A High-Flow Traffic-Counting Distribution

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Although many observations have been made on intervehicle headways and traffic volumes, it is important to improve the theoretical bases for predicting a number of flow and density characteristics from a limited number of observations. Whereas considerable attention has been given to the theoretical and experimental evaluation of the statistical distributions of intervehicle spacings, there has been much less information available about the discrete counting distributions. The principal effort has been devoted to Poisson-like counting distributions.

The purpose of this paper is to review and present counting distributions which take into account two fundamental characteristics of medium- and high-density traffic flows: (a) platooning or bunching, and (b) minimum spacing, jam-density of the so-called maximum-pack situations. These counting distributions are derived from intervehicle spacing distributions, which have been studied both theoretically and experimentally; in the low-density or low-flow case it is shown that these distributions have the limits of the well-known Poisson case.

• IN AN ATTEMPT to understand conditions that affect traffic flow, engineers have applied probability theory to the analysis of many traffic-counting problems. Although it has sometimes been difficult to predict the exact behavior of any one vehicle or driver, experiments have demonstrated that departures from an average behavior may follow predictable and relatively stable patterns.

In the theory of traffic flow, several authors have studied the probability distributions of spacings between vehicles and the related problem of the distribution of vehicle counts in an interval of time or space. With a reasonably accurate description of intervehicle headways and the distribution of vehicle counts, it should be possible to answer a large number of flow and congestion problems that arise in and around traffic streams. The relations between flow, density, road capacities, delays, and the effect of queueing on the velocity distributions of free-moving vehicles will undoubtedly depend on the basic assumptions about intervehicle spacings.

In studying the arrangement of cars on a road, early writers discussed the combinatorial aspects of random arrangements of points on a line. The well-known Poisson counting law was then derived as a limiting (low-density) case. More recently, the counting problems have been studied as time-dependent processes. By formulating the probability that an intervehicle spacing lies between certain limits, it is theoretically possible to find the probability distribution of spacings between nonadjacent vehicles and from them the discrete distributions of vehicle counts in an interval of time or space.

A cursory review of the literature suggests that the theoretical as well as the experimental work in this area has focused on at least two major problems. The first of these is the effect of bunching or queueing within the traffic stream. It is not uncommon to find several vehicles following a slow or unusually large vehicle; the probability that spacings between cars lie between limits that are of the order of several car lengths has been observed to be higher than that predicted by the exponential distribution. Equivalently, the probability of counting several vehicles close to one another is higher than terms of the Poisson distribution would predict.

The second of these problems has to do with the size of the vehicles; this size forbids them from occupying the same road space. To replace cars on a road by points on a line is not always realistic because high flow or jam-density situations inevitably lead to the conclusion that there is an upper bound to the number of vehicles that can be counted in an interval. If vehicles also have an upper limit to their velocities this statement applies as well to time counts as it does to counts over a length of road. In these cases, the probability of finding vehicles within a fraction of their respective lengths is zero and is, of course, smaller than the Poisson law would predict.

Although there has not been complete agreement, either theoretically or experimentally, as to the structure of the distributions of intervehicle spacings at their origins, there does seem to be general acceptance of the exponential shape of the distribution for large arguments; that is to say, the probability of finding intervehicle spacings greater than a large value decreases exponentially with the size of the spacing.

This paper is divided into seven sections. The second section briefly reviews the historical background of the statistical analysis of intervehicle headways. The third section describes a limiting form of Schuhl's double-exponential distribution. The fourth section reviews some of the mathematical properties of the geometrically compounded Poisson process (Stuttering Poisson); these results are then used in the fifth section to obtain a discrete counting distribution. The sixth section discusses the probability of "maximum pack" and the final section presents some numerical results and a discussion of qualitative features of these distributions.

HISTORICAL BACKGROUND

As early as 1936 Adams (1) pointed out that the distribution of cars on a road could be formulated mathematically. By assuming that the vehicles were randomly distributed points on a line and by making certain limiting assumptions he and at least two other authors (6, 7) showed that the Poisson distribution was applicable to some traffic counting experiments.

By 1955 several distributions of intervehicle spacings had been proposed. One of these is the double-exponential distribution (see Eqs. 1, 2, and 3) derived from geometrical arguments by Schuhl (22, 23, 24). He also obtained certain relations for the discrete counting distributions associated with an arbitrary distribution of intervehicle spacings. An important aspect of Schuhl's distribution is that one limiting case represents the exponential distribution, whereas a second limiting case represents the class of distributions found in certain high-density situations.

To study the flow of traffic through a signalized intersection Newell (18) in 1956 discussed a translated exponential distribution for intervehicle headways. The main feature of this distribution was that it could account for the size and finite velocity of a vehicle as well as some experimental evidence which supported maximum or capacity flow rates. Under certain medium flow conditions, Kinzbruner (13) obtained further experimental evidence to support this distribution. Oliver (19) published some theoretical results for the various counting distributions associated with this translated exponential distribution. Feller (3) in 1948 had already formulated the basic problems associated with the count of nuclear particles. The so-called type I counter resulted in a counting distribution which, except for the distribution of spacings to the first count, was in many respects identical to that one posed in the context of traffic flows.

In 1958 Haight and several collaborators (10) analyzed traffic flow data and came to the conclusion that realistic distributions could be classified as intermediate between (a) random and (b) equally spaced models. In the former case, the exponential intervehicle spacings led to the Poisson counting distributions; the second, to a deterministic count that is just equal to the integral part of the interval of interest divided by the fixed spacing between vehicles. Haight showed that a family of distributions satisfying certain theoretical and experimental requirements were the Erlang or Pearson type III distributions. Counting distributions which correspond to this assumption for intervehicle spacings are the generalized Poisson functions described by Haight (9) or various state probabilities calculated by Morse (17) and Jewell (11). Whittlesey and Haight (30) have also obtained certain approximations and numerical results for these counting distributions.

In 1959 Kell (12) produced experimental evidence to show that the double exponential distribution suggested by Schuhl accurately described intervehicle spacings in certain medium flow situations. An extensive number of experiments was made and four unknown parameters in the Schuhl distribution were expressed in terms of the flow rate or volume of traffic. Extrapolation of these parameters for high volumes indicate that a limiting form of the double exponential distribution may be appropriate for high-flow situations (see Eqs. 4 and 5).

By 1960 Miller (16) had reached the important conclusion that the random variables describing successive intervehicle spacings might not be independently sampled. He proposed a model of traveling queues which took specific account of bunching or queueing effects. Mathematically, this was a generalization of a special bunching configuration suggested by Bartlett (2) and derived independently from overtaking rules by Oliver (20). The important new consideration brought into all of these studies was the dependence of gaps between adjacent vehicles. Not only is it necessary to resolve the distributions of gaps between queued vehicles, but also one must discuss the spacings between queues, the distribution of queue lengths, and the formation of queues as the result of flow around slow-moving vehicles.

In 1960 May and Wagner (14) published an extensive list of data gathered in the vicinity of Detroit and Lansing, Mich. In the case of extremely high flow rates, the probability density distributions of intervehicle headways showed a marked tendency to rise sharply from zero and then decrease exponentially from this peak or modal value. Minimum headways were seldom evident for flow rates exceeding 30 per min but were almost always present for flow rates less than this value.

In the same year Weiss and Maradudin (24) published some new results in the theory of vehicle delays at the stop-sign type of intersection. In deriving numerical results they made use of a probability distribution of intervehicle spacings which was a translated version of the geometric-exponential distribution discussed by Jewell (11) and which, as shown later, is the same limiting case of Schuhl's distribution observable in some high-volume samples of Kell's data.

Although vehicles in a dense traffic stream are obviously restrained by each other's movements and although the independence assumption of spacings between successive vehicles may be unrealistic in some respects, a large body of theoretical and experimental research has supported Schuhl's distribution. Either in its own right or as a limiting version of more general cases, the mixture of two exponentials has been used to describe vehicle behavior in medium density traffic streams. A limiting version of Schuhl's distribution is discussed in the following section. The counting distributions that correspond to it are the major subject of the remainder of this paper.

DISTRIBUTION OF INTERVEHICLE SPACINGS

Schuhl obtained a description of the distribution of spacings between vehicles on purely theoretical grounds. By considering two types of vehicles—slow and fast— and by requiring that the sum of their respective flow rates equal the total vehicular flow rate, he obtained the probability distribution of the spacings between adjacent vehicles as the mixture of two exponential functions, each with its own decay constant. An observer picks a fast vehicle with probability α and a slow vehicle with probability $1 - \alpha$; if the choice results in a fast vehicle the probability that the spacing to the next vehicle (either slow or fast) is greater than t is just $e^{-\lambda_1 t}$ if the choice results in a slow vehicle the probability that the spacing to the next vehicle is greater than t being equal to $e^{-\lambda_2 t}$. The mixture of these probabilities results in

$$A(t) = \alpha e^{-\lambda_1 t} + (1 - \alpha) e^{-\lambda_2 t} \qquad (1)$$

for the probability that the spacing between any two vehicles is greater than or equal to t. Though the words "slow" and "fast" may not be appropriate in the sense that vehicle velocities may themselves be distributed over a wide range of values, it may be helpful to think in terms of retarded and unretarded vehicles. That is to say, the slow vehicles travel at their free or desired speed, whereas the fast are restricted in their ability to maneuver; because of heavy flows in an adjacent lane, the latter may not have opportunities to perform the passing maneuvers that lead to unrestrained flow conditions.

Morse (17) has called the distribution of Eq. 1 the hyper-exponential distribution, and one of its counting distributions the hyper-Poisson.

Although Eq. 1 might apply to a set of points with restricted motion along a line, it is clear that vehicles occupy a finite amount of space in a traffic stream; if there is an upper bound to the free velocity then there is at least this same upper limit to the velocity of the constrained or retarded group of vehicles and the minimum time or headway between successive vehicles is simply the ratio of the minimum spacing between vehicles to the maximum velocity at which they can travel. Even if the domain of definition of velocity values were $(0, \infty)$ and the lower bound on intervehicle headways were zero, many experimental results indicate that near-zero headways are highly improbable; hence, the assumption of a lower bound on intervehicle headways serves as an approximation of a real probability distribution where the density function is small for small headways, increases sharply to a maximum, and then decreases exponentially for large values of the argument.

To account for this feature of minimum headways, Schuhl modified Eq. 1 to include a term for minimum separations between vehicles. In this translated version,

$$\begin{aligned} A^{\Delta}(t) &= 1 && 0 \leq t < \Delta \\ &= \alpha e^{-\lambda_1(t-\Delta)} + (1-\alpha) e^{-\lambda_2(t-\Delta)} && \Delta \leq t \end{aligned} \qquad (2)$$

Δ refers to the minimum headway or spacing between vehicles. By a simple change of the constant terms in Eq. 2, it is possible to write the translated probability distribution as

$$\begin{aligned} A^{\Delta}(t) &= 1 && 0 \leq t < \Delta \\ &= c_1 e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t} && \Delta \leq t \end{aligned} \qquad (3)$$

The constants c_1 and c_2 must add to a number greater than one.

Experiments by Kell (12) and May and Wagner (14) indicate that as traffic volumes increase there will be an increase in the fraction of restrained vehicles relative to the free-moving group. This seems reasonable because passing maneuvers generally become difficult as traffic volumes and densities increase. With very high-flow conditions the constrained vehicles travel closer and closer to the free-flowing leader of a platoon or bunch. Kell's data indicate that the exponential decay constant, λ_1 in Eq. 2, increases sharply with increasing flow rates, whereas that of the free-moving group (λ_2 in Eq. 2 is less than λ_1). The composite curve $A^{\Delta}(t)$ tends to look like an exponential with a large decay constant for small headways and like an exponential with a much smaller decay constant for large headways.

In the limit as $\lambda_1 \longrightarrow \infty$ and $\lambda_2 \longrightarrow 0$ one obtains a probability distribution of intervehicle spacings

$$\begin{aligned} A^{\Delta}(t) &= 1 && 0 \leq t < \Delta \\ &= (1-\alpha) e^{-\lambda(t-\Delta)} && \Delta \leq t \end{aligned} \qquad (4)$$

where λ replaces λ_2 in Eq. 2. The probability density distribution

$$a^{\Delta}(t) = -\frac{dA(t)}{dt} = \alpha\delta(t-\Delta) + \lambda(1-\alpha) e^{-\lambda(t-\Delta)} \qquad (5)$$

also points out the fact that the probability of finding vehicles queued at the minimum separation is α.

The probability distribution of Eqs. 4 and 5 forms the basis for the counting distributions obtained in this paper. When $\Delta = 0$, one obtains the special case—called the geometric exponential distribution (11)—of the hyper-exponential distribution which has been discussed by Jewell (11). Although many results have been published on the counting distributions associated with this special case, it may help to review some of their properties in the following section. It is important to point out that many of the analytical expressions obtained for the $\Delta > 0$ case can be obtained in terms of those obtained for the $\Delta = 0$ case; hence, numerical computations made for the $\Delta = 0$ case can be used as building blocks for the $\Delta > 0$ cases.

The mean spacing, ν_Δ, of the translated distribution of Eq. 4 is Δ plus the mean spacing of the untranslated case, and the variance, σ_Δ^2, of the former is identical to the variance of the latter. If constants in Eq. 4 are renormalized so that $\mu = \lambda/1 - \alpha$,

$$\nu_\Delta = \int_0^\infty A^\Delta(t)\,dt = \Delta + \mu^{-1} \tag{6a}$$

$$\sigma_\Delta^2 = \int_0^\infty 2(t - \bar{t})\,A(t)\,dt = \frac{1 + \alpha}{\mu^2(1 - \alpha)} \tag{6b}$$

It has been shown by several authors (11, 24, 25) that if one randomly selects a point in time, the probability density distribution of spacings to the first car, $u^\Delta(t)$, is the product of the stationary flow rate and the probability that the spacing between two cars is greater than t. Because the stationary flow rate, μ_Δ, is the reciprocal of the average intervehicle headway in Eq. 6a the "starting-at-random" density distribution is obtained:

$$u^\Delta(t) = \mu_\Delta \qquad 0 \leq t < \Delta \tag{7a}$$

$$= (1 - \alpha)\,\mu_\Delta e^{-\frac{\mu_\Delta(1 - \alpha)(t - \Delta)}{1 - \Delta\mu_\Delta}} \qquad \Delta \leq t \tag{7b}$$

μ is the stationary vehicle flow rate for the special case where minimum headways are zero. The expected wait to the first vehicle from a random origin is

$$\int_0^\infty t u^\Delta(t)\,dt = \frac{\sigma_\Delta^2 + \nu_\Delta^2}{2\nu_\Delta}$$

GEOMETRICALLY COMPOUNDED OR STUTTERING POISSON PROCESS

Several authors (4, 5, 11) have studied the geometrically-compounded or Stuttering Poisson process which corresponds to the special case $\Delta = 0$ in Eq. 4. As already mentioned, the mathematical structure of the discrete counting distributions which corresponds to the more general case $\Delta > 0$ is similar to that of the Stuttering Poisson.

The Stuttering Poisson distribution may arise in the following way: consider vehicles replaced by points on a road. One group of these vehicles is a free-flowing or unrestrained group, the probability density distribution of spacings (headways) between vehicles in this group is exponential with mean value λ^{-1}. This is equivalent to the statement that the count of vehicles in an interval is Poisson-distributed with average flow rate equal to λ. The second group of vehicles are queued behind unrestrained vehicles. The probability that the queue of restrained vehicles is of length n-1 is a geometric distribution;

$$a_n = (1 - \alpha)\,\alpha^{n-1} \qquad n = 1, 2, \ldots \tag{8}$$

in which $(1 - \alpha)$ is the probability of no restrained vehicle following an unrestrained one. The probability $p(n/m)$ of finding n - m restrained vehicles behind m unrestrained vehicles or a total of n vehicles with m unrestrained is the m-fold convolution of Eq. 7, the negative binomial,

$$p(n/m) = \binom{n-1}{m-1}(1 - \alpha)^m\, a^{n-m} \qquad n \geq m \geq 1 \tag{9a}$$

$$= 1 \qquad n = m = 0 \tag{9b}$$

Consequently, the probability $g_n(t)$ of finding n vehicles in an interval t chosen at random is the sum over the possible number of unrestrained vehicles:

$$g_n(t) = \sum_{m=1}^{n} \binom{n-1}{m-1} (1-\alpha)^m \alpha^{n-m} \frac{e^{-\lambda t}(\lambda t)^m}{m!} \qquad n \geq 1 \tag{10a}$$

The probability that no vehicles are observed in t is just $e^{-\lambda t}$, the probability that the spacing to the first unrestrained vehicle is greater than t. For $n \geq 1$ the distribution $g_n(t)$ can be written in terms of the associated Laguerre polynomials of order 1,

$$g_n(t) = \frac{\lambda t(1-\alpha)\alpha^{n-1}}{n} e^{-\lambda t} L_{n-1}^{(1)}\left(\frac{(\alpha-1)\lambda t}{\alpha}\right) \tag{10b}$$

in which the Laguerre polynomial of order a is

$$L_n^{(a)}(x) = \sum_{i=0}^{n} \binom{n+a}{i+a} \frac{(-x)^i}{i!} \tag{11}$$

The distribution of Eqs. 10a and 10b corresponds to the case where the counting of traffic begins at random. If one starts to count just after one vehicle has passed, the discrete counting distribution differs only slightly from $g_n(t)$. This new distribution is labeled $h_n(t)$; it can also be derived by arguments similar to those used for Eq. 8 through 10.

The distinction between these two cases can be well illustrated by means of their generating functions. In the "starting-at-random" case, the generating function

$$G(z,t) = \sum_{n=0}^{\infty} g_n(t)\, z$$

of the Stuttering Poisson is just that of the geometrically compounded Poisson process discussed by Feller (3). By substituting the generating function of Eq. 8 for the variable z in the generating function $e^{-\lambda t(1-z)}$ of the Poisson process,

$$G(z,t) = e^{-\frac{\lambda t(1-z)}{1-\alpha z}} \tag{12}$$

Expansion of Eq. 12 in powers of z leads to formulas for the counting distribution. By substituting the stationary vehicle flow rate $\mu = \lambda/1 - \alpha$,

$$g_0(t) = e^{-\mu t(1-\alpha)} \tag{13a}$$

$$g_1(t) = \mu t(1-\alpha)^2 e^{-\mu t(1-\alpha)} \tag{13b}$$

$$g_2(t) = \left[\alpha(1-\alpha)^2 \mu t + \frac{(1-\alpha)^4(\mu t)^2}{2}\right] e^{-\mu t(1-\alpha)} \tag{13c}$$

These are identical to the expressions already obtained in Eq. 10a and 10b.

When the counting experiment starts with the passing of a vehicle, the generating function

$$H(z;t) = \sum_{n=0}^{\infty} h_n(t) z^n$$

is obtained by a slight modification of the geometrically-compounded Poisson process previously discussed. The total count can now be expressed as the sum of two random variables X and Y. X is the number of vehicles in the first bunch exclusive of the vehicle which begins the counting experiment. Hence, from Eq. 8

$$\Pr\left\{x = n\right\} = a_{n+1} \qquad n \geq 0 \tag{14}$$

Because the first bunch of vehicles is located at the origin, Y represents the remaining count with probability distribution $g_n(t)$. Hence, the distribution of X + Y is the convolution of Eq. 14 with $g_n(t)$ in Eq. 10:

$$h_n(t) = \sum_{m=0}^{n} (1-\alpha)\alpha^m g_{n-m}(t) \qquad n \geqq 0 \tag{15}$$

Its generating function is therefore equal to the product of the generating function of Eq. 14 and G(z;t):

$$H(z;t) = \frac{1-\alpha}{1-\alpha z}\, e^{-\frac{\mu t(1-\alpha)(1-z)}{1-\alpha z}} \tag{16}$$

By summing Eq. 15 or by expanding H(z;t) in powers of z, the coefficient of z^n becomes

$$h_n(t) = \sum_{m=1}^{n} \sum_{i=0}^{m-1} \binom{m-1}{i} \frac{(1-\alpha)^{i+2}}{(i+1)!}\, \alpha^{n-i-1} (\lambda t)^{i+1} e^{-\lambda t} + (1-\alpha)\alpha^n e^{-\lambda t} \tag{17a}$$

By making use of Eq. 11 $h_n(t)$ can also be expressed in terms of Laguerre polynomials as

$$h_n(t) = \alpha^n (1-\alpha)\, e^{-\mu t(1-\alpha)} L_n^{(0)}\left(-\frac{(1-\alpha)^2 \mu t}{\alpha}\right) \tag{17b}$$

These discrete counting distributions can also be obtained by considering the probability distribution of spacings between nonadjacent vehicles. If a(t) is the probability density distribution of intervehicle spacings, $a_n(t)$ is used to denote the probability density distribution of spacings between every nth vehicle. (Throughout this paper the absence of the subscript n is identical to the n=1 case.) The probability that the spacing between every nth vehicle is greater than or equal to n will be denoted by $A_n(t)$. It is fortunate that the discrete counting distribution, $h_n(t)$, can always be expressed in terms of $A_n(t)$ and $A_{n+1}(t)$. Arguments formally developed by Feller (3) show that

$$h_n(t) = A_{n+1}(t) - A_n(t) \tag{18a}$$

Because this is a linear first-order difference equation in n,

$$A_{n+1}(t) = \sum_{i=0}^{n} h_i(t) \tag{18b}$$

provided one starts with the boundary condition $h_0(t) = A_1(t) = A(t)$. This last equation again points up a result that can be obtained by a direct line of argument: the probability that n or fewer vehicles are observed in t equals the probability that the spacing between n+1 vehicles is greater than or equal to t.

Further, the random variable that measures the spacing from the time origin to the nth vehicle is the sum of the spacing to the first vehicle plus the spacing between the first and the (n-1)th. The probability density distribution $a_n(t)$ is therefore the convolution of the probability that the first lies between r and r+dr with the probability that the spacing to the (n-1)th vehicle lies between (t-r) and (t-r+dt). Hence,

$$A_n(t) = \int_t^{\infty} \int_0^{t} a(r)\, a_{n-1}(t-r)\, dr\, dt \tag{19}$$

Equating the Laplace transform of $A_n(t)$,

$$\tilde{A}_n(s) = \int_0^{\infty} e^{-st} A_n(t)\, dt \tag{20a}$$

to the Laplace transform of the right-hand side of Eq. 19 gives

$$\tilde{A}_n(s) = \frac{1 - [\tilde{a}(s)]^n}{s} \tag{20b}$$

in which $\tilde{a}(s)$ is the Laplace transform of the intervehicle density distribution. When $\Delta = 0$ in Eq. 4,

$$\tilde{a}(s) = \int_0^\infty a(t)\, e^{-st}\, dt = \alpha + \frac{\mu(1-\alpha)^2}{s + \mu(1-\alpha)} \tag{21}$$

Expressing the nth power of $\tilde{a}(s)$ by the binomial expansion and making use of the transform pair,

$$f(t) = c^{-n} \sum_{j=n}^{\infty} \frac{e^{-ct}(ct)^j}{j!}$$

$$\tilde{f}(s) = \frac{1}{s} \frac{1}{(s+c)^n}$$

one can invert Eq. (20b):

$$A_n(t) = \sum_{i=1}^{n} \sum_{j=0}^{j-1} \binom{n}{j} (1-\alpha)^j \alpha^{n-j} \frac{[\mu t(1-\alpha)]^{i}}{i!} e^{-\mu t(1-\alpha)} \tag{22a}$$

$A_n(t)$ can be expressed in several alternate forms: (a) in terms of Laguerre polynomials of order (0)

$$A_n(t) = (1-\alpha)\alpha^{n-1} \sum_{j=0}^{n} \binom{n+1}{j} \left(\frac{\alpha-1}{\alpha}\right)^{n-j} \left(\frac{1}{\alpha}\right)^{j} e^{-\mu t(1-\alpha)} L_{n-j}^{(0)} \left[\mu t(1-\alpha)\right] \tag{22b}$$

or (b) in terms of an incomplete integral, of the Laguerre polynomial of order (1)

$$A_n(t) = (1-\alpha)\alpha^{n-1} \int_{\mu t(1-\alpha)}^{\infty} e^{-x} L_{n-1}^{(1)} \left[\frac{(\alpha-1)x}{\alpha}\right] dx \tag{22c}$$

Substitution of Eq. 22a into 18a and use of Eq. 11 and the identity

$$\binom{n+1}{j+1} - \binom{n}{j+1} = \binom{n}{j} \qquad j+1 \leq n$$

again leads to Eq. 17b.

COUNTING DISTRIBUTION

In this section the discrete counting distributions that correspond to the intervehicle distribution of Eq. 4 are derived. If $A_n^{\Delta}(t)$ is labeled as the probability distribution of intervehicle spacings greater than or equal to t when the minimum headway is Δ, and $\Delta_n(t)$ for the $\Delta = 0$ case, Eq. 4 can be rewritten

$$A^{\Delta}(t) = 1 \qquad 0 \leq t < \Delta \tag{23a}$$

$$= A(t-\Delta) \qquad \Delta \leq t \tag{23b}$$

It follows from the defining Eq. 19 for the distribution of spacings between nonadjacent vehicles that

$$A_n^{\Delta}(t) = 1 \qquad 0 \leq t < n\Delta \tag{24a}$$

$$= A_n(t - n\Delta) \qquad n\Delta \leq t \tag{24b}$$

Hence, the discrete counting distribution

$$p_n(t) = A_{n+1}^{\Delta}(t) - A_n^{\Delta}(t) \tag{24c}$$

which is obtained by substituting $p_n(t)$ for $h_n(t)$ and $A_n^{\Delta}(t)$ for $A_n(t)$ in Eq. 18a can be written in terms of $A_n(t)$:

$$p_n(t) = 0 \qquad 0 \leq t < n\,\Delta \tag{25a}$$

$$= 1 - A_n(t - n\Delta) \qquad n\,\Delta \leq t < (n+1)\,\Delta \tag{25b}$$

$$= A_{n+1}(t - (n+1)\,\Delta) - A_n(t - n\Delta) \qquad (n+1)\,\Delta \leq t \tag{25c}$$

By substituting the translated versions of Eq. 22a into 24

$$A_n^{\Delta}(t) = \sum_{j=1}^{n} \sum_{i=0}^{j-1} \binom{n}{j} (1 - \alpha)^j \alpha^{n-j} \frac{[\mu(1 - \alpha)(t - n\Delta)]^i}{i!} e^{-\mu(1 - \alpha)(t - n\Delta)} \tag{26}$$

and

$$p_n(t) = 0 \qquad 0 \leq t < n\,\Delta \tag{27a}$$

$$= 1 - \sum_{j=1}^{n} \sum_{i=0}^{j-1} \binom{n}{j} (1-\alpha)^j \alpha^{n-j} \frac{[\mu(1-\alpha)(t-n\Delta)]^i}{i!} e^{-\mu(1-\alpha)(t-n\Delta)}$$

$$n\,\Delta \leq t < (n+1)\,\Delta \tag{27b}$$

$$= \sum_{j=1}^{n+1} \sum_{i=0}^{j-1} \binom{n+1}{j} (1-\alpha)^j \alpha^{n+1-j} \frac{[\mu(1-\alpha)(t-(n+1)\Delta)]^i}{i!} e^{-\mu(1-\alpha)[t-(n+1)\Delta]} -$$

$$\sum_{j=1}^{n} \sum_{i=0}^{j-1} \binom{n}{j} (1-\alpha)^j \alpha^{n-j} \frac{[\mu(1-\alpha)(t-n\Delta)]^i}{i!} e^{-\mu(1-\alpha)(t-n\Delta)}$$

$$(n+1)\,\Delta \leq t \tag{27c}$$

There are, of course, many equivalent ways of writing these counting distributions. One of the simpler analytic forms, and possibly one that will be computationally useful, expresses the counting distribution in cumulative form:

$$P_n(t) = \sum_{j=0}^{n} P_j(t)$$

$P_n(t)$ is the probability that n or fewer vehicles are counted in the interval t. From Eqs. 18 and 24,

$$P_n(t) = A_{n+1}^{\Delta}(t) = A_{n+1}[t - (n+1)\,\Delta]$$

can be written in the form of Eq. 26. By making use of the incomplete gamma function notation,

$$\gamma(u;x) = \int_0^x t^{n-1} e^{-t} dt = \sum_{j=n}^{\infty} \epsilon^{-x} x^j$$

one can write $P_n(t)$ in the form

$$P_n(t) = \sum_{j=1}^{n+1} \sum_{i=0}^{j-1} \binom{n+1}{j} (1-\alpha)^j \alpha^{n+1-j} \frac{x^i}{i!} \epsilon^{-x}$$

$$= \sum_{k=0}^{n} \binom{n+1}{k} (1-\alpha)^{n+1-k} \alpha^k \frac{\gamma(n+1-k;x)}{(n-k)!} + \alpha^{n+1} \tag{28}$$

in which $x = \mu(1-\alpha)(t-(n+1)\,\Delta)$. This alternate form for the counting distribution may be useful in view of the well-known properties of the incomplete gamma function and programs which are currently available for high-speed computation (30).

PROBABILITY OF MAXIMUM-PACK

The probability of maximum pack equals the probability that in an interval chosen at random the maximum number of vehicles are counted in that interval. The probability that exactly N vehicles are counted in an interval $t = N\Delta$, which begins just after the passing of a vehicle, equals α^N. This is the probability that a queue of N restrained vehicles follows the car that began the counting experiment. This probability falls off rapidly with small α. However, if one randomly picks the origin of an interval $N\Delta$ units long and asks for the probability that exactly N vehicles will be counted, a number which is larger than α^N is obtained because the first vehicle can occupy any position from the moment after counting begins to an instant just before the end of the first interval Δ. From the definition of the starting-at-random density distribution in Eq. 7 the probability of counting n vehicles in a randomly chosen interval of length t is

$$q_n(t) = \int_0^t u^{\Delta}(r)\, p_{n-1}(t - r)\, dr \tag{29}$$

Because $p_n(t)$ and $u^{\Delta}(t)$ can be expressed in terms of $A_n(t)$ for the $\Delta = 0$ case, one can rewrite $q_N(N\Delta)$, the probability that N vehicles will be observed in a randomly located interval exactly $N\Delta$ units long as

$$q_N(N\Delta) = \mu_{\Delta} \int_0^{\Delta} \left[1 - A_{N-1}(\Delta - t)\right] dt$$

The probability of maximum-pack is obtained by substituting Eq. 22a for $1 - A_{N-1}(\Delta - t)$:

$$q_N(N\Delta) = \frac{\mu\Delta}{1 + \mu\Delta} - \frac{1}{1 - \alpha + \beta} \sum_{j=1}^{N-1} \sum_{i=0}^{j-1} \sum_{k=i+1}^{\infty} \binom{N-1}{j} (1 - \alpha)^j \alpha^{N-1-j} \frac{e^{-\beta}\beta^k}{k!} \tag{30}$$

in which $\beta = \mu\Delta(1 - \alpha)$. When $N = 1$ one obtains the probability that a vehicle is observed in an interval equal to the minimum spacing:

$$q_1(\Delta) = \frac{\mu}{1 + \mu\Delta}$$

This is equal to the average flow rate of Eq. 6a divided by the maximum flow rate Δ^{-1} which would be observed if all vehicles were spaced regularly Δ units apart.

SUMMARY

Figure 1 shows the fraction of intervehicle headways greater than or equal to the values indicated on the horizontal scale. In Figure 1a the experimental data are denoted by the circled points and the solid line is a plot of the theoretical curve of Eq. 2 with parameters calculated from observed flow rates. These data were observed and fitted to Schuhl's distribution by Kell (12) in experiments that observed single-lane flow rates ranging from 150 to 1,200 vehicles per hour. In Figure 1b the theoretical curve is decomposed in two separate exponential terms; the large decay constant refers to the restrained vehicles and the smaller decay constant refers to the free-moving group.

Figure 2 shows the probability distribution $p_n(t)$ of Eq. 28 for several values of α when $t = 10\Delta$; i.e., if the minimum headway were $\Delta = 2$ sec, the period of observation would be 20 sec. In Case a, the probability of finding no restrained vehicle following a free-moving vehicle is 0.1, in Case b, 0.3 and in Case c, 1.0. Case c is also the counting distribution which corresponds to the translated exponential distribution discussed earlier by Newell (18) and Oliver (19).

In comparison to the Poisson law, these counting distributions point up at least two distinct effects. The first of these is the finite number of terms in the distribution. The probability of a count which exceeds the integral part of $t\Delta^{-1}$ is identically zero. For high flow rates and more particularly for average counts which are close to the integral part of $t\Delta^{-1}$ the difference between these and the Poisson distribution is pronounced.

A second feature is due to the effects of queueing. As the fraction of constrained

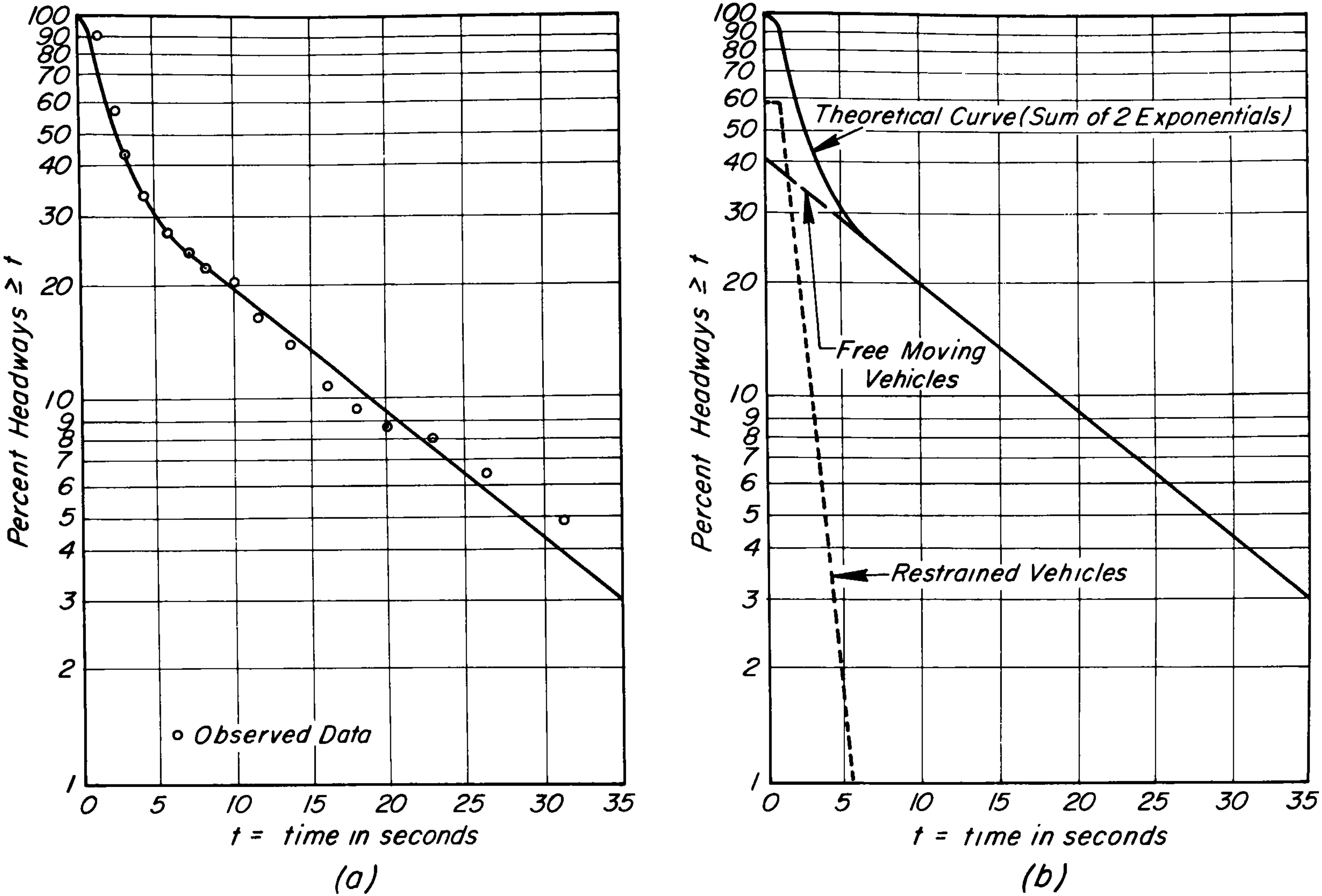

Figure 1. Distribution of intervehicle headways.

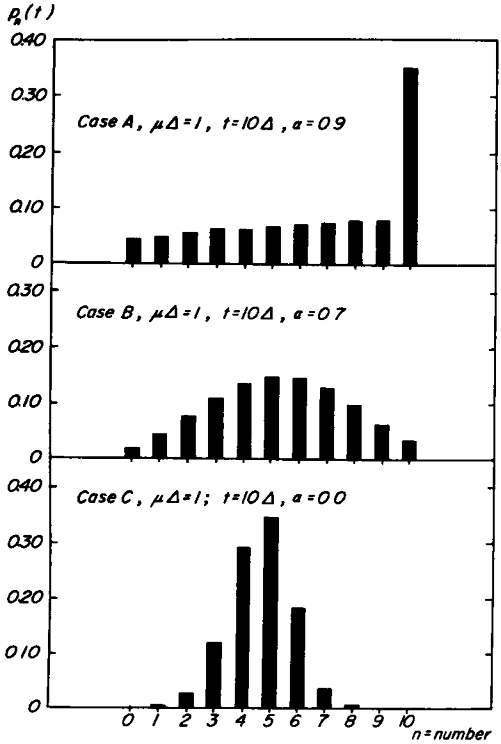

Figure 2. Counting distribution.

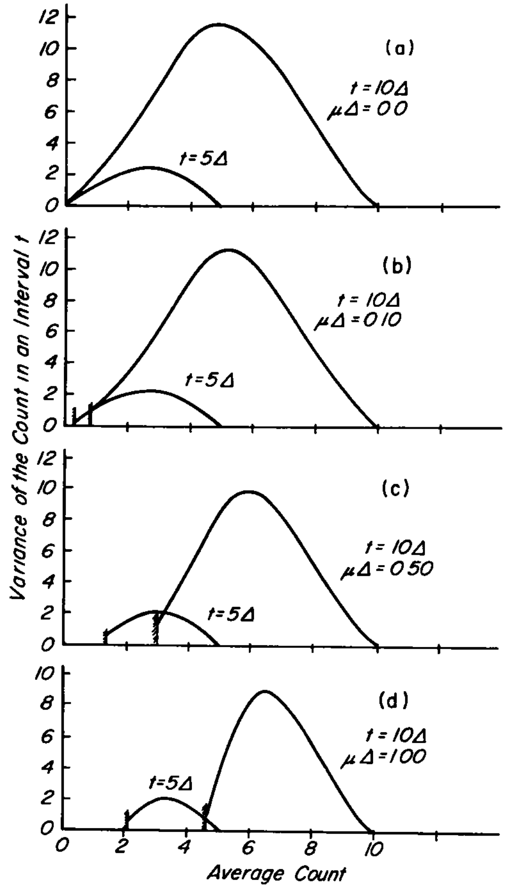

Figure 3. Variance vs average count.

vehicles increases, the average count in a small interval increases even though the stationary flow rate remains constant. This effect is due to the increase in the average size of the first bunch of restrained vehicles located at the counting origin.

Figure 3 shows the variance, Var n(t), as a function of the average count, $\bar{n}(t)$, for several values of the fraction of constrained vehicles and t = 5 Δ, 10 Δ. The parameter $\mu\Delta$ rather than α is varied. With each curve, specification of a minimum headway, Δ, automatically specifies the counting interval, a value of μ, and the steady-state flow rate, $\mu/1 + \mu\Delta$. At least one author (16) has argued that in certain regions traffic counting distributions should have a larger variance than the Poisson due to the vehicles which concentrate in random queues. This feature can be observed in Figure 3 because a straight line Var n(t) = $\bar{n}(t)$ would intersect the peaked curves corresponding to t = 10 Δ. In those cases where small average counts are observed, the variance approaches that of the Poisson distribution. For average counts close to the integral part of $t\Delta^{-1}$ the variance decreases because the probability of maximum-pack increases. In the limit, this probability equals one, vehicles are spaced regularly Δ units apart, and the variance is zero. The slanted lines in Figure 3 indicate that it is not possible to obtain certain average counts for the given values of μ and Δ. This should not be interpreted, however, as a statement that small average counts cannot be observed in t, but rather that the probability of queueing must lie below certain values if small average counts are observed.

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