A Three-Dimensional Calculus Model Of Urban Settlement

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•THIS PAPER essentially has the single purpose of presenting a simplified mathematical description of urban settlement which the author believes may be a useful tool to those working on various aspects of the simulation of urban transportation demands, rent theory, and spatial interaction by analytical models.

There are two basic premises to this descriptive theory which have been verified time and again in the literature on urban analysis:

1. The density of urban settlement can be reasonably well described by a hyperbolic-parabola:

$$\rho = \frac{K}{r^n}$$

in which

ρ = density of settlement in terms of people per unit area of land;

r = distance from the focus of the urban centrality (characteristically the heart of the central business district); and

K and n = empirically derived values (see Curve A, Fig. 1).

2. The density-distance function just described is skewed by the higher density development that traditionally follows radial transportation routes (whether they be the old streetcar lines of the 1920's, the land-service arterials of the 1930's and 1940's, or the limited-access freeways of the 1950's). This point is shown by Figures 1 and 2.

The three curves of Figure 1 may be thought of as representing the density-distance relationships along three separate cross-sections of the urban area. Curve A, the lowest, represents a cut through an interstitial area of the land between major transportation radials; whereas B and C represent cuts along or adjacent to major radial transportation routes (Fig. 2, which represents the urban area in plan form).

The different shapes to curves B and C merely indicate two possibilities for increased density adjacent to well-developed radial routes—both would obviously have to be higher than curve A, reflecting the increased densities that inevitably follow transportation routes. However, only empirical evidence could ascertain whether improved transportation would simply translate the curve vertically, as in the case of curve B, or skew the slope of the curve also, as in the case of curve C.

One additional word of explanation is necessary regarding Figures 1 and 2. The curves are obviously discontinuous at the margins of the CBD core because of the relative absence of residential population in the core itself. For most American metropolises, this discontinuity occurs at about one-half mile from the point of greatest retail activity within the core.

Three assumptions are now made to keep the mathematical development simplified: (a) the exponential value of r is always 2; (b) the density-distance equation for an intensive transportation corridor is only differentiated from that of a less intensively developed sector by a higher value of K; and (c) all of the transportation intensive sec-

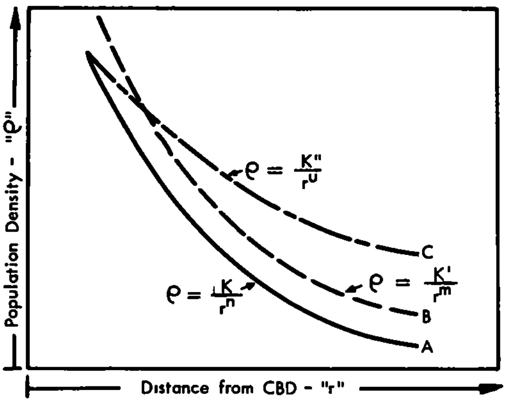


Figure 1.

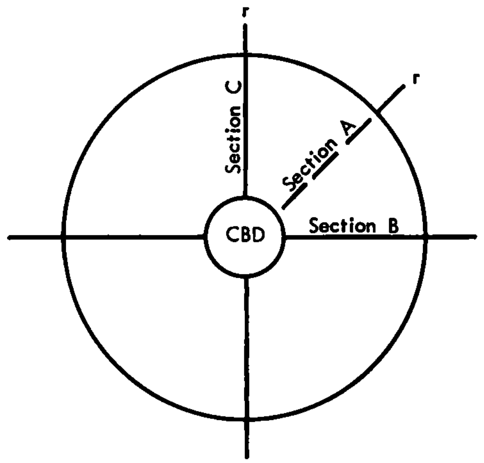


Figure 2.

tors may be characterized by one value of K, and those of the interstitial areas by another value, K'. These assumptions are shown by Figures 3 and 4. Existing knowledge of the space-density function shows all of these assumptions to be relatively valid.

It is now assumed that Figure 5 represents the plan form of an urban area with a four-radial major arterial system, with arterials along the coordinate axes. The most intensive land use of the CBD is at the origin. Distance from the origin is here represented as r.

The population residing in a small portion of the city, dA, is represented by dP, and

$$dP = \rho \frac{\text{Persons}}{\text{unit area}} \times dA \text{ (units of Area)}$$

$$= (\rho)\ (dr)\ (rd\theta)$$

$$= \frac{k}{r^2}\ (dr)\ (rd\theta) \quad \text{in which k is a constant somewhere between K and K'}$$

$$= \frac{k}{r^2}\ (dr)\ (d\theta) \tag{1}$$

To integrate the entire population of the urban area from the derivative expression it is necessary to make some assumption as to the circumferential variation of k between the arterial values of K^1 and the interstitial values of K. This variation could be represented by many functions, but to preserve a surface continuum for the urban model a cosinusoidal function will be assumed. In other words, as one would traverse the urban area in a circumferential manner within the annulus area described by dr, density would vary cosinusoidally, reaching a maximum amplitude as the circumferential route crosses the four radial arterials. This circumferential function of density is shown in Figure 6.

The significance of the expression for density as shown in Figure 6 is that it represents a method of simulating population density very readily from either known or assumed values of K', K, r, and θ. In other words, for assumed density-distance characteristics one can readily obtain density for any differentially small area of the universe of urban space.

For example, for a moderate-sized metropolitan area in the United States (say,

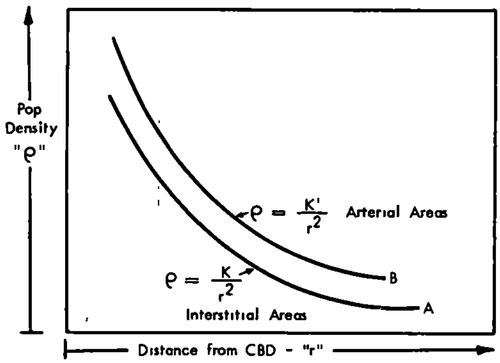


Figure 3.

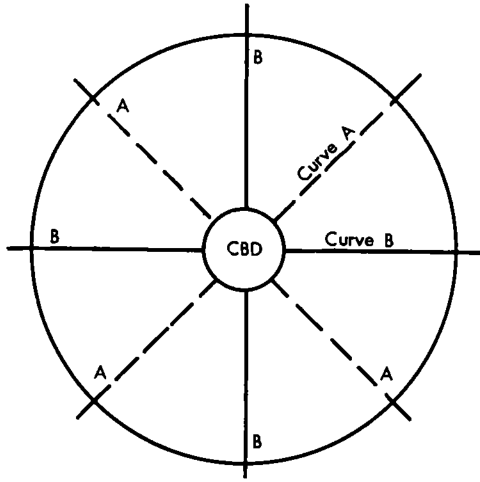


Figure 4.

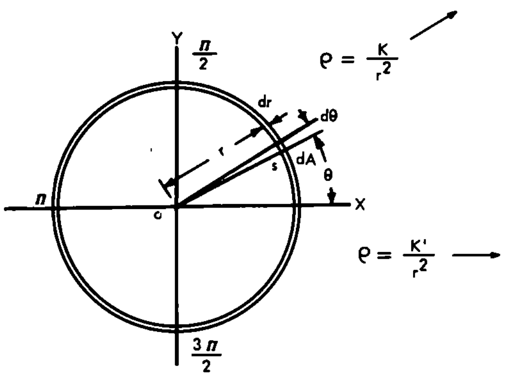
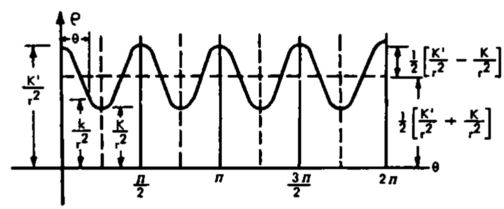


Figure 5.



Circumferential Density, ρ

$$= \frac{k}{r^2} = \frac{1}{2}\left[\frac{K'}{r^2} + \frac{K}{r^2}\right] + \left[\frac{K'}{r^2} - \frac{K}{r^2}\right] \cos 4\theta$$

$$= \frac{1}{2r^2}\left[K'(1 + \cos 4\theta) + K(1 - \cos 4\theta)\right]$$

Figure 6.

between 400,000 and 800,000 people), the constants of the density-distance curves have values of approximately 16,000 for K and 32,000 for K', when distance is expressed in miles and density in persons per square mile (ppsm). Thus, in a typical interstitial area between arterial routes at four miles distance out (r = 4, and θ = 45°, 135°, 225°, and 315°, respectively, for a four-radial model), for the smallest value of θ, as well as all other values,

$$\rho = \frac{1}{2\,(4)^2}\left[32,000\,(\,1 - \cos 180^\circ) - 16,000\,(1 - \cos 180^\circ)\right]$$

$$= 1,000 \text{ ppsm.}$$

The next significant step in the use of this theory is the evaluation of population within and segment of the urban area. If the segment is small enough so that density variation is insignificant, population can simply be obtained by multiplying density by area. Thus, in the foregoing example, the number of people on one acre of land would be the density value of 1,000 people per square mile divided by 640 acres, or approximately 1½ persons per gross acre. If, on the other hand the size of the area is significantly large so that density variance is an importance, then an integrable expression may be obtained from Eq. 1 in the following way:

$$dP = \frac{k}{r}\,(dr)\,(d\theta)$$

$$= \tfrac{1}{2}\left\{\left[\frac{K'}{r^2} + \frac{K}{r^2}\right] + \left[\frac{K'}{r^2} - \frac{K}{r^2}\right]\cos 4\theta\right\}(dr)\,(d\theta)$$

$$= \frac{1}{2r^2}\left[(K' + K) + (K' - K)\cos 4\theta\right](dr)\,(d\theta)$$

$$= \left[\frac{K' + K}{2}\right]\frac{dr}{r^2}\,d\theta + \left[\frac{K' - K}{2}\right]\frac{dr}{r^2}\cos 4\theta\,d\theta$$

thus

$$P_a = \int_{R_1}^{R_2}\int_{\theta_1}^{\theta_2}\left[\frac{K' + K}{2}\right]d\theta\,\frac{dr}{r^2} + \int_{R_1}^{R_2}\int_{\theta_1}^{\theta_2}\left[\frac{K' - K}{2}\right]\cos 4\theta\,d\theta\,\frac{dr}{r^2} \qquad (2)$$

in which P_a represents the population residing between the radial limits of R_1 and R_2, and the angular limits of θ_1 and θ_2.

In a similar fashion, a value for the entire urban population can be obtained by integrating Eq. 2 over the entire plane comprising the urban area. Assuming the radial limits of the integration to be A for the inner value and R for the outer, the entire population P is given by

$$P = \pi\left[\frac{1}{A} - \frac{1}{R}\right]\left[K' - K\right] \qquad (3)$$

The following points of significance may be attached to this model:

1. Unlike previous models involving density-distance relationships, it allows for the higher density of land development which almost universally follows alongside radial traffic arteries.
2. It describes the density-distance relationship as a continuous space function, permitting rapid calculation of density for any point on the urban surface.
3. As traffic origins and destinations can be related to population density, and as interzonal travel demand can be related to population and distance in a gravity-type model, this theory should enable rough predictions to be made for interzonal travel.
4. Stemming from the continuity of the model and the previous point, it should be relatively easy to program traffic demand by the use of a computer that could be set to integrate discreet areas and develop interzonal demand between these areas.
5. The model should give some reasonably good idea of the allocation of population growth to an urban area under varying assumptions of the density-distance relationship.
6. Subnodal concentrations should suggest of treatment in a similar way and permit simulation to be made of the compounding effect of more than one center of activity concentration in an urban area by simply adding the density ordinates.