Some Mathematical Aspects of The Problem of Merging

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• AS ROADS and highways become capable of carrying higher and higher traffic volumes, the perturbations introduced by vehicles traveling at speeds or in paths that differ substantially from the norm become increasingly harmful to safe and efficient operation of the road network. Some of this individual variation is undoubtedly fortuitous and can be removed, or at least diminished, by sensible efforts to educate and control drivers.

Even in the best of circumstances, however, there remains the necessity for accelerating, decelerating, weaving, and merging; namely, the need of each car to enter the system and leave the system where it wishes. Not only is this each driver's prerogative, but it is also one that in many cases he exercises without specific traffic control.

Perhaps the most important example of such a situation is the freeway on-ramp and acceleration lane. At these points, which must be provided fairly frequently in urban areas, the smooth flow of traffic is perpetually harassed by new arrivals.

Although it does not seem practical at the moment to imagine an automatic merging control device having the ability to synchronize effectively the multitude of individual merges that occur in a day, this does not mean that the traffic engineer need go to the other extreme and abandon any idea of controlling the merging process.

Indeed, the literature contains ample evidence that location and design of on-ramps and acceleration lanes are closely connected with the influence they exert on traffic stability. If the exact nature of this influence is imperfectly understood today, it is only because the relative complexity of the merging situation has made a completely scientific treatment of the subject too difficult.

A complete mathematical model for merging cannot be claimed, but it is hoped instead to point out the problems in formulation of such a model, and solve a few of them. From the purely mathematical point of view, the merging problem has some interest beyond the simple question of waiting for a suitable gap in traffic. It might be supposed, for example, that a car traveling along an acceleration lane while waiting for the opportunity to merge is mathematically equivalent to a car waiting at a stop sign, or that the difference resides only in the moving coordinate system. However, the driver on the acceleration lane is able to control the traffic stream with which he wishes to merge by changing his own speed, thereby increasing or decreasing his headway and spacing relative to the main stream. The stop sign problem (which has been very fully analyzed by mathematicians) does not contain this important ingredient, and therefore questions of driving policy do not arise. There is only one possible policy at a stop sign: wait for a suitable gap. Therefore, a mathematical model for a stop sign is purely descriptive, and its principal result consists of a probability distribution for delay.

It is hoped to show that there are a much more varied and interesting collection of problems available when the driver is allowed to alter (within limits) his attitude with respect to the main stream.

NOTATION AND TERMINOLOGY

Three fundamental maneuvers performed by vehicles in traffic may be identified as follows:

1. Weaving. The process of changing lanes within a flow where more than one flow lane exists.

2. Branching. The process of leaving a flow. It does not include any preliminary weaving necessary to get into position for the maneuver.

3. Merging. The process of entering and establishing constituency within a flow. In this part, a vehicle is said to merge during the time when it moves from one acceleration lane to the lane of principal flow, excluding its travel time in the acceleration lane. The particular vehicle under study is called the merging vehicle.

In actual practice, the merging lane can be regarded as an extreme case of the uncontrolled intersection, as shown in Figures 1 and 2. In an idealized model, this is simplified as in Figure 3, where the car shown is said to be in entry position. The length of the merging lane is called L, and of the merging vehicle D. In some circumstances, it will be assumed that L is infinite, or that D is zero.

At any moment, the merging car will define its leading car and following car, meaning simply those cars in the flow lane nearest to the merging car, and respectively ahead of and behind it. The possibility of merging depends largely on these two cars, but may also be influenced by other cars in the main stream. Therefore, the n^{th} car ahead of the merging car is defined as his n^{th} leader, and similarly his n^{th} follower. The merging car cannot have any effect on his leaders, but can compel deceleration among his followers if he wishes to do so. Also, these definitions refer to different cars whenever the merging car passes or is passed.

There are several categories of merging problems:

1. What information is the merging driver assumed to possess? Is he instantly aware of the dynamic characteristics of all cars in the flow lane, or only of his leader and follower, or perhaps only of certain cars' positions, or positions and velocities, or positions, velocities and accelerations? By varying slightly the degrees of information available to the merging driver, new variations can be created on the merging problem.

2. What is assumed the merging driver is attempting to do? Is he trying to merge as quickly as possible, or as far downstream as possible, or as safely as possible? How much deceleration among the following cars is he willing to tolerate?

3. What constraints exist on the merging driver's behavior? Clearly it must be assumed that he cannot accelerate or decelerate his own car beyond the known range of vehicle performance; also, that he will not collide with other cars. This last point is slightly ambiguous, however. If he is not permitted to collide, can it be said that he can merge in such a way as to produce a collision between other cars? It is a well-known consequence of several theories of traffic flow that fairly modest interference with high density traffic can produce shock waves which in certain ranges of parameter values lead quickly to a rear-end collision. Is one to assume the merging driver's familiarity with such theories?

4. What stochastic process shall be assumed governs the flow of traffic in the main stream? An answer to this question might vary from the specification of a separate function x(t) for each car in that stream to some relatively simple idea such as random arrangement and equal velocities.

5. Description of the best policy for a

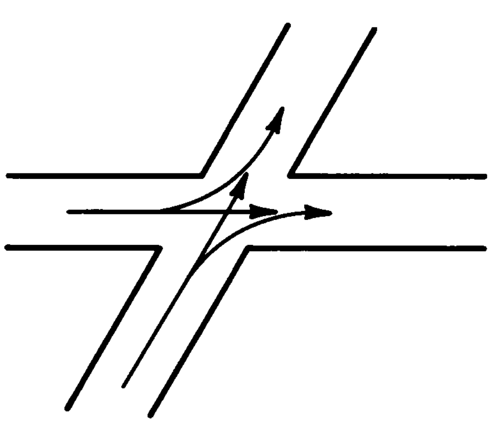

Figure 1. Single lanes intersecting.

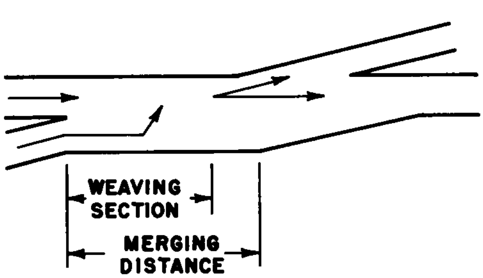

Figure 2. High flow merge.

driver to follow in order to satisfy some particular merging criterion.

6. Description of the probable effect (distribution of delay, for example) on a driver who pursues such a policy.

7. Description of the operation of the system if each driver pursues such a policy.

It is easy to see that by varying 1 to 4, different answers should obtain for 5 to 7. This paper deals with certain special cases.

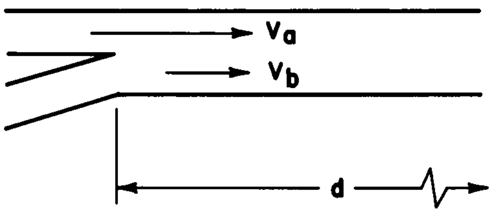

Figure 3. Infinite merging lane.

SAFE MERGING

Before entering the flow of traffic on a freeway, a driver must select the right moment for merging. This selection is based on his judging whether a gap which he intends to enter is large enough for a safe merge.

It is assumed that the driver's primary concern is the distance between his vehicle and the one in front. This distance should be large enough so that, in the event that the vehicle in front makes an emergency stop, there is enough room for the second car to make a safe stop. The distance between two cars could be small if the driver of the following vehicle had all information (i.e., position, velocity, and acceleration) about the vehicle in front and had the means of controlling the acceleration of his vehicle instantaneously. However, this is not the case in practice.

Knowledge about the vehicle in front is limited. The gap and the rate at which it opens or closes can only be roughly estimated. Then, the responses are delayed. The most helpful information in case of emergency stop is the instantaneous appearance of the tail lights on the car in front. This light indicates that the brakes are applied, yet, it is not known how hard. In defining the "safe distance," all of these facts should be taken into consideration. Before defining this "safe distance," the basic mechanics of a vehicle on a straight path should be reviewed.

According to Newton's first law, every body continues in its state of rest or in uniform motion in a straight line until it is compelled by force to change that state. In the case of an automobile traveling on a straight road at a constant speed, the sum of all forces acting on it is equal to zero. Two types of forces are distinguished here: (a) driving forces and (b) motion-resisting forces. The driving force usually is derived from the torque generated by the power plant; sometimes it may be a grade of highway (actually the gravity force) or a wind. The motion resisting forces are caused by friction, rolling friction, wind, highway grade, etc. For a car to travel at a speed of, say, 50 mph, a certain amount of power has to be delivered to overcome the resisting forces. A decrease in supply of power will make a vehicle slow down, until it reaches a new velocity for which the driving and motion-resisting forces are in equilibrium (steady state). In some cases, for example in highway driving, such a control (supply of power) of speed is sufficient for extended periods of time. However, for changes in speed as encountered in city driving, brakes are used to slow the vehicle at a much greater rate than the motion resisting forces would do. In either case (i.e., whether using or not using gas or brake pedal), the driver actually does not control the speed of the car directly; he controls some forces (driving and braking) in such a way that the resultant of all forces makes the vehicle accelerate or decelerate. Both these controlling forces are limited by car and road characteristics. Knowing these characteristics, all forces acting on the vehicle can be evaluated and thus its motion defined. However, in this limited scope of defining the safe distance, it will be sufficient to consider only the maximum values of the controlling forces; in other words, vehicle performance limits. Further, it will be much more convenient to express these limits in terms of acceleration or deceleration rather than in terms of forces. The values of acceleration can be easily measured and are convenient to use in equations of motion. In further discussions, it is assumed that all cars considered have the same acceleration and deceleration capabilities.

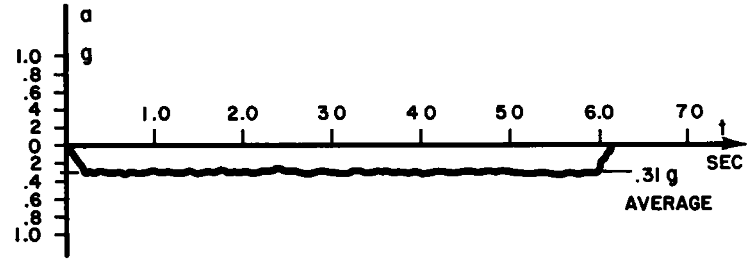

Figure 4. Maximum acceleration-time curve, up to 40 mph, dry surface.

Figures 4 and 5 show typical recordings of maximum acceleration and maximum deceleration taken by the Institute. To simplify the analysis, the average values are used. Thus, in Figure 4, the average value of the maximum acceleration is 0.31 g = 10 ft per sq sec, and in Figure 5, the average maximum deceleration is 0.63 g = 20 ft per sq sec.

An absolute safe distance is a gap between two cars in a lane which will allow the following car to stop safely, even if deceleration of the car in front is maximum. Also, it is assumed that the driver of the merging vehicle will use his brakes to full capacity in order to avoid collision.

For example, two cars on a straight and level path can be represented in the mathematical model as the x-axis (see Fig. 6). The position of car 1 is denoted as x_1 and that of car 2 as x_2. The quantities $\dot{x}_1$, $\ddot{x}_1$ and $\dot{x}_2$, $\ddot{x}_2$ are the respective velocities and accelerations (or decelerations). It is assumed that at time $t = 0$, the driver of car 1 applies brakes and at the same instant, the tail lights light up. Further, it is assumed that at $t = 0$, $x_2 = 0$ and therefore, $x_1(0) = y(0) = y_0$ (see Fig. 6), also $\dot{x}_1(0) = v_{1_0}$ and $\dot{x}_{2_0} = v_{2_0}$

The driver of vehicle 2 will respond to the signal (tail lights) and will apply his brakes. However, there is always some time required for a driver to move his foot from the gas to brake pedal. This amount of time, called a "time delay," or "reaction time," varies greatly for different people. Figure 7 shows a distribution of reaction times for a group of drivers. The average reaction time according to this figure is 0.73 sec. This time can be defined by $T = 0.73$ sec. Therefore, at time $t = T$, car 2 will start to decelerate (neglecting a small variation in speed due to the removal of foot from the gas pedal, an action that precedes the application of brakes by a fraction of a second).

The positions of the cars, for time $t > T$, are defined by:

$$x_1 = y_0 + v_{1_0} t - \frac{\ddot{x}_1 t^2}{2} \tag{1}$$

and

$$x_2 = v_{2_0} t - \frac{\ddot{x}_2 (t - T)^2}{2} \tag{2}$$

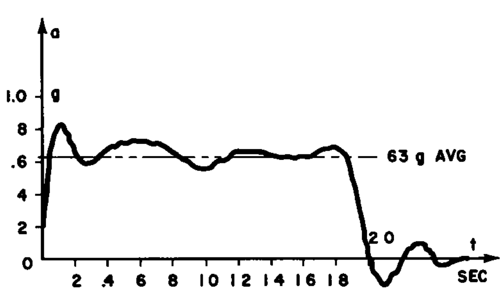

Figure 5. Deceleration-time curve for emergency stop from 30 mph on dry surface.

Because only the emergency stop is considered $\ddot{x}_1 = \ddot{x}_2 = a = 20$ ft per sq sec to find $x_{1max.}$ and $x_{2max.}$, or in other words, the positions at which vehicles 1 and 2 come to a full stop, Eqs. 1 and 2 are differentiated and equated to zero. Thus, $dx_1/dt = v_{1_0} - at = 0$ and $dx_2/dt = v_{2_0} - a(t-T) = 0$, which gives $t = v_{1_0}/a = t_1$ and $t = v_{2_0}/a + T = t_2$ in which t_1 and t_2 are stopping times of vehicles 1 and 2. Substituting t_1 and t_2 for t Eqs. 1 and 2,

$$x_{1max.} = y_0 + \frac{v_{1_0}^2}{2a} \tag{3}$$

and

$$x_{2max.} = v_{2o} T + \frac{v_{2o}^2}{2a} \qquad (4)$$

Subtracting Eq. 4 from 3,

$$x_{1max.} - x_{2max.} =$$

$$y_o - v_{2o}T - \frac{v_{2o}^2 - v_{1o}^2}{2a} \qquad (5)$$

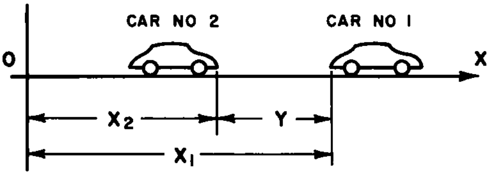

Figure 6. Model of two cars in lane.

For a safe stop (no collision),

$$x_{1max.} - x_{2max.} \geqq 0 \qquad (6a)$$

Therefore,

$$y_o - v_{2o}T - \frac{v_{2o}^2 - v_{1o}^2}{2a} \geqq 0 \qquad (6b)$$

For y_o to be a safe distance,

$$y_o \geqq v_{2o}T + \frac{v_{2o}^2 - v_{1o}^2}{2a} \qquad (6c)$$

Case 1

If

$$v_{2o} = v_{1o} = v_o, \text{ then,}$$

$$y_o \geqq v_oT \qquad (7a)$$

The "safe distance" here, y_o, is a function of the initial velocity v_o and the response time T. It is independent of the decelerations as long as both $\ddot{x}_1$ and $\ddot{x}_2$ are equal. As an example, if v_o = 50 mph = 73 ft per sec and T = 0.73 sec, then $y_o \geqq$ 53.3 ft.

Case 2

In the case when the following car is "catching up" with the car in front, or $v_{2o} > v_{1o}$, Eq. 6c obtains. In comparison with Case 1, the "safe distance," y_o, is increased by $(v_{2o}^2 - v_{1o}^2)/2a$. As an example, if v_{2o} = 50 mph = 73 ft per sec, v_{1o} = 40 mph = 58.4 ft per sec, a = 20 ft per sec, and T = 0.73 sec, then substituting these values in Eq. 6, gives $y_o \geqq$ 101.3 ft. As seen, y_o, is nearly doubled.

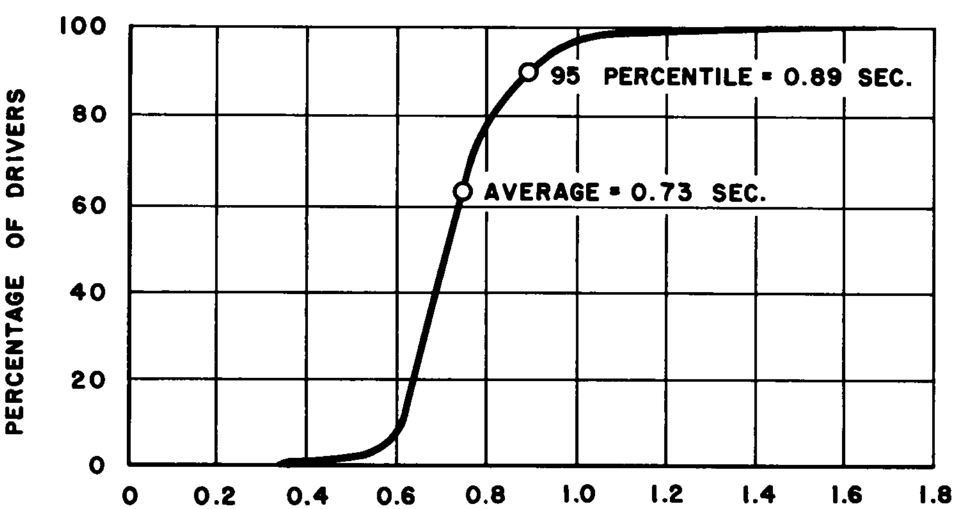

Figure 7. Reaction time in seconds (first stop).

For more general cases, Eqs. 1 and 2 would be used and in similar manner the safe distance, y_o, derived for various $\ddot{X}_1(t)$ and $\ddot{X}_2(t)$.

So far, the safe distance discussed here refers to the gap between two vehicles when one is following the other. In case of merging into oncoming traffic on a freeway, the gap must be large enough to include safe distances between the merging car and the cars in front and behind, and the length of the merging car. Assuming again marginal conditions—i.e., use of brakes to their full capacity on all cars (emergency stop)—the safe gap for merging is

$$s_o = y_{1o} + y_{2o} + L \tag{7b}$$

in which L is the length of the merging car, and y_{1o} and y_{2o} are safe distances (front and rear) computed in the same way as y_o.

In the selection of gap for merging and "placing" the vehicle within this gap, the following three conditions have to be satisfied: $s \geqq s_o$, $y \geqq y_{1o}$, and $y \geqq y_{2o}$.

If $s = s_o + b$, then b is a distance within which the merging vehicle should be placed (see Fig. 8).

From previous discussion (Eqs. 6 and 7) it follows that, for the same value of velocities of vehicles 1 and 3, $y_{1o} \geqq y_{2o}$ for $v_{2o} \geqq v_{1o}$, and $y_{1o} \leqq y_{2o}$ for $v_{2o} \leqq v_{1o}$; when $v_{2o} = v_{1o}$, then $y_{2o} = y_{1o}$. These facts should be remembered by the driver so he can place his car at the right distances, depending on whether his velocity is greater or smaller than that of traffic.

MERGING AS A SIMPLE DELAY PROBLEM

This section makes the simplifying assumptions that the merging vehicle maintains the constant speed v with which it arrives at the entry position, and that the vehicles in the flow lane travel with constant speed V and random placement. This means that the spacing or headway between consecutive vehicles in the flow lane will be governed by the negative exponential distribution. There is then a flow which is Poisson for a moving or stationary observer in either the number of vehicles passing in a given time, or the number of vehicles contained in a given length of road.

When a gap appears that is large enough to allow the entering vehicle to merge safely, taking into account the difference in velocity between the merging vehicle and its leading and following vehicles, then the merge is executed. The distance traveled while waiting for this gap is simply

$$d = vt \tag{8}$$

in which t is the time elapsed after passing the entry point P_1 until a gap appears. The distance d is measured from P_1. A gap sufficient for a safe merge will be at least T time units in length.

It is the intent to construct a theory of merging based on known results in delay theory. These latter treat the wait that a vehicle must endure to enter or cross a stream of traffic when the entering vehicle is at a stop. Under these conditions, the probability distribution of waiting time W(t) has been discussed and is well known (6, 7, 8). However, when the merging vehicle is moving, two difficulties arise. In the first place, a safe gap must be defined more carefully because at some relative velocity for the entering vehicle with respect to the major stream velocity, a time criterion for merge must give way to a space criterion. At a very low relative velocity between major stream and entering vehicle, the time between the transits of two successive vehicles past the entering vehicle may be very long without the existence of sufficient physical space between them for a merge.

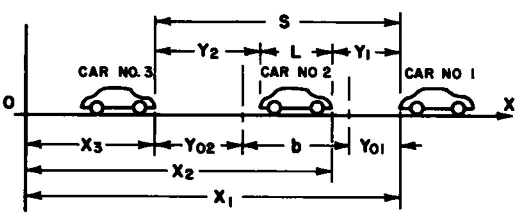

Figure 8. Model of three cars in lane.

The second difficulty arises in the changed rate of flow of gaps past a moving vehicle from the rate of gap flow past a stationary vehicle. It is necessary to be able to characterize the velocity of the vehicles in the flow lane which cannot be done from a mere statement of the flow rate.

As the merging vehicle arrives at the point P_1 imagine all traffic to be stopped instantly as in a photograph. Two points, P_2 and P_3 in the flow lane, in the upstream and downstream direction, respectively, are defined: P_2 is the first point upstream from P_1 for which the distance to the next upstream vehicle is greater than, or equal to, a value S (see Fig. 9). P_3 is defined similarly. If s is the distance between P_1 and P_2, then s has a distribution which is called g(s). The distance P_1 to P_3 also has the same distribution g. If the arrival of the merging vehicle at P_1 occurs at an arbitrary time, then at that instant the location of the other vehicles with respect to P_1 is also arbitrary.

The distance S is more explicitly defined in terms of the safe gap T. If a value is assumed for the quantity T, then to a stationary observer at P_1, the safe gap T can be transformed to a minimum distance VT, and to an observer in the merging vehicle, the distance is (V - v) T. The foregoing expression is only valid in the case where v is less than V. If a safe gap T is required at a relative velocity V - v, then spacing $|V - v| T$ is required.

When relative velocities are small and approach zero, then the physical requirement of a certain minimum space must be accounted for. If S_0 is the length of a vehicle plus minimum maneuvering clearance, the expression for S may now be written,

$$S = \max \left[|V - v| T, \; S_0 \right] \tag{9}$$

In Figure 10, S is plotted against relative velocity.

It is important to relate the well-known distribution of wait for a gap, w(t), with the distribution g(s). If a minimum gap time is assumed,

$$W(t;T) = \text{Prob} \left[\text{Wait for (gap} \geq T) \text{ is } \geq t \right] \tag{10}$$

and assuming a minimum gap distance,

$$G(s;S) = \text{Prob} \left[\text{Distance to (gap} \geq S) \text{ is } \geq s \right] \tag{11}$$

When each unit of the traffic stream has a velocity V, the minimum time gap T = S/V and, t = s/V. Then,

$$\begin{aligned} G(s;S) &= \text{Prob} \; [\text{Distance to (gap} \geq S) \text{ is } \geq s] \\ &= \text{Prob} \; [v \cdot \text{time to (gap} \geq S) \text{ is } \geq s] \\ &= \text{Prob} \; [\text{Time to (gap} \geq S/V) \text{ is } \geq s/V] \\ &= W(s/V;S/V) \end{aligned} \tag{12}$$

The same result may be obtained by making a change of variable in the density and integrating:

$$g(s;S) = w(t;T) \; |dt/ds| = (1/V) \; w(s/V;S/V) \tag{13a}$$

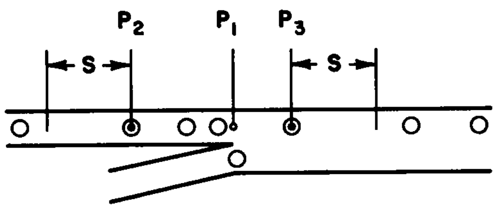

Figure 9. Safe merging space.

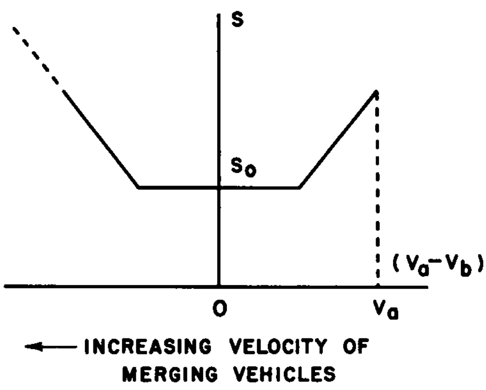

Figure 10. Safe merging space vs relative velocity.

$$G(s;S) = \int_s^{\infty} g(u;S)\, du = (1/V) \int_s^{\infty} w(u/V;S/V)\, du = W(s/V;S/V) \tag{13b}$$

If the probability that the merging vehicle travels a distance greater than or equal to d before being able to merge is F(d), the time which the merging vehicle waits for a safe gap is just

$$t = s\, |V - v|^{-1} \tag{14}$$

and the distance traveled by the merging vehicle while waiting this time, assuming V - v is not small, is

$$d = vt = \left(\frac{v}{|V - v|}\right) s \tag{15}$$

The merging vehicle may travel faster or slower than V, but if V = v, then the distance traveled before merge is zero with probability e^{-aS_o}, where a is the average flow rate, and infinite with probability $1 - e^{-aS_o}$.

Substituting in F(d),

$$\begin{aligned} F(d) &= \text{Prob}\ [\text{Distance to merge} > d] \\ &= \text{Prob}\ [(v/\ |V - v|\ )s > d] \\ &= \text{Prob}\ [s > (\ |V - v|\ /v)d] \\ &= G(\ |V - v|\ d/v) \\ &= W\left(\frac{|V - v|\ d}{V_v}\right) \end{aligned} \tag{16}$$

For the particular case of exponential spacings, the distribution of wait is given by the following expression which has been tabulated by Raff (6). The distribution of F(d) then proceeds from the substitution indicated in Eq. 16.

$$\begin{aligned} W(t;T) &= \text{Prob}\ [\text{Wait} > t] \\ &= \sum_{i=0}^{j-1} (-1)^i e^{-(i+1)aT} \left\{ \frac{[a(t - iT)]^{i}}{i!} = \frac{[a(t - iT)]^{i+1}}{(i+1)!} \right\} \end{aligned} \tag{17}$$

$$\text{for } (j - 1)T \leqq t \leqq jT$$

The gap criterion enters as a parameter in this distribution. F(d) has been plotted in Figure 11. The probability of zero wait has also been plotted (see Fig. 12).

In Figure 13 the most interesting results of this section are plotted. The length of the merging lane is considered fixed at 500 ft and merging vehicles travel at constant velocity on the merging lane until either a merge is completed or the lane ends. The stopping behavior considered in the next section is simplified here to instantaneous braking in zero distance. The figure shows the probability of success in merging with the elementary policy of constant velocity and two features are of interest; the anomaly due to the minimum distance requirement in the vicinity of the flow lane velocity V, and the minimum probability of success for moderate merging velocity.

For this simple merging policy, Figure 11 shows the point of view of design length of merging lane so that a given fraction of vehicles will merge before stopping, and Figure 13 shows the alternate point of view, which is the best constant velocity to choose for an existing merging lane and given flow lane velocity.

VARIABLE SPEEDS, COORDINATE SYSTEMS

If the length of the merging lane is L, and the position of the leading point of the merging car is denoted by x, $0 \leq x \leq L$, where the origin is taken at the beginning of the merging lane, the time origin at the moment when the merging car appears in the lane may also be conveniently taken so that t = 0 when x = 0. If the velocity of the merging car is v (no longer constant), if v_0 be the value on entering the lane, and v_m the largest value obtainable in the distance L, then with exponential acceleration, the

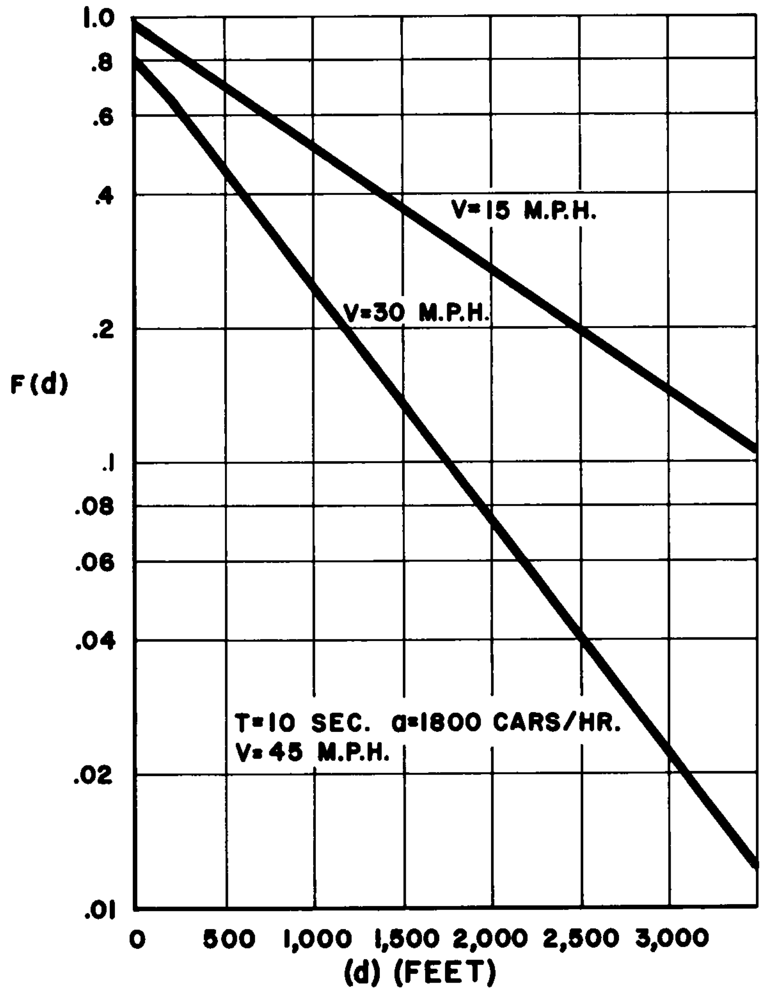

Figure 11. Probability that merge requires at least d feet.

best velocity achievable in time t in the merging lane would be

$$v = v_m - (v_m - v_o)\, e^{-\beta t} \tag{18}$$

in which β is either a constant or the mean value of a random variable. Some information on values of β and v_m could be obtained from the drag races, which are now widely held. In these contests, both v_m and elapsed time are announced in every case, apparently in recognition of the independence of these quantities. In fact, if Eq. 18 is integrated,

$$x = v_m t + (1/\beta)(v_m - v_o)(e^{-\beta t} - 1) \tag{19}$$

Setting $x = L$ in this equation yields a relationship involving the elapsed time, denoted by t_L. In the drag races, $v_o = 0$, and $L = \frac{1}{4}$ mi; therefore, β could be conveniently computed for various given values of v_m and t_L.

The minimum velocity permitted the merging car is v_o. The reason for this apparently arbitrary restriction is quite simple; if that car were permitted to have very small velocities, this would be equivalent to allowing an infinitely long acceleration

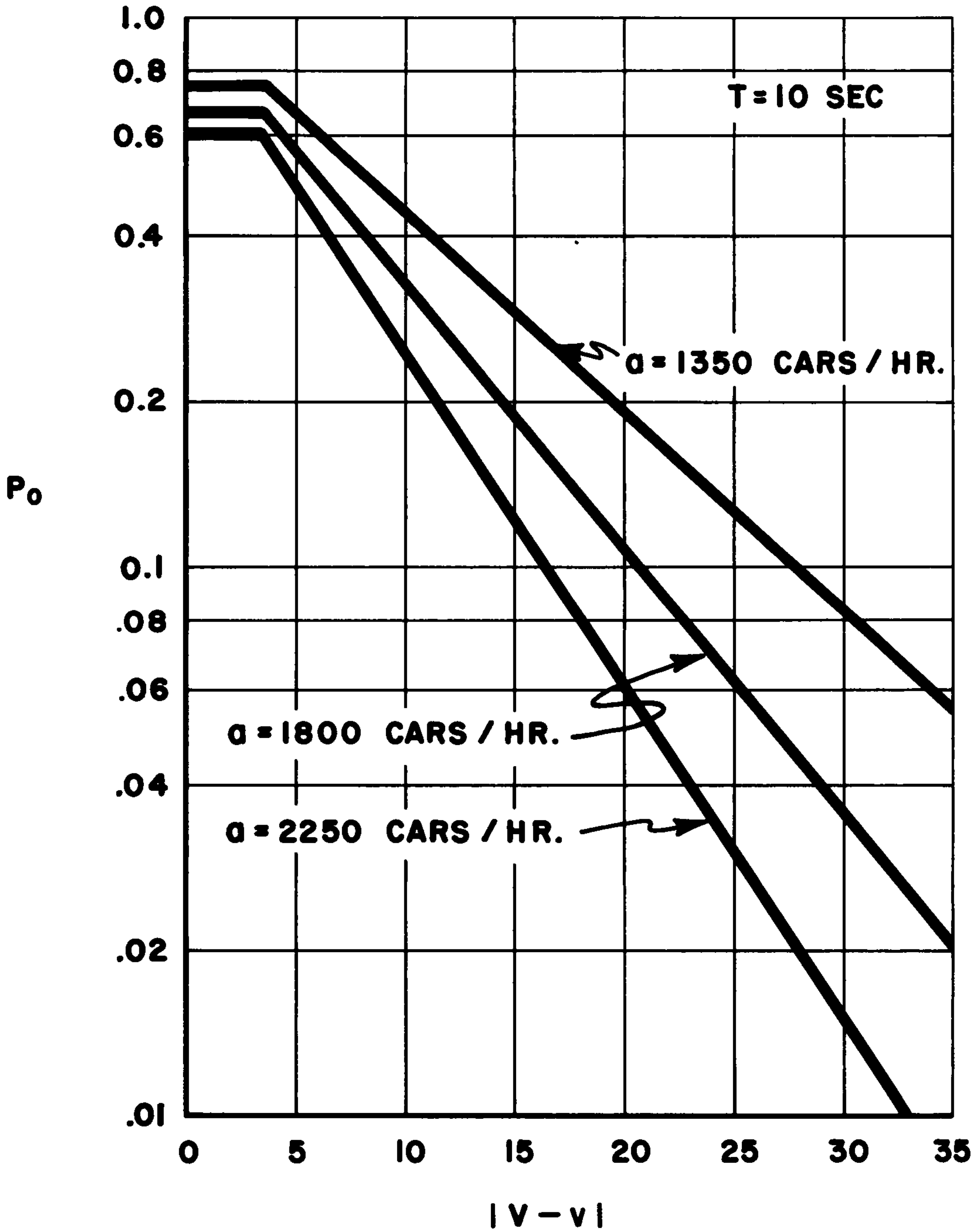

Figure 12. Probability of zero wait (P_0) vs relative velocity $|V-v|$.

lane. Because it is intrinsic to problem 5 that L is finite (for with an infinitely long merging lane, a best policy might be never to merge), the velocities allowed must be bounded away from zero, and v_0 is a convenient and not wholly unrealistic bound.

There is however, an exception to this statement. When the merging car approaches the end of the available lane, he must stop if he has been unsuccessful in merging. Assuming linear braking, the allowed velocity variation for the merging driver is shown in Figure 14, where the equation of the curved portion of the boundary is Eq. 18. The driver will be permitted to cross from the vertically shaded region into the horizontally shaded region only if he is able to merge before coming to the end of the merging lane. Otherwise, he must apply maximum braking at the line of maximum braking, and merge from a standstill at the end of the lane.

If it is assumed that the traffic in the adjoining lane is all going at the same speed, then an auxiliary coordinate system can be defined in the adjoining lane relative to the merging car, and this system can be used to measure the degree of success in merging. If y denotes the position of the merging car relative to the rigidly moving adjoining lane; if the value of y is taken as zero at the point of entry into the acceleration lane when the merging car first arrives there; and if the position of the merging car relative to the adjoining stream is positive downstream from $y = 0$ and negative upstream from $y = 0$; then one can refer to a positive or negative merge, depending on whether at the instant of merging, the merging car has improved his position relative to the adjoining stream or not.

If the constant speed of the adjoining stream is V, then the merging car moves at speed $v - V$ relative to that stream, and in time t changes his y coordinate by an amount $t(v - V)$. Therefore, if he obtains maximum acceleration in the merging lane, and is able to merge at the last moment, he will have gained on the traffic stream an amount,

$$t_L (v - V),$$

which represents the best value y can have. If the criterion of success that he shall merge as far downstream as possible, is adopted, this can be expressed numerically

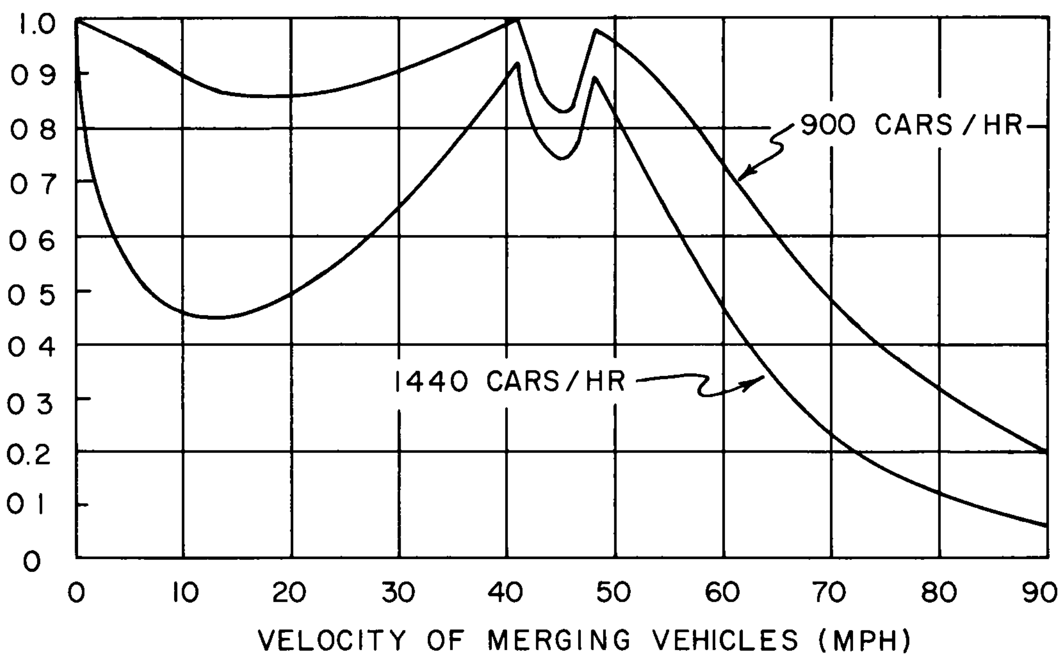

Figure 13. Probability of merging in 500 ft or less for flow lane velocity of 45 mph.

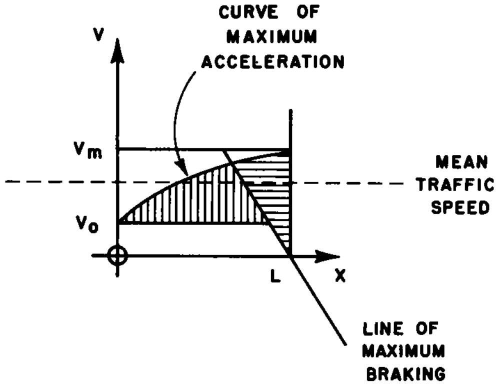

Figure 14. Allowed velocity variance for merging driver.

according to the distance of the y value obtained from its maximum given by $t_L(v-V)$.

PROBABILITY OF MERGING IN NTH GAP

If at the moment when the merging car arrives at the entry position, the length of the gap from L back to the first vehicle flow lane is d_1, from that one to the next is d_2, etc., what conditions need to be fulfilled for the merging vehicle to merge into d_r, and what is the probability of these conditions being fulfilled? First, considering d_1, which is supposed to extend back from $x = L$ to $x = L - d_1$, if the merging vehicle is barely going to fit into this gap, then its nose must arrive at $x = L$ exactly when the gap has shrunk to a length D (a car length). This has taken an elapsed time t_L, during which the traffic flow has traveled Vt_L, and this is the amount by which d_1 has shrunk. Therefore,

$$L - d_1 + t_L V = L - D \qquad (20)$$

or

$$d_1 = D + t_L V \qquad (21)$$

for the barely possible merge into the first gap. Consequently, if Eq. 20 is improved by enlarging d_1, merge in that gap will be possible. Therefore, if it is said that

$$C = D + t_L V \qquad (22)$$

then it follows that merging into d_1 is possible when $d \geqq C$ and impossible otherwise. Because the distribution of gaps is negative exponential, the probability of a merge into d_1 is

$$P_1 = \int_C^\infty \lambda e^{-\lambda x} dx = e^{-\lambda C} \qquad (23)$$

in which λ is the traffic density in the flow lane.

The probability that the first gap d_1 will be unsatisfactory but the second one d_2 will be satisfactory is the probability that all the following inequalities will be satisfied: $d_1 < C$, $d_1 + d_2 \geqq C$, and $d_2 \geqq D$, which, in the d_1-d_2 plane represents the area shaded in Figure 15. Setting up the integral for this area,

$$P_2 = \int_C^\infty \int_0^C \lambda^2 e^{-\lambda(x+y)} dx\, dy + \int_D^C \int_{C-y}^C \lambda^2 e^{-\lambda(x+y)} dx\, dy$$

$$= e^{-\lambda C} \left[1 - e^{-\lambda D} + \lambda(C - d)\right] \qquad (24)$$

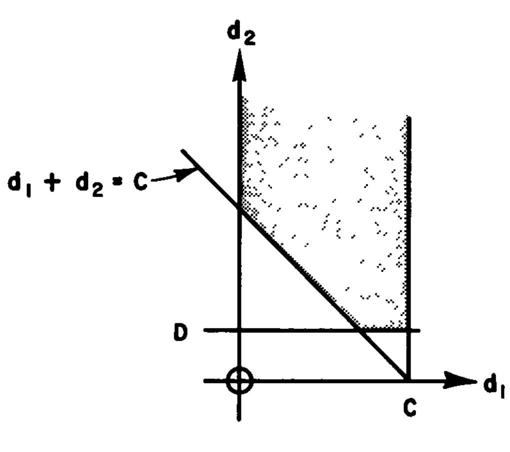

Figure 15.

It is easy to calculate successive forms of this equation, using in each case the previous form, together with the fact that the distribution of sums of exponential variables obey the gamma-type distribution. For example, the third stage uses the inequalities $d_1 + d_2 < C$, $d_1 + d_2 + d_3 \geq C$, and $d_3 \geq D$, which can be reduced to $x < C$, $x + y \geq C$, and $y \geq D$, in which $x = d_1 + d_2$ and $y = d_3$. In this way, the probability of fitting into the n^{th} slot is given by

$$p_n = \int_C^\infty \int_0^C \lambda e^{-\lambda y} \frac{\lambda^{n-1} x^{n-2}}{(n-2)!} e^{-\lambda x} \, dx \, dy + \int_D^C \int_{C-y}^C \lambda e^{-\lambda y} \frac{\lambda^{n-1} x^{n-2}}{(n-2)!} \, dx \, dy \qquad (25)$$

So far the constraints mentioned in the other parts of this paper have not been applied to Eq. 25. It appears that the best way to proceed from this point would be to use programed digital computers to analyze and compare the various approaches proposed in this paper, and hope to present further results in this direction. Meanwhile, the analysis does not seem to have been completely worn out, and the authors wish to encourage other workers to carry it further.

REVIEW OF MERGING LITERATURE

There have not been many studies of merging as a distinct model from delay at traffic lights to stop signs. The game-theory aspect of the problem, in which the merging driver is able to control to some extent the process in which he wishes to merge, has been recognized by Huemer (2) but not carried far except by computer simulation.

As long ago as 1954, Ho (1) proposed a primitive merging model in which n_2 cars in the merging lane are waiting to merge with n_1 cars in the flow lane separated randomly with mean headway $1/\lambda$. If T is the time required for a single car to merge, then Ho gives the density function of the total time to complete the merging by

$$f(t) = C e^{-\lambda t} \sum a_i (t + b_i)^n \qquad (26)$$

in which C, a_i, b_i, and n are functions of the parameters defining the system. When n_1 and n_2 are equal, and some approximations are used, Eq. 26 simplifies drastically to

$$f(t) = \frac{\lambda^{n-1} (t - nT)^{n-2}}{(n-2)!} \exp\left[-(t - nT)\lambda\right] \qquad (27)$$

Little (3) compares the advantages of merging just before and just after the main stream has passed through a signalized intersection, and obtains formulas for the average delay in each case. He also treats a number of other maneuvers in Poisson traffic near intersections.

By far the best mathematical treatment of merging is due to Oliver (5). He first considers equally important lanes merging with each other and allows the possibility of queues in either branch. He then develops the classical queue probability equations for both branches and solves these to obtain the steady state queue levels. The system considered here is called by Oliver "priority merging," in the sense that vehicles in the merging lane are always at the mercy of traffic in the flow lane. In this case, Oliver finds both the stationary queue length probabilities and the distribution of delay.

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