.21195	0.00106
0.8	0.26149	6.23424	0.31172	0.00331	0.79583	0.00398
1.6	0.45078	14.11490	0.70574	0.03070	2.67578	0.01338
3.2	0.57074	29.95815	1.49791	0.10470	7.61457	0.03807

H = 0.5

k_1 = 200.0

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						k_2 = 0.2
0.1	0.00363	0.22388	0.00112	0.00256	0.00033	0.00163
0.2	0.01414	0.81903	0.00410	0.01021	0.00130	0.00648
0.4	0.05256	2.52558	0.01263	0.04014	0.00506	0.02529
0.8	0.18107	6.11429	0.03057	0.15048	0.01844	0.09221
1.6	0.53465	10.82705	0.05414	0.48201	0.05399	0.26993
3.2	1.04537	9.34212	0.04671	1.00671	0.08624	0.43121
						k_2 = 2.0
0.1	0.00215	0.26620	0.00133	0.00094	0.00128	0.00064
0.2	0.00826	0.98772	0.00494	0.00373	0.00509	0.00254
0.4	0.02946	3.19580	0.01598	0.01474	0.01996	0.00998
0.8	0.09508	8.71973	0.04360	0.05622	0.07434	0.03717
1.6	0.27135	20.15765	0.10079	0.19358	0.23838	0.11919
3.2	0.62399	34.25229	0.17126	0.52912	0.54931	0.27466
						k_2 = 20.0
0.1	0.00149	0.31847	0.00159	0.00023	0.00257	0.00013
0.2	0.00564	1.19598	0.00598	0.00094	0.01025	0.00051
0.4	0.01911	4.02732	0.02014	0.00372	0.04047	0.00202
0.8	0.05574	12.00835	0.06004	0.01453	0.15452	0.00773
1.6	0.13946	32.77028	0.16385	0.05399	0.53836	0.02692
3.2	0.30247	77.62943	0.38815	0.18091	1.56409	0.07820
						k_2 = 200.0
0.1	0.00133	0.37065	0.00185	0.00005	0.00387	0.00002
0.2	0.00498	1.40493	0.00702	0.00022	0.01544	0.00008
0.4	0.01649	4.86215	0.02431	0.00086	0.06118	0.00031
0.8	0.04553	15.33902	0.07670	0.00340	0.23698	0.00118
1.6	0.10209	45.93954	0.22970	0.01315	0.86345	0.00432
3.2	0.18358	128.13051	0.64065	0.04854	2.80877	0.01404

H = 1.0

k_1 = 0.2

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						k_2 = 0.2
0.1	0.02090	0.00464	0.02320	0.00541	0.00128	0.00638
0.2	0.08023	0.01773	0.08865	0.02138	0.00503	0.02515
0.4	0.27493	0.05976	0.29878	0.08125	0.01903	0.09516
0.8	0.67330	0.13818	0.69092	0.26887	0.06192	0.30960
1.6	0.92595	0.15978	0.79888	0.60229	0.13002	0.65010
3.2	0.95852	0.09722	0.48612	0.82194	0.14348	0.71742
						k_2 = 2.0
0.1	0.02045	0.00410	0.02052	0.00356	0.00687	0.00343
0.2	0.07845	0.01561	0.07805	0.01410	0.02713	0.01357
0.4	0.26816	0.05166	0.25828	0.05427	0.10351	0.05175
0.8	0.65090	0.11111	0.55555	0.18842	0.34703	0.17351
1.6	0.88171	0.10364	0.51819	0.48957	0.79986	0.39993
3.2	0.94153	0.06967	0.34335	0.81663	0.99757	0.49879
						k_2 = 20.0
0.1	0.01981	0.00306	0.01529	0.00118	0.01591	0.00080
0.2	0.07587	0.01145	0.05726	0.00471	0.06310	0.00315
0.4	0.25817	0.03540	0.17702	0.01846	0.24396	0.01220
0.8	0.61544	0.05163	0.25817	0.06839	0.86114	0.04306
1.6	0.73884	− 0.07218	− 0.36091	0.21770	2.36054	0.11803
3.2	0.82936	− 0.25569	− 1.27847	0.53612	4.28169	0.21408
						k_2 = 200.0
0.1	0.01952	0.00214	0.01068	0.00028	0.02412	0.00012
0.2	0.07473	0.00777	0.03883	0.00110	0.09587	0.00048
0.4	0.25368	0.02076	0.10382	0.00436	0.37417	0.00187
0.8	0.59853	− 0.00538	− 0.02690	0.01679	1.36930	0.00685
1.6	0.73387	− 0.28050	− 1.40250	0.06020	4.23805	0.02119
3.2	0.70243	− 0.90965	− 4.54826	0.19189	10.36507	0.05183

$H = 1.0$

$k_1 = 2.0$

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						$k_2 = 0.2$
0.1	0.01241	0.02186	0.01093	0.00490	0.00096	0.00478
0.2	0.04816	0.08396	0.04198	0.01943	0.00378	0.01890
0.4	0.17203	0.28866	0.14433	0.07496	0.01448	0.07241
0.8	0.48612	0.71684	0.35842	0.26193	0.04924	0.24620
1.6	0.91312	0.97206	0.48603	0.67611	0.11558	0.57790
3.2	1.04671	0.60091	0.30046	0.95985	0.12527	0.62637
						$k_2 = 2.0$
0.1	0.01083	0.02179	0.01090	0.00241	0.00453	0.00227
0.2	0.04176	0.08337	0.04169	0.00958	0.01797	0.00899
0.4	0.14665	0.28491	0.14246	0.03724	0.06934	0.03467
0.8	0.39942	0.71341	0.35670	0.13401	0.24250	0.12125
1.6	0.71032	1.02680	0.51340	0.38690	0.63631	0.31815
3.2	0.92112	0.90482	0.45241	0.75805	0.97509	0.48754
						$k_2 = 20.0$
0.1	0.00963	0.02249	0.01124	0.00061	0.00920	0.00046
0.2	0.03697	0.08618	0.04309	0.00241	0.03654	0.00183
0.4	0.12805	0.29640	0.14820	0.00950	0.14241	0.00712
0.8	0.33263	0.76292	0.38146	0.03578	0.51815	0.02591
1.6	0.52721	1.25168	0.62584	0.12007	1.56503	0.07825
3.2	0.65530	1.70723	0.85361	0.33669	3.51128	0.17556
						$k_2 = 200.0$
0.1	0.00925	0.02339	0.01170	0.00013	0.01319	0.00007
0.2	0.03561	0.09018	0.04509	0.00051	0.05252	0.00026
0.4	0.12348	0.31470	0.15735	0.00202	0.20609	0.00103
0.8	0.31422	0.83274	0.41637	0.00783	0.76955	0.00385
1.6	0.46897	1.53521	0.76760	0.02874	2.53100	0.01265
3.2	0.51161	2.76420	1.38210	0.09751	6.99283	0.03496

H = 1.0

k_1 = 20.0

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						k_2 = 0.2
0.1	0.00417	0.04050	0.00202	0.00271	0.00039	0.00195
0.2	0.01641	0.15675	0.00784	0.01080	0.00155	0.00777
0.4	0.06210	0.55548	0.02777	0.04241	0.00606	0.03028
0.8	0.21057	1.53667	0.07683	0.15308	0.02198	0.10991
1.6	0.58218	2.77359	0.13868	0.49705	0.06327	0.31635
3.2	1.06296	2.55195	0.12760	1.00217	0.09906	0.49525
						k_2 = 2.0
0.1	0.00263	0.04751	0.00238	0.00100	0.00160	0.00080
0.2	0.01029	0.18481	0.00924	0.00397	0.00637	0.00319
0.4	0.03810	0.66727	0.03336	0.01565	0.02498	0.01249
0.8	0.12173	1.97428	0.09871	0.05938	0.09268	0.04634
1.6	0.31575	4.37407	0.21870	0.20098	0.29253	0.14626
3.2	0.66041	6.97695	0.34885	0.53398	0.65446	0.32723
						k_2 = 20.0
0.1	0.00193	0.05737	0.00287	0.00024	0.00322	0.00016
0.2	0.00751	0.22418	0.01121	0.00098	0.01283	0.00064
0.4	0.02713	0.82430	0.04121	0.00387	0.05063	0.00253
0.8	0.08027	2.59672	0.12984	0.01507	0.19267	0.00963
1.6	0.17961	6.77014	0.33851	0.05549	0.66326	0.03316
3.2	0.34355	15.23252	0.76163	0.18344	1.88634	0.09432
						k_2 = 200.0
0.1	0.00176	0.06733	0.00337	0.00006	0.00478	0.00002
0.2	0.00683	0.26401	0.01320	0.00022	0.01908	0.00010
0.4	0.02443	0.98346	0.04917	0.00088	0.07557	0.00038
0.8	0.06983	3.23164	0.16158	0.00348	0.29194	0.00146
1.6	0.14191	9.28148	0.46407	0.01339	1.05385	0.00527
3.2	0.22655	24.85236	1.24262	0.04911	3.37605	0.01688

$H = 1.0$

$k_1 = 200.0$

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						$k_2 = 0.2$
0.1	0.00117	0.05507	0.00028	0.00097	0.00010	0.00051
0.2	0.00464	0.21467	0.00107	0.00388	0.00041	0.00203
0.4	0.01814	0.78191	0.00391	0.01538	0.00160	0.00801
0.8	0.06766	2.38055	0.01190	0.05952	0.00607	0.03037
1.6	0.22994	5.57945	0.02790	0.21214	0.02028	0.10140
3.2	0.62710	9.29529	0.04648	0.60056	0.04847	0.24236
						$k_2 = 2.0$
0.1	0.00049	0.06883	0.00034	0.00029	0.00035	0.00017
0.2	0.00195	0.26966	0.00135	0.00116	0.00138	0.00069
0.4	0.00746	1.00131	0.00501	0.00460	0.00545	0.00273
0.8	0.02647	3.24971	0.01625	0.01797	0.02092	0.01046
1.6	0.08556	8.92442	0.04462	0.06671	0.07335	0.03668
3.2	0.25186	20.83387	0.10417	0.22047	0.21288	0.10644
						$k_2 = 20.0$
0.1	0.00027	0.08469	0.00042	0.00007	0.00062	0.00003
0.2	0.00104	0.33312	0.00167	0.00028	0.00248	0.00012
0.4	0.00384	1.25495	0.00627	0.00110	0.00985	0.00049
0.8	0.01236	4.26100	0.02130	0.00436	0.03825	0.00191
1.6	0.03379	12.91809	0.06459	0.01683	0.13989	0.00699
3.2	0.08859	36.04291	0.18021	0.06167	0.45544	0.02277
						$k_2 = 200.0$
0.1	0.00021	0.10075	0.00050	0.00002	0.00037	0.00000
0.2	0.00082	0.39741	0.00199	0.00006	0.00347	0.00002
0.4	0.00298	1.51234	0.00756	0.00025	0.01331	0.00007
0.8	0.00893	5.28939	0.02645	0.00100	0.05403	0.00027
1.6	0.02065	17.01872	0.08509	0.00392	0.20250	0.00101
3.2	0.04154	52.23615	0.26118	0.01505	0.70098	0.00350

H = 2.0

k_1 = 0.2

s_1	σ_{s_1}	$\sigma_{s_1}-\sigma_{r_1}$	$\sigma_{s_1}-\sigma_{r_2}$	σ_{s_2}	$\sigma_{s_2}-\sigma_{r_2}$	$\sigma_{s_2}-\sigma_{r_3}$
						k_2 = 0.2
0.1	0.00540	0.00121	0.00604	0.00242	0.00060	0.00302
0.2	0.02138	0.00477	0.02386	0.00964	0.00240	0.01202
0.4	0.08209	0.01821	0.09106	0.03770	0.00939	0.04695
0.8	0.28150	0.06106	0.30531	0.13832	0.03422	0.17112
1.6	0.68908	0.13660	0.68299	0.40830	0.09826	0.49131
3.2	0.93103	0.12809	0.64493	0.73496	0.15705	0.78523
						k_2 = 2.0
0.1	0.00502	0.00098	0.00494	0.00180	0.00339	0.00170
0.2	0.01986	0.00389	0.01953	0.00716	0.01350	0.00675
0.4	0.07630	0.01485	0.07449	0.02815	0.05288	0.02644
0.8	0.26196	0.04977	0.24875	0.10523	0.19467	0.09733
1.6	0.63535	0.10924	0.54641	0.33075	0.57811	0.28905
3.2	0.87025	0.12296	0.61462	0.68388	1.00199	0.50100
						k_2 = 20.0
0.1	0.00444	0.00056	0.00282	0.00065	0.00825	0.00041
0.2	0.01756	0.00221	0.01105	0.00260	0.03286	0.00164
0.4	0.06706	0.00819	0.04097	0.01030	0.12933	0.00647
0.8	0.22561	0.02431	0.12153	0.03956	0.48595	0.02430
1.6	0.51929	0.03070	0.15352	0.13743	1.55804	0.07790
3.2	0.65700 −	0.00926 −	0.04632	0.37409	3.39883	0.16994
						k_2 = 200.0
0.1	0.00414	0.00032	0.00160	0.00015	0.01234	0.00006
0.2	0.01635	0.00124	0.00621	0.00058	0.04922	0.00025
0.4	0.06231	0.00436	0.02180	0.00231	0.19450	0.00097
0.8	0.20757	0.00955	0.04774	0.00905	0.74256	0.00371
1.6	0.45550 −	0.02172 −	0.10861	0.03363	2.52847	0.01264
3.2	0.48642 −	0.15589 −	0.77944	0.11105	6.69835	0.03349

$H = 2.0$

$k_1 = 2.0$

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						$k_2 = 0.2$
0.1	0.00356	0.00545	0.00272	0.00216	0.00041	0.00203
0.2	0.01415	0.02155	0.01078	0.00861	0.00162	0.00809
0.4	0.05493	0.08266	0.04133	0.03386	0.00634	0.03172
0.8	0.19661	0.28226	0.14113	0.12702	0.02349	0.11744
1.6	0.55306	0.67844	0.33922	0.40376	0.07109	0.35545
3.2	0.96647	0.79393	0.39696	0.83197	0.12583	0.62913
						$k_2 = 2.0$
0.1	0.00250	0.00555	0.00278	0.00100	0.00188	0.00094
0.2	0.00991	0.02199	0.01099	0.00397	0.00750	0.00375
0.4	0.03832	0.08465	0.04231	0.01569	0.02950	0.01475
0.8	0.13516	0.29365	0.14683	0.05974	0.11080	0.05540
1.6	0.36644	0.75087	0.37542	0.20145	0.35515	0.17757
3.2	0.67384	1.17294	0.58647	0.51156	0.77434	0.38717
						$k_2 = 20.0$
0.1	0.00181	0.00652	0.00326	0.00025	0.00378	0.00019
0.2	0.00716	0.02586	0.01293	0.00099	0.01507	0.00075
0.4	0.02746	0.10017	0.05007	0.00394	0.05958	0.00298
0.8	0.09396	0.35641	0.17821	0.01535	0.22795	0.01140
1.6	0.23065	1.00785	0.50392	0.05599	0.78347	0.03917
3.2	0.37001	2.16033	1.08017	0.17843	2.13215	0.10661
						$k_2 = 200.0$
0.1	0.00164	0.00778	0.00389	0.00005	0.00542	0.00003
0.2	0.00647	0.03090	0.01544	0.00021	0.02163	0.00011
0.4	0.02470	0.12030	0.06014	0.00085	0.08578	0.00043
0.8	0.08326	0.43693	0.21847	0.00335	0.33214	0.00166
1.6	0.19224	1.32870	0.66434	0.01283	1.19190	0.00596
3.2	0.25526	3.40664	1.70332	0.04612	3.67558	0.01838

$H = 2.0$

$k_1 = 20.0$

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						$k_2 = 0.2$
0.1	0.00134	0.00968	0.00048	0.00108	0.00014	0.00068
0.2	0.00533	0.03839	0.00192	0.00429	0.00055	0.00273
0.4	0.02100	0.14845	0.00741	0.01702	0.00216	0.01078
0.8	0.07950	0.52414	0.02621	0.06576	0.00820	0.04101
1.6	0.26613	1.41720	0.07085	0.23186	0.02740	0.13698
3.2	0.67882	2.38258	0.11913	0.63006	0.06384	0.31919
						$k_2 = 2.0$
0.1	0.00059	0.01219	0.00061	0.00033	0.00051	0.00025
0.2	0.00235	0.04843	0.00242	0.00130	0.00203	0.00101
0.4	0.00922	0.18857	0.00943	0.00513	0.00803	0.00401
0.8	0.03412	0.68382	0.03419	0.02023	0.03093	0.01547
1.6	0.10918	2.04134	0.10207	0.07444	0.10864	0.05432
3.2	0.29183	4.60426	0.23021	0.23852	0.30709	0.15354
						$k_2 = 20.0$
0.1	0.00033	0.01568	0.00078	0.00008	0.00094	0.00005
0.2	0.00130	0.06236	0.00312	0.00031	0.00374	0.00019
0.4	0.00503	0.24425	0.01221	0.00123	0.01486	0.00074
0.8	0.01782	0.90594	0.04530	0.00485	0.05739	0.00289
1.6	0.05012	2.91994	0.14600	0.01862	0.21190	0.01060
3.2	0.11331	7.95104	0.39755	0.06728	0.67732	0.03387
						$k_2 = 200.0$
0.1	0.00027	0.01927	0.00096	0.00002	0.00131	0.00001
0.2	0.00106	0.07675	0.00384	0.00007	0.00524	0.00003
0.4	0.00406	0.30182	0.01509	0.00028	0.02085	0.00010
0.8	0.01397	1.13555	0.05673	0.00110	0.08180	0.00041
1.6	0.03538	3.83254	0.19163	0.00431	0.30676	0.00153
3.2	0.06182	11.55403	0.57770	0.01644	1.04794	0.00524

$H = 2.0$

$k_1 = 200.0$

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						$k_2 = 0.2$
0.1	0.00036	0.01350	0.00007	0.00033	0.00003	0.00015
0.2	0.00144	0.05366	0.00027	0.00130	0.00012	0.00058
0.4	0.00572	0.20911	0.00105	0.00518	0.00046	0.00232
0.8	0.02231	0.76035	0.00380	0.02038	0.00180	0.00901
1.6	0.08215	2.29642	0.01148	0.07675	0.00649	0.03244
3.2	0.26576	5.28589	0.02643	0.25434	0.01912	0.09562
						$k_2 = 2.0$
0.1	0.00011	0.01737	0.00009	0.00008	0.00009	0.00004
0.2	0.00045	0.06913	0.00035	0.00033	0.00036	0.00018
0.4	0.00179	0.27103	0.00136	0.00131	0.00142	0.00071
0.8	0.00685	1.00808	0.00504	0.00520	0.00553	0.00277
1.6	0.02441	3.27590	0.01638	0.02003	0.02043	0.01021
3.2	0.08061	9.02195	0.04511	0.07248	0.06638	0.03319
						$k_2 = 20.0$
0.1	0.00005	0.02160	0.00011	0.00002	0.00014	0.00001
0.2	0.00018	0.08604	0.00043	0.00007	0.00058	0.00003
0.4	0.00071	0.33866	0.00169	0.00030	0.00229	0.00011
0.8	0.00261	1.27835	0.00639	0.00119	0.00901	0.00045
1.6	0.00819	4.35311	0.02177	0.00467	0.03390	0.00170
3.2	0.02341	13.26873	0.06634	0.01734	0.11666	0.00583
						$k_2 = 200.0$
0.1	0.00003	0.02587	0.00013	0.00000	0.00019	0.00000
0.2	0.00012	0.10310	0.00052	0.00002	0.00075	0.00000
0.4	0.00047	0.40676	0.00203	0.00007	0.00300	0.00002
0.8	0.00165	1.54951	0.00775	0.00026	0.01183	0.00006
1.6	0.00445	5.43705	0.02719	0.00104	0.04515	0.00023
3.2	0.00929	17.58810	0.08794	0.00409	0.16107	0.00081

$H = 4.0$

$k_1 = 0.2$

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						$k_2 = 0.2$
0.1	0.00139	0.00028	0.00141	0.00086	0.00023	0.00114
0.2	0.00555	0.00112	0.00562	0.00345	0.00091	0.00454
0.4	0.02198	0.00444	0.02220	0.01371	0.00360	0.01801
0.8	0.08435	0.01636	0.08428	0.05323	0.01394	0.06968
1.6	0.28870	0.05529	0.27647	0.19003	0.04909	0.24545
3.2	0.70074	0.11356	0.56778	0.51882	0.12670	0.63352
						$k_2 = 2.0$
0.1	0.00133	0.00026	0.00131	0.00071	0.00130	0.00065
0.2	0.00491	0.00104	0.00521	0.00283	0.00518	0.00259
0.4	0.01942	0.00412	0.02059	0.01126	0.02057	0.01028
0.8	0.07447	0.01574	0.07869	0.04383	0.07977	0.03989
1.6	0.25449	0.05311	0.26554	0.15904	0.28357	0.14178
3.2	0.62074	0.12524	0.62622	0.45455	0.75651	0.37825
						$k_2 = 20.0$
0.1	0.00087	0.00018	0.00090	0.00028	0.00325	0.00016
0.2	0.00346	0.00072	0.00358	0.00111	0.01298	0.00065
0.4	0.01367	0.00283	0.01417	0.00443	0.05159	0.00258
0.8	0.05207	0.01089	0.05444	0.01741	0.20134	0.01007
1.6	0.17367	0.03790	0.13949	0.06525	0.73322	0.03666
3.2	0.39955	0.10841	0.54203	0.20965	2.13666	0.10683
						$k_2 = 200.0$
0.1	0.00069	0.00019	0.00097	0.00006	0.00487	0.00002
0.2	0.00274	0.00078	0.00389	0.00024	0.01947	0.00010
0.4	0.01079	0.00309	0.01544	0.00095	0.07752	0.00039
0.8	0.04074	0.01199	0.05995	0.00373	0.30432	0.00152
1.6	0.13117	0.04352	0.21753	0.01456	1.13373	0.00567
3.2	0.26403	0.14445	0.72224	0.05161	3.59608	0.01798

$H = 4.0$

$k_1 = 2.0$

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						$k_2 = 0.2$
0.1	0.00103	0.00123	0.00064	0.00078	0.00014	0.00071
0.2	0.00411	0.00511	0.00256	0.00312	0.00057	0.00284
0.4	0.01631	0.02022	0.01011	0.01241	0.00226	0.01129
0.8	0.06319	0.07722	0.03861	0.04842	0.00877	0.04384
1.6	0.22413	0.25955	0.12977	0.17617	0.03133	0.15666
3.2	0.60654	0.58704	0.29352	0.50917	0.08500	0.42501
						$k_2 = 2.0$
0.1	0.00057	0.00147	0.00074	0.00034	0.00065	0.00032
0.2	0.00223	0.00587	0.00293	0.00137	0.00260	0.00130
0.4	0.00905	0.02324	0.01162	0.00544	0.01032	0.00516
0.8	0.03500	0.08957	0.04479	0.02135	0.04031	0.02015
1.6	0.12354	0.31215	0.15603	0.07972	0.14735	0.07368
3.2	0.34121	0.81908	0.40954	0.25441	0.43632	0.21816
						$k_2 = 20.0$
0.1	0.00030	0.00201	0.00101	0.00008	0.00128	0.00006
0.2	0.00119	0.00803	0.00402	0.00034	0.00510	0.00026
0.4	0.00469	0.03191	0.01596	0.00134	0.02032	0.00102
0.8	0.01700	0.12427	0.06213	0.00532	0.07991	0.00400
1.6	0.06045	0.45100	0.22550	0.02049	0.29991	0.01500
3.2	0.14979	1.36427	0.68214	0.07294	0.97701	0.04885
						$k_2 = 200.0$
0.1	0.00023	0.00263	0.00131	0.00002	0.00180	0.00001
0.2	0.00091	0.01050	0.00525	0.00007	0.00720	0.00004
0.4	0.00360	0.04179	0.02000	0.00029	0.02870	0.00014
0.8	0.01360	0.16380	0.08190	0.00115	0.11334	0.00057
1.6	0.04409	0.60898	0.30449	0.00451	0.43251	0.00216
3.2	0.00323	1.98899	0.99449	0.01705	1.49306	0.00747

H = 4.0

k_1 = 20.0

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						k_2 = 0.2
0.1	0.00042	0.00233	0.00012	0.00037	0.00004	0.00021
0.2	0.00166	0.00932	0.00047	0.00143	0.00017	0.00085
0.4	0.00663	0.03692	0.00185	0.00588	0.00068	0.00340
0.8	0.02603	0.14242	0.00712	0.02319	0.00266	0.01331
1.6	0.09718	0.49826	0.02491	0.08758	0.00983	0.04914
3.2	0.31040	1.31627	0.06581	0.28747	0.02990	0.14951
						k_2 = 2.0
0.1	0.00013	0.00312	0.00016	0.00010	0.00015	0.00007
0.2	0.00054	0.01245	0.00062	0.00039	0.00059	0.00029
0.4	0.00214	0.04944	0.00247	0.00154	0.00235	0.00117
0.8	0.00837	0.19247	0.00962	0.00610	0.00924	0.00462
1.6	0.03109	0.69749	0.03487	0.02358	0.03488	0.01744
3.2	0.10140	2.09049	0.10452	0.03444	0.11553	0.05776
						k_2 = 20.0
0.1	0.00005	0.00213	0.00021	0.00002	0.00025	0.00001
0.2	0.00021	0.01651	0.00083	0.00009	0.00099	0.00005
0.4	0.00083	0.06569	0.00328	0.00035	0.00396	0.00020
0.8	0.00321	0.25739	0.01287	0.00138	0.01565	0.00078
1.6	0.01130	0.95622	0.04781	0.00542	0.05993	0.00300
3.2	0.03258	3.10980	0.15549	0.02061	0.20906	0.01045
						k_2 = 200.0
0.1	0.00003	0.00515	0.00026	0.00000	0.00033	0.00000
0.2	0.00014	0.02056	0.00103	0.00003	0.00131	0.00001
0.4	0.00054	0.08191	0.00410	0.00008	0.00524	0.00003
0.8	0.00206	0.32231	0.01612	0.00030	0.02077	0.00010
1.6	0.00683	1.21587	0.06079	0.00120	0.08034	0.00040
3.2	0.01590	4.14395	0.20720	0.00468	0.28961	0.00145

$H = 4.0$

$k_1 = 200.0$

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						$k_2 = 0.2$
0.1	0.00010	0.00334	0.00002	0.00010	0.00001	0.00004
0.2	0.00042	0.01333	0.00007	0.00039	0.00003	0.00016
0.4	0.00167	0.05295	0.00026	0.00157	0.00013	0.00065
0.8	0.00663	0.20621	0.00103	0.00625	0.00051	0.00256
1.6	0.02562	0.74824	0.00374	0.02427	0.00195	0.00975
3.2	0.09166	2.25046	0.01125	0.08799	0.00660	0.03298
						$k_2 = 2.0$
0.1	0.00003	0.00437	0.00002	0.00002	0.00002	0.00001
0.2	0.00011	0.01746	0.00009	0.00009	0.00009	0.00005
0.4	0.00042	0.06947	0.00035	0.00036	0.00036	0.00018
0.8	0.00168	0.27221	0.00136	0.00142	0.00144	0.00072
1.6	0.00646	1.01140	0.00506	0.00560	0.00553	0.00277
3.2	0.02332	3.23913	0.01645	0.02126	0.01951	0.00975
						$k_2 = 20.0$
0.1	0.00001	0.00545	0.00003	0.00000	0.00003	0.00000
0.2	0.00003	0.02178	0.00011	0.00002	0.00014	0.00001
0.4	0.00013	0.08673	0.00043	0.00003	0.00054	0.00003
0.8	0.00050	0.34131	0.00171	0.00031	0.00215	0.00011
1.6	0.00186	1.28773	0.00644	0.00124	0.00833	0.00042
3.2	0.00612	4.38974	0.02195	0.00483	0.03010	0.00150
						$k_2 = 200.0$
0.1	0.00000	0.00652	0.00003	0.00000	0.00004	0.00000
0.2	0.00002	0.02606	0.00013	0.00000	0.00017	0.00000
0.4	0.00007	0.10389	0.00052	0.00002	0.00068	0.00000
0.8	0.00025	0.40997	0.00205	0.00007	0.00269	0.00001
1.6	0.00086	1.56284	0.00781	0.00027	0.01049	0.00005
3.2	0.00225	5.48870	0.02744	0.00107	0.03866	0.00019

$H = 8.0$

$k_1 = 0.2$

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						$k_2 = 0.2$
0.1	0.00035	0.00006	0.00028	0.00027	0.00007	0.00036
0.2	0.00142	0.00023	0.00113	0.00108	0.00028	0.00142
0.4	0.00566	0.00090	0.00449	0.00432	0.00113	0.00567
0.8	0.02240	0.00354	0.01769	0.01711	0.00449	0.02246
1.6	0.08589	0.01335	0.06673	0.06610	0.01725	0.08624
3.2	0.29318	0.04270	0.21350	0.23182	0.05907	0.29533
						$k_2 = 2.0$
0.1	0.00030	0.00008	0.00038	0.00023	0.00041	0.00021
0.2	0.00120	0.00030	0.00152	0.00091	0.00165	0.00083
0.4	0.00479	0.00121	0.00606	0.00364	0.00660	0.00330
0.8	0.01894	0.00480	0.02399	0.01446	0.02616	0.01308
1.6	0.07271	0.01841	0.09206	0.05601	0.10080	0.05040
3.2	0.24933	0.06307	0.31534	0.19828	0.35008	0.17504
						$k_2 = 20.0$
0.1	0.00016	0.00010	0.00049	0.00009	0.00105	0.00005
0.2	0.00065	0.00040	0.00198	0.00037	0.00421	0.00021
0.4	0.00260	0.00158	0.00790	0.00149	0.01679	0.00084
0.8	0.01026	0.00629	0.03143	0.00594	0.06664	0.00333
1.6	0.03926	0.02463	0.12314	0.02320	0.25871	0.01294
3.2	0.13335	0.09123	0.45615	0.08510	0.92478	0.04624
						$k_2 = 200.0$
0.1	0.00009	0.00015	0.00074	0.00002	0.00162	0.00001
0.2	0.00036	0.00059	0.00294	0.00008	0.00648	0.00003
0.4	0.00145	0.00235	0.01176	0.00032	0.02587	0.00013
0.8	0.00573	0.00938	0.04690	0.00127	0.10287	0.00051
1.6	0.02160	0.03710	0.18549	0.00503	0.40238	0.00201
3.2	0.06938	0.14226	0.71130	0.01912	1.48097	0.00740

H = 8.0

k_1 = 2.0

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						k_2 = 0.2
0.1	0.00028	0.00028	0.00014	0.00024	0.00004	0.00022
0.2	0.00113	0.00111	0.00056	0.00096	0.00017	0.00087
0.4	0.00451	0.00444	0.00222	0.00384	0.00069	0.00347
0.8	0.01736	0.01752	0.00876	0.01522	0.00275	0.01373
1.6	0.06895	0.06662	0.03331	0.05900	0.01060	0.05298
3.2	0.24127	0.22014	0.11007	0.20949	0.03693	0.18466
						k_2 = 2.0
0.1	0.00013	0.00039	0.00020	0.00010	0.00020	0.00010
0.2	0.00053	0.00157	0.00079	0.00041	0.00078	0.00039
0.4	0.00213	0.00628	0.00314	0.00164	0.00311	0.00156
0.8	0.00844	0.02487	0.01244	0.00653	0.01237	0.00618
1.6	0.03269	0.09597	0.04798	0.02556	0.04802	0.02401
3.2	0.11640	0.33606	0.16803	0.09405	0.17138	0.08594
						k_2 = 20.0
0.1	0.00005	0.00061	0.00030	0.00002	0.00037	0.00002
0.2	0.00019	0.00242	0.00121	0.00010	0.00149	0.00007
0.4	0.00076	0.00967	0.00484	0.00040	0.00596	0.00030
0.8	0.00300	0.03845	0.01922	0.00159	0.02369	0.00118
1.6	0.01154	0.15010	0.07505	0.00630	0.09274	0.00464
3.2	0.04003	0.54942	0.27471	0.02409	0.34233	0.01712
						k_2 = 200.0
0.1	0.00003	0.00082	0.00041	0.00001	0.00052	0.00000
0.2	0.00011	0.00328	0.00164	0.00002	0.00206	0.00001
0.4	0.00042	0.01310	0.00655	0.00008	0.00825	0.00004
0.8	0.00167	0.05216	0.02608	0.00034	0.03287	0.00016
1.6	0.00629	0.20491	0.10245	0.00135	0.12933	0.00065
3.2	0.02030	0.76769	0.38384	0.00527	0.48719	0.00244

H = 8.0

k_1 = 20.0

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						k_2 = 0.2
0.1	0.00012	0.00056	0.00003	0.00011	0.00001	0.00006
0.2	0.00047	0.00223	0.00011	0.00044	0.00005	0.00025
0.4	0.00190	0.00839	0.00044	0.00176	0.00020	0.00099
0.8	0.00754	0.03522	0.00176	0.00701	0.00079	0.00393
1.6	0.02947	0.13569	0.00678	0.02746	0.00306	0.01528
3.2	0.10817	0.47240	0.02362	0.10145	0.01105	0.05524
						k_2 = 2.0
0.1	0.00003	0.00079	0.00004	0.00003	0.00004	0.00002
0.2	0.00013	0.00316	0.00016	0.00011	0.00016	0.00008
0.4	0.00050	0.01260	0.00063	0.00043	0.00064	0.00032
0.8	0.00200	0.05007	0.00250	0.00170	0.00253	0.00127
1.6	0.00736	0.19496	0.00975	0.00673	0.00993	0.00496
3.2	0.02944	0.70709	0.03535	0.02579	0.03678	0.01839
						k_2 = 20.0
0.1	0.00001	0.00106	0.00005	0.00001	0.00006	0.00000
0.2	0.00004	0.00425	0.00021	0.00002	0.00025	0.00001
0.4	0.00014	0.01696	0.00085	0.00009	0.00100	0.00005
0.8	0.00056	0.06751	0.00338	0.00037	0.00398	0.00020
1.6	0.00217	0.26466	0.01323	0.00147	0.01565	0.00078
3.2	0.00791	0.98450	0.04922	0.00576	0.05892	0.00295
						k_2 = 200.0
0.1	0.00000	0.00133	0.00007	0.00000	0.00003	0.00000
0.2	0.00002	0.00531	0.00027	0.00000	0.00032	0.00000
0.4	0.00006	0.02122	0.00106	0.00002	0.00128	0.00001
0.8	0.00025	0.08453	0.00423	0.00008	0.00509	0.00003
1.6	0.00096	0.33268	0.01663	0.00032	0.02009	0.00010
3.2	0.00319	1.25614	0.06281	0.00125	0.07660	0.00033

H = 8.0

k_1 = 200.0

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
						k_2 = 0.2
0.1	0.00003	0.00083	0.00000	0.00003	0.00000	0.00001
0.2	0.00011	0.00330	0.00002	0.00011	0.00001	0.00005
0.4	0.00046	0.01320	0.00007	0.00044	0.00004	0.00018
0.8	0.00182	0.05242	0.00026	0.00175	0.00014	0.00072
1.6	0.00720	0.20411	0.00102	0.00693	0.00056	0.00282
3.2	0.02751	0.74013	0.00370	0.02656	0.00212	0.01058
						k_2 = 2.0
0.1	0.00001	0.00109	0.00001	0.00001	0.00001	0.00000
0.2	0.00003	0.00438	0.00002	0.00002	0.00002	0.00001
0.4	0.00010	0.01748	0.00009	0.00009	0.00009	0.00005
0.8	0.00041	0.06956	0.00035	0.00038	0.00037	0.00018
1.6	0.00162	0.27262	0.00136	0.00149	0.00145	0.00072
3.2	0.00625	1.01322	0.00507	0.00584	0.00547	0.00273
						k_2 = 20.0
0.1	0.00000	0.00136	0.00001	0.00000	0.00001	0.00000
0.2	0.00001	0.00546	0.00003	0.00001	0.00003	0.00000
0.4	0.00002	0.02181	0.00011	0.00002	0.00013	0.00001
0.8	0.00010	0.08687	0.00043	0.00008	0.00052	0.00003
1.6	0.00039	0.34202	0.00171	0.00032	0.00204	0.00010
3.2	0.00149	1.29190	0.00646	0.00127	0.00777	0.00039
						k_2 = 200.0
0.1	0.00000	0.00163	0.00001	0.00000	0.00001	0.00000
0.2	0.00000	0.00654	0.00003	0.00000	0.00004	0.00000
0.4	0.00001	0.02613	0.00013	0.00000	0.00016	0.00000
0.8	0.00003	0.10417	0.00052	0.00002	0.00063	0.00000
1.6	0.00013	0.41121	0.00206	0.00007	0.00249	0.00001
3.2	0.00047	1.56843	0.00784	0.00027	0.00957	0.00005

Appendix B

A MATHEMATICAL DESCRIPTION OF THE PROBLEM

A diagrammatic representation of the system was given in Figure 1. The system consists of a semi-infinite elastic solid bounded at the surface by two finite elastic layers of constant thickness. The surface is loaded uniformly over a circular area and so, if cylindrical polar coordinates (r, θ, z) are used, the stresses and strains in the system will be independent of the angle θ.

Assumptions

1. The Applicability of the Theory of Elasticity.—The materials involved are assumed to be homogeneous and isotropic, the stress/strain relationship to be linear and the modulus to be the same in compression as in tension.
2. Description of System.—The three layers are horizontal and have different elastic properties. The two upper layers are weightless and of infinite extent in the horizontal plane. The bottom layer is infinite in the horizontal plane and semi-infinite in the vertical plane.
3. Boundary Conditions.—The surface of the top layer is free of shear stress. Outside the circular loaded area the surface is free of normal stress and inside the load is uniformly distributed. All stresses and displacements are zero at infinite depth.
4. Continuity Conditions.—The three layers are continuously in contact and act together as a composite medium. At the interfaces the normal and shear stresses and the vertical and horizontal displacements must be the same in both layers.

TABLE 2

NOTATION FOR STRESSES

Stress	Symbol
Vertical normal	σ_z
Radial normal	σ_r
Circumferential normal	σ_θ
Shear	τ_{rz}

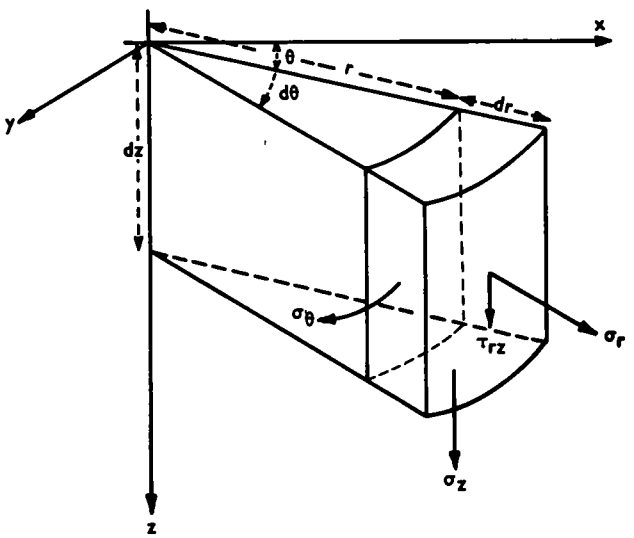

Figure 2. Components of stress in cylindrical co-ordinates.

5. Poisson's Ratio. —This is assumed to be equal to 0.5 in all three layers, the main reasons being that the algebra involved in developing the stress expression is considerably simplified. This value of Poisson's ratio corresponds to an incompressible medium.

Derivations of the Expressions for the Stresses in a Three-Layered System

The notation for the stresses is presented in Table 2. To take advantage of the symmetry of the problem cylindrical polar coordinates have been used. The components of stress are shown in Figure 2.

The equations of equilibrium are obtained by considering the total force for a small element in both the radial and vertical directions.

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \qquad (1a)$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0 \qquad (1b)$$

(Because of the axial symmetry in the problem, $\tau_{r\theta}$ and $\tau_{\theta z}$ are zero, and there is no equilibrium condition for the θ-direction).

In the general three-dimensional case in rectangular coordinates, each of the six components of strain (three normal and three shear) may be expressed in terms of the three components of displacement. Thus these equations are not independent and three relationships exist between the six components of strain. These are called the compatibility conditions, and they may be expressed in terms of components of stress by using Hooke's law. In cylindrical polar coordinates with axial symmetry the compatibility conditions become

$$\nabla^2 \sigma_r - \frac{2}{r^2}(\sigma_r - \sigma_\theta) + \frac{1}{1+\mu}\frac{\partial^2}{\partial r^2}(\sigma_r + \sigma_\theta + \sigma_z) = 0 \qquad (2a)$$

$$\nabla^2 \sigma_\theta + \frac{2}{r^2}(\sigma_r - \sigma_\theta) + \frac{1}{1+\mu}\frac{1}{r}\frac{\partial}{\partial r}(\sigma_r + \sigma_\theta + \sigma_z) = 0 \qquad (2b)$$

where μ is Poisson's ratio and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

is the Laplacian operator.

The components of stress and displacement may be expressed in terms of a stress function φ in such a way that Eqs. 1a and 1b are identically satisfied. These expressions are

Stress

$$\sigma_z = \frac{\partial}{\partial z}\left[(2-\mu)\nabla^2\varphi - \frac{\partial^2 \varphi}{\partial z^2}\right] \qquad (3a)$$

$$\sigma_r = \frac{\partial}{\partial z}\left[\mu\nabla^2\varphi - \frac{\partial^2 \varphi}{\partial r^2}\right] \qquad (3b)$$

$$\sigma_\theta = \frac{\partial}{\partial z}\left[\mu\nabla^2\varphi - \frac{1}{r}\frac{\partial \varphi}{\partial r}\right] \qquad (3c)$$

$$\tau_{rz} = \frac{\partial}{\partial r}\left[(1-\mu)\nabla^2\varphi - \frac{\partial^2 \varphi}{\partial z^2}\right] \qquad (3d)$$

Displacement

(vertical) $$w = \frac{1+\mu}{E}\left[(1-2\mu)\nabla^2\varphi + \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r}\right] \qquad (3e)$$

$$\text{(horizontal)} \quad u = -\frac{(1+\mu)}{E} \frac{\partial^2 \varphi}{\partial r \partial z} \tag{3f}$$

in which E is the modulus of elasticity.

It may be shown quite easily that Eqs. 2a and 2b are satisfied provided φ is a solution of the biharmonic equation:

$$\nabla^4 \varphi = 0 \tag{4}$$

in which ∇^2 is again the Laplacian operator.

The problem is thus reduced to the solution of this partial differential equation subject to the boundary conditions at the surface, the interfaces and at infinite depth. The boundary conditions are expressed next; the numerical subscripts refer to the different layers.

Interface 1 ($z = -h_1$)	Interface 2 ($z = -h_1-h_2$)	
$\sigma_{z_1} = \sigma_{z_2}$	$\sigma_{z_2} = \sigma_{z_3}$	(normal stresses)
$\tau_{rz_1} = \tau_{rz_2}$	$\tau_{rz_2} = \tau_{rz_3}$	(shear stresses)
$w_1 = w_2$	$w_2 = w_3$	(vertical displacements)
$u_1 = u_2$	$u_2 = u_3$	(horizontal displacements)

$\tau_{rz_1} = 0$ ($z = 0$, surface free of shear stress)

$w_3 = u_3 = \sigma_{z_3} = \tau_{rz_3} = 0$ at infinite depth

$$\left.\begin{aligned} \sigma_z &= 1 \quad 0 \le r \le a \\ \sigma_z &= 0 \quad r > a \end{aligned}\right\} \quad \text{(surface loading)}$$

The radial stresses σ_r are not continuous across the interface because the horizontal displacements are equal; they are determined by the relevant modulus of elasticity.

The partial differential Eq. 4 may be solved by using the Hankel transform*. However, for purpose here the solution may be expressed in the following form.

The stress functions

$$\varphi_1 \quad \text{(top layer)} = J_o(mr)\left[(A_1 + B_1 z)\, e^{mz} + (C_1 + D_1 z)\, e^{-mz}\right]$$

$$\varphi_2 \text{ (middle layer)} = J_o(mr)\left[(A_2 + B_2 z)\, e^{mz} + (C_2 + D_2 z)\, e^{-mz}\right]$$

$$\varphi_3 \text{ (bottom layer)} = J_o(mr)\left[(C_3 + D_3 z)\, e^{-mz}\right]$$

satisfy Eq. 4 and the conditions at infinite depth**. The ten constants are evaluated by using the boundary conditions at the interfaces, the condition that the surface shear stress is zero and by assuming a stress distribution of the form:

*The Hankel transform $F(m)$ of order zero of the function $f(x)$ is given by

$$F(m) = \int_0^\infty xf(x)J_o(mx)dx$$

for details of the application of this transform to problems in three dimensional elasticity, see (6).

**$J_o(x)$ is the Bessel function of order zero.

$$\sigma_z\,(r,\ m) = -mJ_0(mr)$$

The final expressions for the stresses and displacements are then given by

$$\sigma_z = \int_0^a \int_0^\infty \sigma_z(r,\ m)\ r\ dr\ dm \tag{5a}$$

$$\sigma_r = \sigma_\theta = \int_0^a \int_0^\infty \frac{\sigma_r(r,\ m) + \sigma_\theta(r,\ m)}{2}\ r\ dr\ dm \tag{5b}$$

The total surface load is given by

$$\int_0^a \int_0^\infty \sigma_z(r,\ m)\ r\ dr\ dm$$

$$= -\int_0^a \int_0^\infty mr\ J_0(mr)\ dr\ dm = -\int_0^\infty aJ_1(am)dm = -1 \tag{6}$$

The final expressions for the stresses and displacements are obtained by substituting the stress functions φ into Eqs. 3a through 3f for the components of stress and then using the Eqs. 5a and 5b. The algebra involved is quite considerable but is simplified by assuming that Poisson's ratio is equal to 0.5 in all three layers. The expressions for the components of stress that appear in the tables are given next.

(1) First interface

$$\sigma_{z_1} = -(1-K)\int_0^\infty a_1 J_1(a_1 x)\ \frac{f_1(x)}{D(x)}\ dx$$

$$2\sigma_{r_1} = \int_0^\infty a_1 J_1(a_1 x)\ \frac{g_1(x)}{D(x)}\ dx$$

$$2\sigma_{r_2} = (1-K)\int_0^\infty a_1 J_1(a_1 x)\ \frac{g_2(x)}{D(x)}\ dx$$

(2) Second interface

$$\sigma_{z_2} = (1-K)(1-N)\int_0^\infty a_1 J_1(a_1 x)\ \frac{F_1(x)}{D(x)}\ dx$$

$$2\sigma_{r_2} = -(1-K)\int_0^\infty a_1 J_1(a_1 x)\ \frac{G_1(x)}{D(x)}\ dx$$

$$2\sigma_{r_3} = -(1-K)(1-N)\int_0^\infty a_1 J_1(a_1 x)\,\frac{G_2(x)}{D(x)}\,dx$$

The symbols and functions in these expressions are defined as follows:

$$a_1 = a/h_2,\ H = h_1/h_2,\ k_1 = E_1/E_2,\ \text{and } k_2 = E_2/E_3,$$

$$K = \frac{k_1-1}{k_1+1},\quad N = \frac{k_2-1}{k_2+1}$$

$$f_1(x) = (1+Hx)\left\{e^{-Hx} - KN^2 e^{-(4+H)x}\right\} - (1-Hx)\left\{Ke^{-3Hx} - N^2 e^{-(3H+4)x}\right\}$$

$$-Np_1(x)e^{-(3H+2)x} - Np_2(x)e^{-(H+2)x}$$

$$g_1(x) = \frac{4}{1+k_1}\ (1-Hx)\left\{Ke^{-3Hx} - N^2 e^{-(3H+4)x}\right\} - \frac{4}{1+k_1}\ (1+Hx)\left\{e^{-Hx}\right.$$

$$\left.-KN^2 e^{-(H+4)x}\right\} + \frac{6k_1 Hx}{1+k_1}\left\{e^{-Hx} + KN^2 e^{-(H+4)x} - Ke^{-3Hx} - N^2 e^{-(3H+4)x}\right\}$$

$$-Np_3(x)e^{-(3H+2)x} - Np_4(x)e^{-(H+2)x}$$

$$g_2(x) = K(2-5Hx)e^{-3Hx} - (2-Hx)e^{-Hx} - N^2(2+Hx)e^{-(3H+4)x}$$

$$+KN^2(2+5Hx)e^{-(H+4)x} - Np_5(x)e^{-(3H+2)x} - Np_6(x)e^{-(H+2)x}$$

$$F_1(x) = -(1+x+Hx)e^{-(H+1)x} - KN(1-x+Hx-2Hx^2)e^{-(H+3)x}$$

$$+K(1+x-Hx-2Hx^2)e^{-(3H+1)x} + N(1-x-Hx)e^{-(3H+3)x}$$

$$G_1(x) = Kq_1(x)e^{-(3H+1)x} - q_2(x)e^{-(H+1)x} + Nq_3(x)e^{-(3H+3)x} - KNq_4(x)e^{-(H+3)}$$

$$G_2(x) = K(-2+x+5Hx-2Hx^2)e^{-(3H+1)x} + (2-x-Hx)e^{-(H+1)x}$$

$$- N(2-5x-5Hx)e^{-(3H+3)x} + KN(2-5x-Hx-10Hx^2)e^{-(H+3)x}$$

$$D(x) = 1+2KN(1+2x^2)e^{-2x} + K^2N^2e^{-4x} + K^2e^{-4Hx} + 2KN(1+2x^2)e^{-(4H+2)x}$$

$$+ N^2e^{-(4H+4)x} - 2K(1+2H^2x^2)e^{-2Hx} - K^2Np_7(x)e^{-(2H+2)x} - Np_8(x)e^{-(2H+2}$$

$$- 2KN^2(1+2H^2x^2)e^{-(2H+4)x}$$

and

$$p_1(x) = p_2(-x) = -K(1+2x-Hx+2x^2-2Hx^2-4Hx^3) + (1-2x-Hx+2x^2+2Hx^2)$$

$$p_3(x) = p_4(-x) = K(4+2Hx+8x^2+4Hx^3) - (2-4x-5Hx-2x^2-2Hx^2)$$

$$- K^2(2+4x-5Hx+10x^2+2Hx^2-20Hx^3)$$

$$p_5(x) = p_6(-x) = K(2+4x+Hx-2x^2-10Hx^2+4Hx^3) - (2-4x-5Hx-2x^2-2Hx^2)$$

$$p_7(x) = 2+4x^2-8Hx^2+4H^2x^2+16H^2x^4$$

$$p_8(x) = 2+4x^2+8Hx^2+4H^2x^2$$

$$q_1(x) = N(2+5x+Hx-10Hx^2) - (2-x-5Hx+2Hx^2)$$

$$q_2(x) = N(2+5x+5Hx) - (2-x-Hx)$$

$$q_3(x) = N(2+x+Hx) - (2-5x-5Hx)$$

$$q_4(x) = N(2+x+5Hx+2Hx^2) - (2-5x-Hx-10Hx^2)$$

The expression for $g_1(x)$ differs from that given by Acum and Fox in the fourth term in the third bracket. This was discovered when an unaccountable discrepancy was noted for values of $\sigma_{z_1} = \sigma_{r_1}$. An error was suspected when a certain lack of symmetry in the expression for $g_1(x)$ was noted. Inspection showed that the coefficient of $\exp\left[-(3H+4)x\right]$ should be N^2 on both occasions, whereas the paper of Acum and Fox shows N instead of N^2 in the third bracket. When this change was made the disagreement disappeared. The corrected expression was verified later when it was derived from first principles.

As a further check some results were obtained for comparison with those of Acum and Fox. These are presented in Appendix C (figures in brackets being taken from Acum and Fox's paper).

Appendix C

H = 0.25 k_1 = 5.0 k_2 = 5.0

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
0.5	0.74131	1.39458	0.37892	0.08783	0.38000	0.07600
	(0.7410)	(1.8900)	(0.3790)	(0.0879)	(0.3800)	(0.0761)
1.0	0.94443	1.25311	0.25062	0.26496	1.04512	0.20902
	(0.9470)	(1.2700)	(0.2540)	(0.2650)	(1.0400)	(0.2090)

k_1 = 5.0 k_2 = 100.0

0.5	0.72401	1.69907	0.33931	0.01171	0.70741	0.00707
	(0.7240)	(1.7000)	(0.3400)	(0.0117)	(0.7070)	(0.00707)
1.0	0.89239	0.49183	0.09836	0.04125	2.23183	0.02232
	(0.8970)	(0.5030)	(0.1020)	(0.0413)	(2.2300)	(0.0223)

k_1 = 50.0 k_2 = 5.0

0.5	0.30774	7.47254	0.14915	0.05798	0.23339	0.04768
	(0.3080)	(7.4700)	(0.1500)	(0.0580)	(0.2380)	(0.0477)
1.0	0.64196	12.73759	0.25475	0.19381	0.75223	0.15045
	(0.6420)	(12.7300)	(0.2550)	(0.1940)	(0.7520)	(0.1500)

k_1 = 50.0 k_2 = 100.0

0.5	0.23529	8.73121	0.17462	0.00791	0.46746	0.00467
	(0.2850)	(8.7200)	(0.1740)	(0.00791)	(0.4660)	(0.00466)
1.0	0.56937	17.17838	0.34958	0.02924	1.61943	0.01619
	(0.5700)	(17.1500)	(0.3490)	(0.0292)	(1.6100)	(0.0161)

Continued

H = 2.0 k_1 = 5.0 k_2 = 5.0

a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_1}-\sigma_{r_3}$
0.5	0.02848	0.21464	0.04293	0.00955	0.04100	0.00820
	(0.0285)	(0.2150)	(0.0430)	(0.00950)	(0.0409)	(0.00817)
1.0	0.09503	0.72664	0.14533	0.03649	0.15324	0.03065
	(0.0951)	(0.7270)	(0.1450)	(0.0364)	(0.1530)	(0.0306)
						k_1 = 50.0 k_2 = 5.0
0.5	0.00524	0.38265	0.00765	0.00269	0.00856	0.00171
	(0.00522)	(0.3820)	(0.00765)	(0.00268)	(0.00853)	(0.00170)
1.0	0.01867	1.36457	0.02729	0.01054	0.03237	0.00657
	(0.0187)	(1.3600)	(0.0273)	(0.0105)	(0.0330)	(0.0066)
						k_1 = 50.0 k_2 = 100.0
0.5	0.00280	0.51248	0.01025	0.00039	0.01430	0.00014
	(0.00281)	(0.5230)	(0.0105)	(0.00041)	(0.0147)	(0.00015)
1.0	0.00914	1.88310	0.03766	0.00156	0.05566	0.00056
	(0.00921)	(1.9300)	(0.0385)	(0.00163)	(0.0575)	(0.00058)
						k_1 = 10.0 k_2 = 5.0
0.5	0.01752	0.27077	0.02706	0.00671	0.02701	0.00540
	(0.0176)	(0.2720)	(0.0272)	(0.00670)	(0.0270)	(0.00540)
1.0	0.06004	0.93871	0.09388	0.02588	0.10202	0.02040
	(0.0600)	(0.9390)	(0.0938)	(0.0259)	(0.1020)	(0.0204)

(test 10), the sample expanded in the absence of water as could possibly occur with a sampling technique. It is significant that the slopes of the recompression curves vary considerably. Tests 6, 7, and 8 tend to follow the rebound lines 4, 5, and 9, and 11, 12, and 13 fall far higher. In particular, the slope of 13 is 1½ times greater than that of test 8 which measures the true compressibility of the clay.

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