STRESSES IN CONCRETE PAVEMENT SLABS

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SYNOPSIS

This paper is a preliminary report on a current research project being conducted at Iowa State College, for the purpose of supplying experimental evidence bearing upon some of the assumptions utilized by Clifford Older in his interpretation of the Bates Road Tests and by Dr Westergaard in his analytical solutions for stresses in pavement slabs Loads have been applied to experimental slabs at a corner and the tensile strains in the top surface of the slab have been measured by means of optical lever extensometers

The strain measurements, together with observation of the shape of structural corner breaks, both in the field and the laboratory have led to a new hypothesis relative to the distribution of stress, or the locus of maximum moment in a slab when loaded at a corner It appears that the locus of maximum moment is a curved line which may lie anywhere between a circular curve having the corner of the slab as a center and a line tangent to this curve at right angles to the corner bisector Also, the moment is non-uniformly distributed along this path, being greater near the bisector than at the edges of the slab The factors which control the variations in shape of the locus of maximum moment between the described limits are not discernible.

Since the greatest stress in the slab under this hypothesis will occur when the locus is a circular curve and the least stress will occur when it is a straight line normal to the bisector, analyses incorporating these limiting cases are introduced and two expressions, one for the maximum and the other for the minimum probable tensile stresses along the corner bisector are derived These equations, when evaluated with constants applicable to the first experimental slab studied, define two stress curves, one of which is about 50 per cent greater than the other In general the measured stresses in this slab lie between these limiting curves and are nearer the curve of maximum probable stress

A comparison between both the measured and analytically determined stresses and those obtained by Dr Westergaard's analysis reveals substantial agreement between them as to the distance from the corner at which maximum stresses occur Also his conclusion that the magnitude of subgrade reaction has relatively little effect upon stresses is verified The expression here derived for minimum probable stress coincides with his expression for maximum stress

The studies reveal the need for further experimental evidence relative to the relationship between subgrade reactions and deflections and the distribution of deflections in the corner region of slabs

The stresses induced in concrete pavement slabs by concentrated wheel loads acting upon the pavement surface have been the subject of intense interest and discussion by highway engineers and allied scientists since the advent of this rigid type of highway pavement A number of studies directed toward the solution of this problem have been conducted by several research agencies and these studies, together with observed performance of existing structures, have provided the basis for the design of many thousands of miles of concrete pavement and pavement base Because of the breath taking speed with which highway builders have had to design and construct these rigid slabs in order to keep up with the demand for all year round traffic service, most of the experimental studies have, of necessity, been of the performance type of test and the analytical studies have been based upon hypotheses which are largely *a priori* in character As a result, there remain a number of questions bearing upon this problem, concerning which more knowledge is needed before further refinement in design can be accomplished

The Iowa Engineering Experiment Station established a project several years ago, the objective of which is to measure the stresses in concrete pavement slabs acted upon by concentrated surface loads and to study the distribution of these stresses as well as their magnitude in relation to the applied load Up to the present time, this work has been limited to the single case of a load placed at a right angle corner of a slab and stress measurements have been confined to the corner region in the vicinity of the applied load

This paper is a progress report on that project and will describe the methods of measurement employed and present a limited amount of data in support of a new hypothesis relative to the distribution of stress or the locus of maximum stress in a slab when loaded at a corner The preliminary or tentative conclusions which will be offered are given mainly for the purpose of thought provocation and discussion

Before entering upon an exposition of the technique of this study, it will be of service to review briefly two of the outstanding previous researches in this field The first of these, widely known as the Bates Road Tests, was conducted in 1922 and 1923 by the Illinois State Highway Department under the direction of Clifford Older[1] In these tests a series of concrete pavement slabs and concrete bases for other types of surfacing were constructed in sections 100 and 200 ft long on a continuous tangent under field conditions There were about 71 sections, each of different design All sections were 18 ft wide, but some were provided with center joints of various designs

The test sections were loaded by a truck which traveled up one side and back on the other a large number of round trips The load on the truck was varied so that the rear wheel concentrations ranged from 2500 to 13,000 lb and careful observation of the progressive effect on each pavement section was made and recorded Also, extensive studies were made of the deflections of the slab, the impact effect of the moving wheel load as compared to the same load statically applied, and the curling effect of temperature changes in various parts of the slabs

The data obtained in the Bates Road Tests have been of inestimable value in the design of rigid pavement slabs

These extensive Illinois tests vividly demonstrated the importance of providing adequate strength in the pavement design to resist corner loads, since disintegration and ultimate destruction of the test slabs practically always followed the occurrence of breaks at the corners Transverse cracks, due to temperature changes or traffic, or both, did not seriously impair the structural integrity of the slab unless and until they were followed by corner breaks

From the observation of loads causing

[1] Am Soc Civil Engineers, Trans v 87, 1924, p 1180

corner breaks and the thickness of broken slabs, Older presented the load-thickness relationship for all the broken slabs in the study in a diagram and found that the observed points cluster about a curve which is defined by the equation

$$d = \sqrt{\frac{3W}{S}} \qquad (1)$$

in which d = thickness of slab, W = breaking load, and S = modulus of rupture of companion beams

This formula is based upon a computation which was first proposed by A. T Goldbeck and involves the following assumptions "That the load, W, is applied at the extreme point of a right-angled corner formed by the intersection of an open transverse crack or joint with the edge of the pavement, that the corner is entirely unsupported by the subgrade and, therefore, acts as a simple cantilever, and that fiber stresses are uniform on any section normal to a line bisecting the corner angle" It is known that the first of these assumptions does not truly represent the situation in the Bates tests, since the path of the center of the wheels was carefully held to a distance 6 inches from the edge of the pavement in the most severe case, making the minimum distance along the bisector from the corner of the slab to the center of gravity of the load about 8½ in Also, of course, the load must have been distributed in some manner over the entire contact area between the tire and pavement and not concentrated at a point

The second and third assumptions probably do not represent the exact conditions which prevailed in the tests, although the evidence obtained at Bates is not extensive enough to warrant a definite conclusion on this point

Another valuable attack upon this problem of stresses in pavement slabs is the analytical solutions proposed by Dr H M Westergaard [2] In this study he analyzes the stress situation resulting from loads applied at a corner, at the center of a slab, and at an unsupported edge at some considerable distance from a corner Only the solution of the corner situation will be reviewed here

In this solution, Dr. Westergaard assumes the subgrade reactions to be vertical and proportional to the deflection of the slab and he introduces a quantity called the "radius of relative stiffness" which is a measure of the stiffness of the slab relative to that of the subgrade [3] It is expressed by the formula:

$$l = \sqrt[4]{\frac{Eh^3}{12(1 - \mu^2)k}} \qquad (2)$$

in which l = radius of relative stiffness, E = modulus of elasticity of concrete, μ = Poisson's ratio, and k = modulus of subgrade resistance, stated in lb per sq. in per inch of deflection, that is in lb per in[3] (p c i)

Also, he derives an expression for the deflections of a slab in the neighborhood of the corner (see Equation (3)), in which z = deflection of the slab, P = load at a corner, x = distance from the corner to any point measured in a direction parallel to the axis bisecting the corner, and a_1 = distance from the corner to the center of gravity of the load along the bisector

$$z = \frac{P}{kl^2}\left(1.1\, e^{-\frac{x}{l}} - \frac{a_1}{l}\, .88\, e^{-\frac{2x}{l}}\right) \qquad (3)$$

The reactions of the subgrade are then expressed as kz and the bending moment

[2] Proceedings, Fifth Annual Meeting, Highway Research Board, 1926 Part 1, page 90

[3] Proceedings of the American Society for Testing Materials, Vol 24, Part II, Technical Papers, p 1025

computed in the section $x = x_1$ due to the combined influence of the applied load and the reactions of the subgrade. Then he employs the third assumption used in the Bates Tests ". . this bending moment will be approximately uniformly distributed over the width $2x_1$ of the cross section," and obtains the approximate expression for the greatest tensile stress in the corner region:

$$\sigma_C = \frac{3P}{h^2}\left[1 - \left(\frac{a_1}{l}\right)^6\right] \qquad (4)$$

and the distance from the corner at which this stress occurs is found to be approximately

$$x_1 = 2\sqrt{a_1 l} \qquad (5)$$

It will be noted that when $a = 0$, that is, when the load is placed at the corner, the expression for the tensile stress reduces to

$$\sigma_C = \frac{3P}{h^2} \qquad (6)$$

which is the same as the formula employed by Older in interpreting the Bates Tests.

These studies by Dr. Westergaard have introduced a number of enlightening concepts into the study of stresses in pavement slabs and his solution of the corner load case undoubtedly comes nearer to the actual situation than does the treatment employed by Goldbeck and Older, since he deals with the load in its actual position Also, while some of his assumptions in regard to the subgrade reaction are as yet unverified, it is a distinct forward step to include these pressures in the analysis Both the Westergaard and the Goldbeck and Older analyses, however, employ the same assumption in regard to the distribution of the maximum tensile stresses and this assumption has not as yet been verified by direct measurement of stresses in slabs

The research in this field at Iowa State College has been directed mainly toward a study of the stress distribution situation in the corner region in an attempt to provide an experimental background for the assumed distribution. The plan of the experiments has been to measure the strains in the top surface of the corner region of several concrete slabs supported on an earth subgrade and loaded through a circular area placed at the corner.

The first experimental slab was 6 in. thick and 10 by 12 ft in plan. Some preliminary studies were made with a smaller slab, but it was found that the corner opposite the load tipped up appreciably and the smaller size was deemed unsatisfactory. A 0 0001-in. dial placed on the corner opposite the loaded corner of the 10 by 12 slab showed vertical upward movements ranging from 0 to 0 0004 in , which was considered to be negligible as far as effect on stresses at the loaded corner was concerned.

The subgrade upon which the slabs rested was constructed synthetically by tamping moist yellow clay in thin layers within a plank crib 12 by 14 ft in plan and 2 ft. deep. The whole set up is located in a basement room which is affected very little by outside temperatures so that temperature effects upon measurements of strains in the concrete were eliminated to a large extent Loads were applied to the slab by means of a screw jack acting against the floor beams of the floor above, and were measured by means of a pair of calibrated coil springs mounted between two cast iron plates. The loads were transmitted to the slab through circular cast iron bearing plates with a cushion of cornstalk insulating board between the plate and the slab to aid in securing a uniform distribution of the load over the circular area

The strains in the concrete were measured by means of a number of optical lever extensometers which were placed along elements radiating from the loaded corner at 0°, 22½°, 45°, 67½°, and 90° These extensometers, which had a gauge length of 3 in , were of the type developed by A. N Johnson[3] and consisted of a gauge bar with a fixed knife edge at one end. At the other end, a shallow notch was machined in the lower surface of the gauge bar to receive the upper edge of a hardened steel diamond shaped, rotating knife edge, while the lower edge was in contact with the surface of the concrete The diamond shaped knife edges were attached rigidly to a $\frac{1}{16}$-in steel bar which supported a small polished steel mirror at one end and a counterweight for the mirror at the other end The mirrors were 16 by 19 mm in dimension The fixed and the rotating ends were held in contact with the concrete by a ½ x 3 x 3½-in steel weight attached to the upper side of the gauge bars The extensometers could be moved readily from point to point on the top of the slab and they required no gauge points or other inclusions which might impair the integrity of the slab surface and cause localized stress disturbances

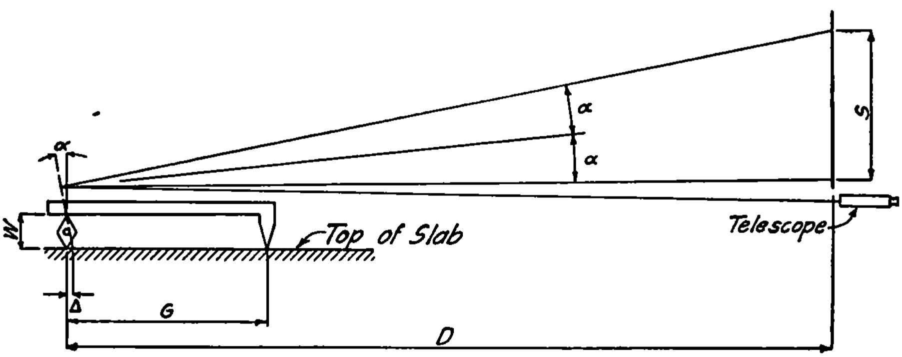
The reading telescopes, which were ordinary surveyors' levels in this case, and the scales upon which movement of the optical lever was observed, were located at points from 25 to 40 ft away from the mirrors, depending upon the space available along the various projected elements The scales were divided into inches and tenths and the images in the instruments were sufficiently clear to warrant estimating the scale readings to $\frac{1}{100}$ of an inch

As the slab was loaded and strain developed in the slab, the movement within the gauge length caused the movable diamond point and the mirror to rotate The line of sight from the instrument to the mirror and back to the scale was thus deflected in accordance with optical laws and the change in scale readings due to this rotation was a measure of the strain in the concrete slab within the gauge length The geometry of the situation, by which the unit strain in the slab was calculated, is shown with the diagram of the extensometer in Figure 1 The line of sight to the scales was affected by two phenomena, the change in length between the gauge points or knife edges and the change in slope of the slab surface as it deflected under load It was necessary, therefore, to use these extensometers in pairs, placed side by side close together, with the mirror of one of the pair arranged so that it would rotate in the opposite direction to that of the other Then the strain in the slab would cause a positive reading on the scale for one mirror and a negative reading for the other, while the change in slope due to deflection of the slab would deflect the line of sight upward and cause a positive reading for both mirrors The algebraic average of the two mirror readings eliminated the scale difference due to the change in slope and gave the true scale reading caused by the strain in the slab alone

These extensometers have operated satisfactorily and have gained the confidence of the users The technique which has been adopted in their use has been to load the slab three times for any given load situation or placement of the extensometers and the average of the three results obtained A uniform period of 5 min has been allowed to elapse after the removal of the load before the final zero reading is taken. If this final reading did not coincide with the initial zero reading of the scale, as sometimes happened, though infrequently, it was assumed that some slippage must have taken place between the knife edges and the concrete and the extensometer was reset and a new series of readings taken

In the early stages of stress measurement, only four pairs of extensometers were used and these were all placed along one radial element at a time. Later eight additional pairs of extensometers were made in order to facilitate the measurement of strains at a larger number of points for each load application Four pairs were placed on each of three radial elements, which required six levels for reading, since four scale images in the field of view of one instrument was about the maximum which could be read In order to obtain four images in one telescope it was necessary to vary the length of bar between the diamond shaped knife edges and the mirrors, the shortest being placed nearest the instrument, so that the forward mirror would not obstruct the line of sight of those behind After the scale readings for one setting of the extensometers had been taken, they were reset either at other points on the same radial element, or on other elements and the load again applied These operations were repeated until the extensometers had been placed at a sufficient number of points to give a complete picture of the distribution of strains in the region of the corner

Vertical deflections of the slab at the loaded corner were measured by means of a 0 0001-in Federal dial. These measurements showed an increasing total downward deflection of the corner under a load of 3000 lb as the number of loading cycles increased. The increase in total deflection was relatively rapid in the early loading cycles, but declined as the number of cycles increased. Roughly it may be said that two-thirds of the total increase in deflection occurred within the first one-third of the loading cycles. From these measurements and visual inspection, it was apparent that the subgrade was



$$\text{Tan } \alpha = \frac{\Delta}{W} = \frac{S}{2D}$$

$$\text{Unit Strain, } \delta = \frac{\Delta}{G}$$

$$\delta = \frac{SW}{2GD}$$

Figure 1

stressed beyond its elastic limit and was permanently deformed under the repeated loading. In fact, toward the end of the load application period a sheet of paper could readily be inserted between the slab and the subgrade when no load was on the slab, indicating that it was entirely unsupported for some undetermined distance back from the corner. This space closed up as load was applied, however, and the slab seemed to be in full

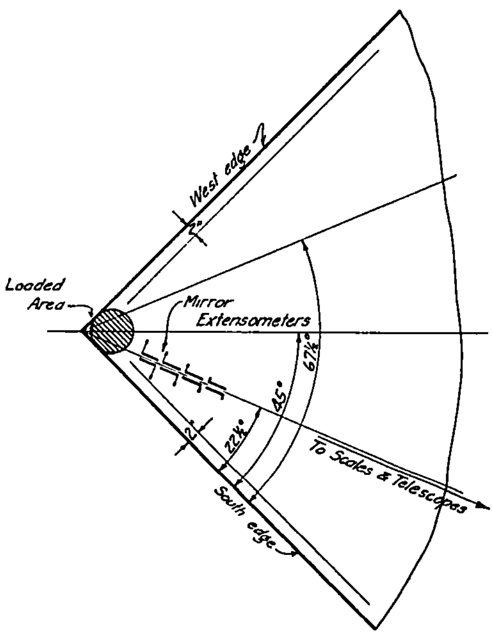


Figure 2

contact with the subgrade at the maximum test load.

These deflection measurements have been utilized in determining an apparent modulus of subgrade resistance for use in analyzing the stresses in the slab, by substituting observed values in Westergaard's formula for corner deflection and solving for k with x = 0. Apparent values of the moduli thus obtained range from about 275 p.c.i., corresponding to a deflection of 0.0232 in., to a minimum of 40 p.c.i., corresponding to a deflection of 0.0620 in. Since, as stated above, the deflections increased rapidly in the earlier loadings, and since the apparent modulus of subgrade resistance decreases at a greater rate than the deflection increases, the apparent modulus decreased very rapidly from about 275 p.c.i. to about 65 p.c.i. in roughly the first one-third repetitions of load. For the later loadings, the modulus remained nearly constant.

The measured unit strains along the various elements in the top surface of the first test slab, under loads of 1000 lb., 2000 lb., and 3000 lb., applied over a circular area 6$\frac{3}{4}$ in. in diameter and placed tangent to each side of the corner are shown in Figures 3 to 7. The observed strains along the corner bisector (Figure 5) are divided into two groups, one of which seems to be consistently lower than the other. The lower group of these two represents the strain measurements obtained for the first 25 load applications on the slab when the apparent modulus of subgrade reaction was relatively high (about 250 p.c.i.). After this group of observations was made, the extensometers were moved to other elements and then later were reset on the bisector and the higher group of readings was obtained. This latter group, therefore, represents a later period of the loading cycles when the apparent modulus had settled down to a slowly changing value. The average value for this group of strains is about 75 p.c.i. The differences in subgrade modulus seem to have had no important effect upon the strains along the other radial elements, and no attempt has been made to divide these data on that basis.

The measured strains are widely scattered at some points along the radial elements, especially in the case of the higher

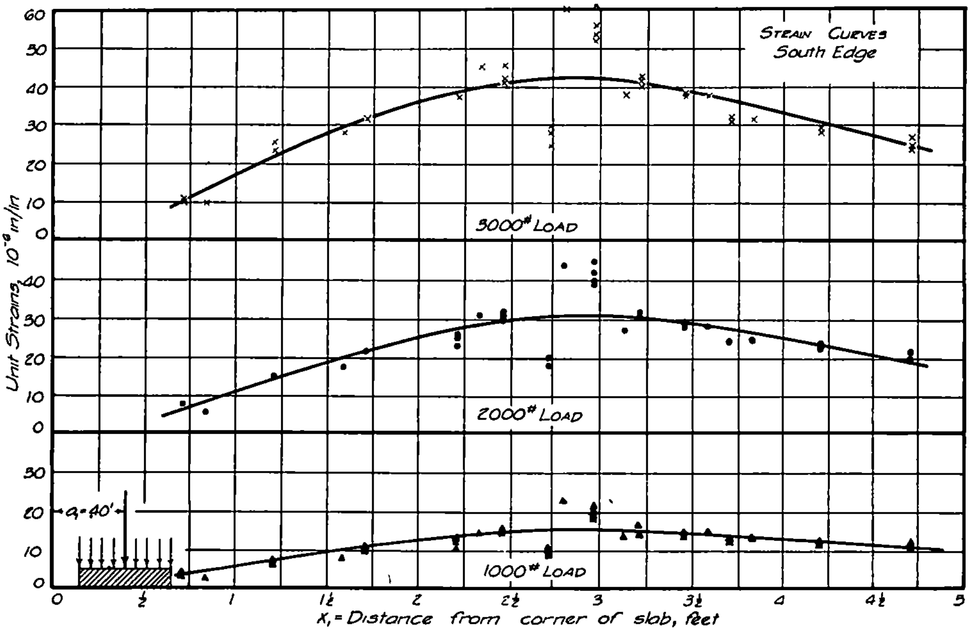


Figure 3

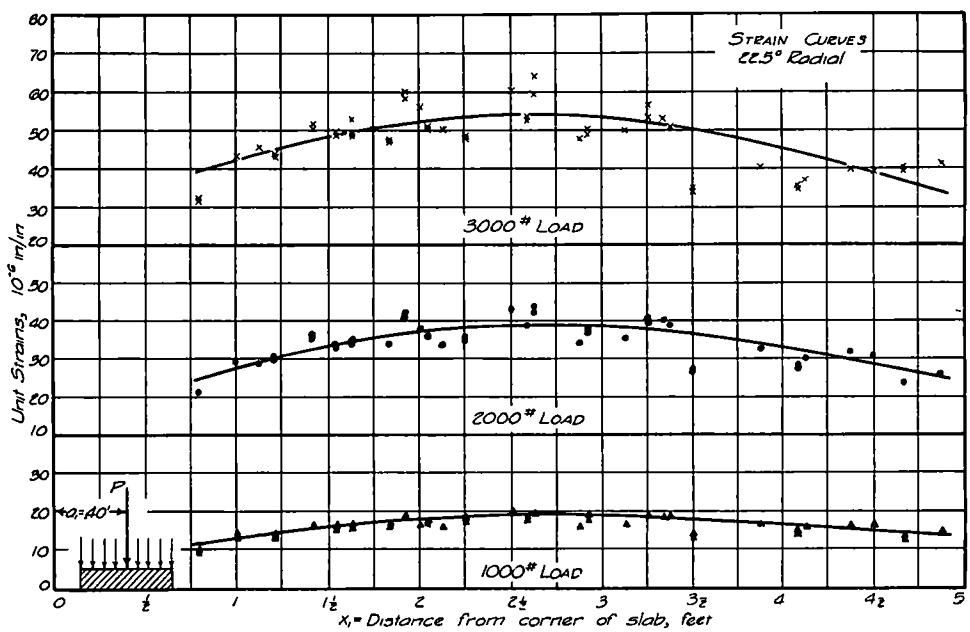


Figure 4

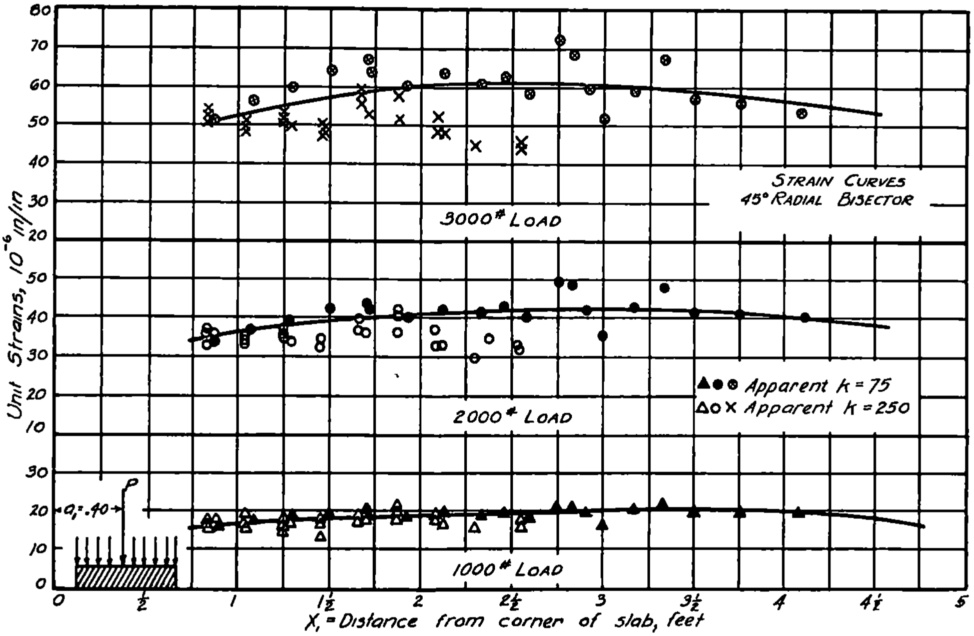


Figure 5

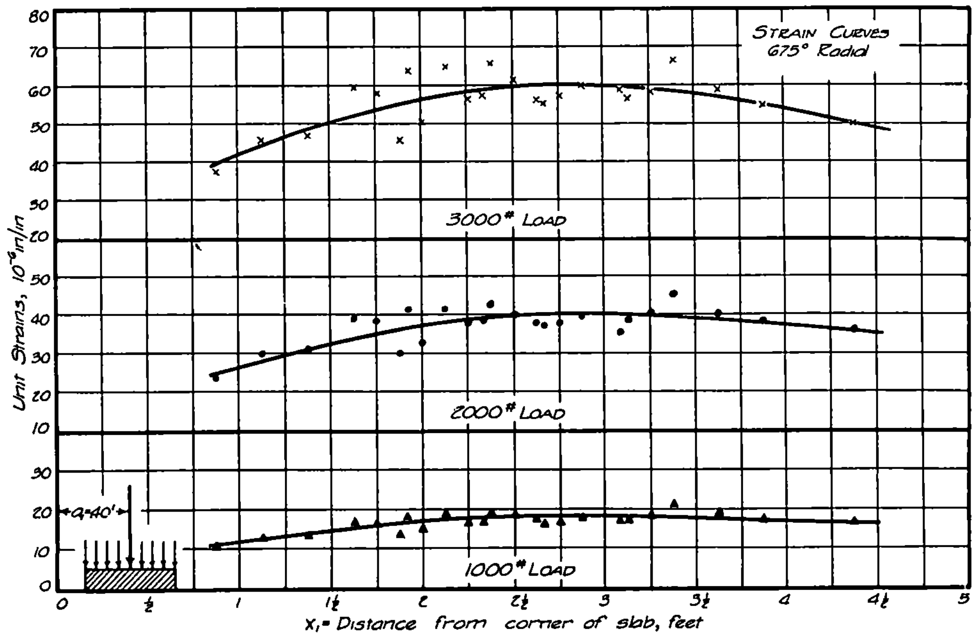


Figure 6

loads, which fact has precipitated some intensive study to determine whether this scattering was due to experimental errors in measurement, or whether it represented the true strain situation for this slab This study has led to the conclusion that the measurements were accurate, within reasonable limits, and that the scattering of the points is due to the fact that the type of extensometer used measured the strains in the top skin of the concrete slab along the element by one-half inch increments, loading the slab for each setting of the gauge, in an attempt to locate the trouble point It was found that as the gauge approached a certain point, the reading fell off considerably in relation to the points nearer the load Then when this point was straddled by the knife edges, the readings took a sudden jump to a value nearly double the low readings As the gauges moved outward along the

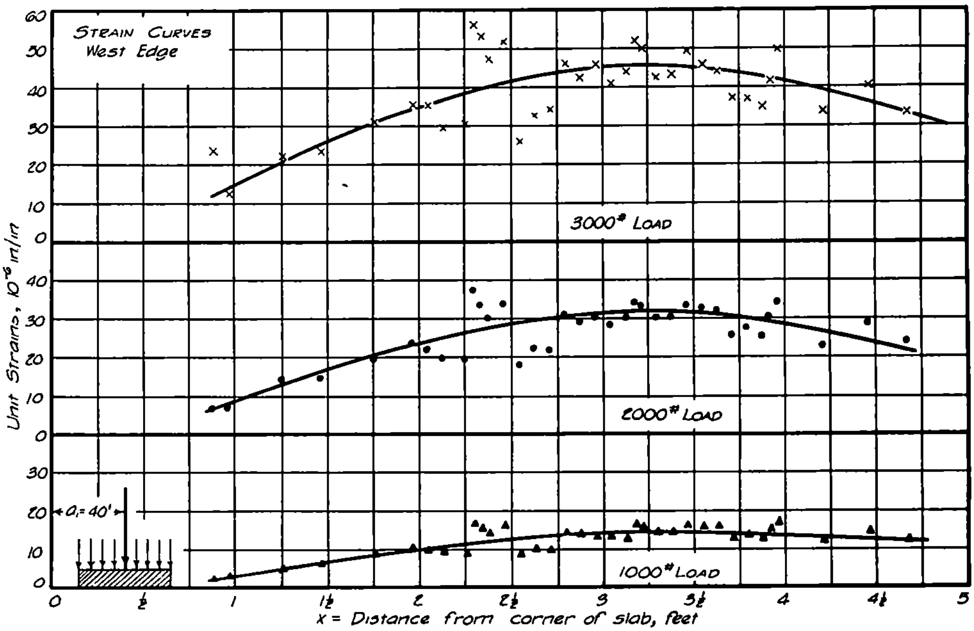


Figure 7

and that this top skin probably contained many microscopic cracks due to surface drying of the slab

As evidence to support this conclusion, it will be of interest to relate an experience of the operators in measuring strains along the 0° element which are shown in Figure 3 The extensometers in the neighborhood of 2 ft 9 in. from the corner gave very erratic results, and after making sure that the extensometers were in order, it was decided to move the gauges element, the readings returned to a value which appeared to be more nearly normal This behavior gave rise to the theory that the surface must be cracked at the trouble point and a microscope was applied to the slab and a very distinct crack was found. It was observed to open and close as the slab was loaded and unloaded, but did not appear to travel or increase in size appreciably, although the slab was loaded between 100 and 200 times

Further evidence that the scattered

measurements obtained were due to surface skin abnormalities due to drying is found in the fact that a later slab which is under observation at the present time, and which has been kept damp on the surface at all times, is yielding much more uniform and consistent strain measurements

strain in the corner region of the slab An examination of this diagram reveals a distribution of strain across the corner which is not uniform along a straight line through the point of maximum strain and normal to the corner bisector, but rather, the locus of maximum strain seems to follow a curved path which bends toward

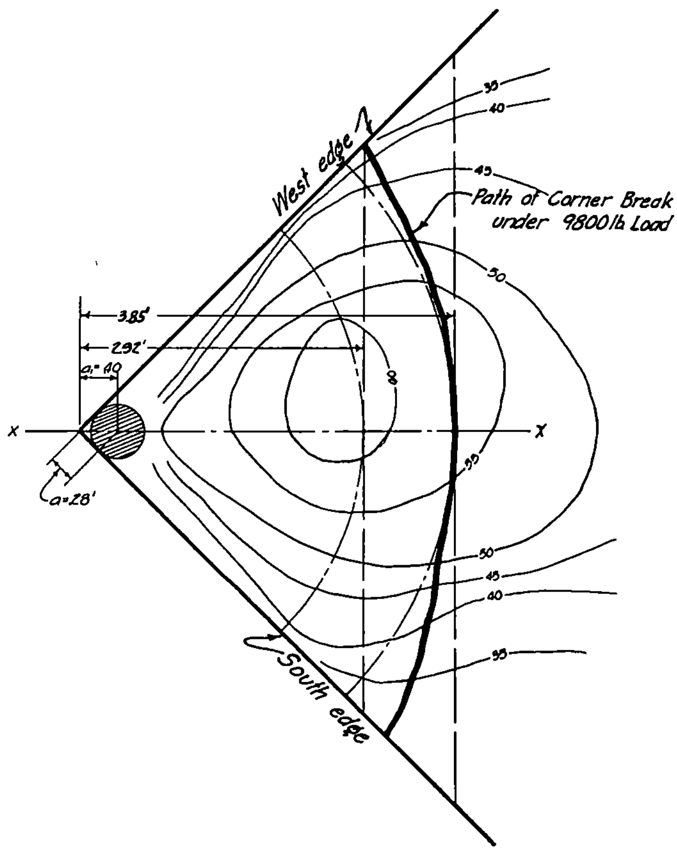


Figure 8 Iso-strain Diagram. Radial Unit Strains, 10^{-6} in /in under 3000 lb Load

These studies have led to the belief that it is justifiable to draw a mean curve through the scattered points to obtain a graph of the distribution of strains along the several elements studied

From these mean curves, then, an isostrain diagram can be drawn, as in Figure 8, showing the lines of equal radial

the corner as the edges of the slab are approached. Also, it appears that the strain is not uniformly distributed along this path, but is less at the edges than in the vicinity of the bisector

Another approach to a determination of the path of the maximum strain in a slab loaded at a corner may be made by observing the shape of corner cracks or breaks in actual slabs in service and in slabs broken in the laboratory experiments It cannot be stated definitely that the shape of a corner break will coincide with the locus of maximum strain in the slab since the shape of the break will be influenced by non-uniform strength characteristics of the concrete, by unevenness of the subgrade resistance, and other factors Also, such a correlation involves the theory of failure of a corner and no such theory has been established For example, if the strain in the neighborhood of the corner bisector is greater than at the edges of the slab, as shown in the iso-strain diagram in Figure 8, in all probability, rupture of the concrete would begin near the bisector and progress to the edges Whether the path which the failure travels would coincide with the locus of maximum strain which existed prior to the initial rupture is purely conjectural However, it seems entirely reasonable to the author that the shape of a corner break would approximate the locus of maximum strain

After completing the strain measurements on the first slab, as outlined above, the corners were loaded to failure and a record of the shape of the breaks was made, as well as the load causing the failure The results of these tests are shown in Figure 9 The loads causing failure were very uniform, having an average value of 9780 lb. The shape of three of the breaks was definitely a curve with the ends bending toward the corner The other break was quite irregular, although it approximated a straight line For convenience, a circular curve and a straight line normal to the bisector and passing through the intersection of the bisector and the break have been drawn.

A number of corner breaks in concrete pavement slabs in and around Ames, Iowa have been surveyed and the data concerning them are shown in Figures 10 to 15 It is, of course, impossible to say definitely that these breaks are due to corner wheel loads, since the author was not present at the time the break occurred However, the circumstances surrounding each case point rather definitely to the conclusion that they are structural breaks, except in the case shown in Figure 15 Here there

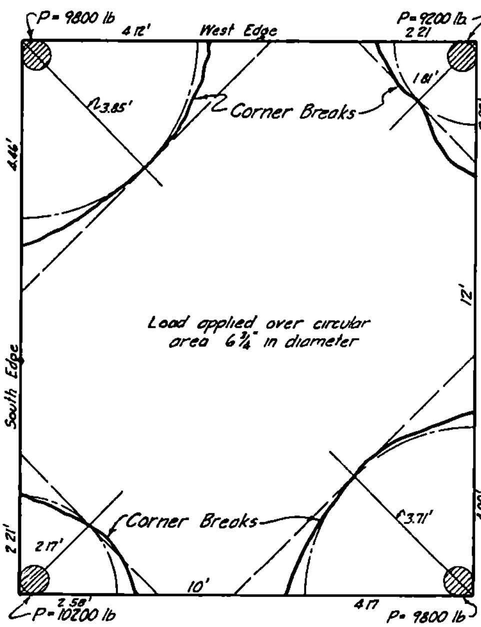


Figure 9. Corner Breaks on First Experimental Slab

is no indication whether the break is structural or not, but it is such a good illustration of a curved corner break, that it is included for what it is worth Further, even though the assumption that the breaks are due to corner wheel loads is accepted, there is no evidence as to the magnitude of such loads, the place of application, or the size and shape of the area of contact between the wheels and the pavement

These limited studies of corner breaks indicate that a slab may break under a corner load along a curved path which lies anywhere between a circular curve having the corner as a center and a straight line corner and to increase in curvature toward the circular shape, as the distance from the corner to the break increases

The laboratory strain measurements conducted so far in this project and the

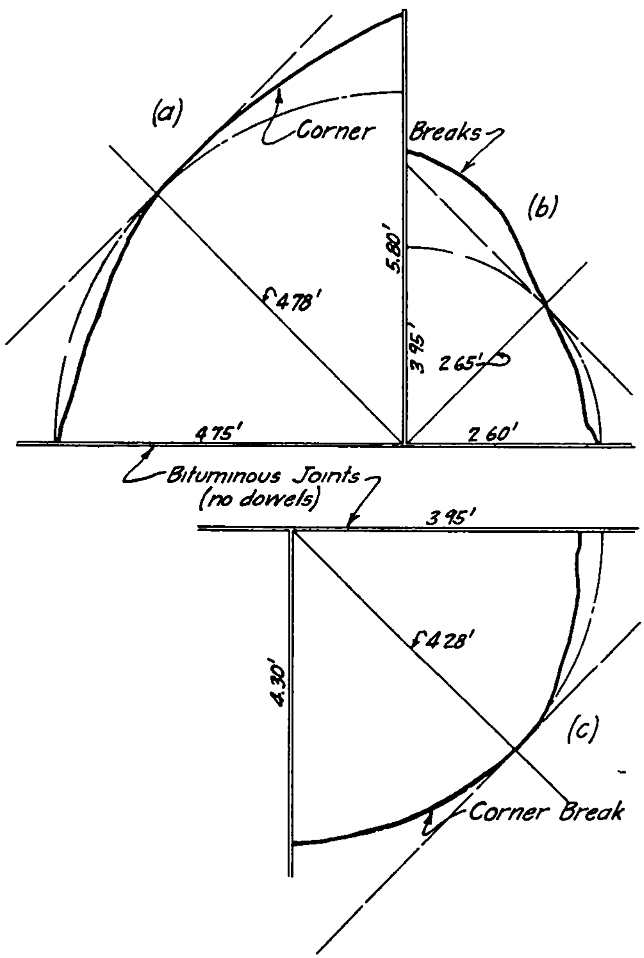


Figure 10. Five-inch Concrete Pavement Reinforced with 6-6-9 wire mesh placed near bottom of slab

tangent to such a curve which is normal to the corner bisector The factors which cause the variation in shape of the break are not discernible. There seems to be a definite tendency for the break to approach a straight line if it occurs near the observation of structural corner breaks both in the laboratory and the field lead to the hypothesis that the locus of maximum moment produced in a concrete pavement slab by a corner load is a curved line which bends toward the corner as it approaches the edges of the slab It appears that the locus may lie anywhere between a straight line normal to the bisector and a circular curve having the corner as a center Under this hypothesis, the maximum stress will occur when the locus is a circular curve since this is the shorter of the two limiting sections

Applying this hypothesis as to maximum stress in a slab under a corner load and utilizing the Westergaard concepts relative to subgrade reactions and deflections in the corner region, it is found that the forces acting upon a corner portion of the slab are as shown in Figure 16 Writing moments about the section a-a gives[4] the results shown in Equation (7).

$$P(x_1 - a_1) - 2\int_0^{707\,x_1} px(x_1 - x)dx - 2\int_{707\,x_1}^{x_1} p(x_1^2 - x^2)^{\frac{1}{2}}(x_1 - x)dx - 2\,x_1^2\int_0^{\frac{\pi}{4}} v(1 - \cos\phi)d\phi - 2\,x_1\int_0^{\frac{\pi}{4}} m\cos\phi d\phi = 0 \qquad (7)$$

This equation may be simplified by including the subgrade reactions on the area between the line a-a and the curve bx_1 b and ignoring the shears on the elements $x_1\,d\phi$ The moment due to each of these groups of forces is small so that adding one and eliminating the other cannot seriously affect the final result Equation (7) may then be written as shown in Equation (8)

$$P(x_1 - a_1) - 2\int_0^{x_1} px(x_1 - x)dx - 2\,x_1\int_0^{\frac{\pi}{4}} m\cos\phi d\phi = 0 \qquad (8)$$

The strain measurements showed that the moment at the edges was about 75% of that at the center, and varied roughly as ϕ^2 Therefore

$$m = M(1 - .4\phi^2) \qquad (9)$$

[4] The author is indebted to Prof V P Jensen of Iowa State College for assistance in setting up this equation

in which M = the maximum moment per unit length at the corner bisector.

Substituting this value of m and Westergaard's value of p in Equation (8) gives Equation (10) Integrating (10) we get Equation (11) In this equation, the third and fifth terms are relatively small and since one is plus and the other minus, they may be disregarded Solving for M gives Equation (12)

Dividing by the section modulus per unit of width, which is $\frac{t^2}{6}$, the tensile stress at any point on the bisector x_1 from the corner is as shown in Equation (13)

Assuming the stress to be uniformly distributed over the length $2x_1$, normal to the bisector, an expression for the minimum probable stress may be derived in similar manner (see Equation (14)) This equation is similar to Dr Westergaard's expression for maximum stress (Equation 4, page 125) except that it is general for all values of x_1

Using values of a_1 = 0.4 ft, E = 3,750,000 and t = 6 in, which are applicable to the slab reported herein, and assuming Poisson's ratio to be 15, the maximum and minimum probable stresses in the slab with subgrade moduli of 250 p c i. and 75 p c i are shown in Table I

The stresses given in Table I are shown graphically in Figures 17 and 18, along with the stresses obtained for the experimental slab by multiplying the measured

$$P(x_1 - a_1) - 2\,2\,\frac{px_1}{l^2}\int_0^{x_1} x\,e^{-\frac{x}{l}}\,dx + 1\,76\,\frac{px_1a_1}{l^3}\int_0^{x_1} x\,e^{-\frac{2x}{l}}\,dx + 2\,2\,\frac{p}{l^2}\int_0^{x_1} x^2\,e^{-\frac{x}{l}}\,dx - 1\,76\,\frac{pa_1}{l^3}\int_0^{x_1} x^2\,e^{-\frac{2x}{l}}\,dx - 2\,x_1\,M\int_0^{\frac{\pi}{4}}(1 - 4\phi^2)\cos\phi\,d\phi = 0 \qquad (10)$$

$$P(x_1 - a_1) - 2\,2\,Px_1\left[1 - e^{-\frac{x_1}{l}}\left(\frac{x_1}{l} + 1\right)\right] + 44\,Px_1\frac{a_1}{l}\left[1 - e^{-\frac{2x_1}{l}}\left(\frac{2x_1}{l} + 1\right)\right] + 4\,4\,Pl\left[1 - e^{-\frac{x_1}{l}}\left(\frac{x_1^2}{2l^2} + \frac{x_1}{l} + 1\right)\right] - 44\,Pa_1\left[1 - e^{-\frac{2x_1}{l}}\left(\frac{2x_1^2}{l^2} + \frac{2x_1}{l} + 1\right)\right] - 1\,3\,Mx_1 = 0 \qquad (11)$$

$$M = 77\,P\left(1 - \frac{a_1}{x_1} - 2\,2\left[1 - e^{-\frac{x_1}{l}}\left(\frac{x_1}{l} + 1\right)\right] + 4\,4\,\frac{l}{x_1}\left[1 - e^{-\frac{x_1}{l}}\left(\frac{x_1^2}{2l^2} + \frac{x_1}{l} + 1\right)\right]\right) \qquad (12)$$

$$\sigma_{max} = \frac{4\,6\,P}{t^2}\left(1 - \frac{a_1}{x_1} - 2\,2\left[1 - e^{-\frac{x_1}{l}}\left(\frac{x_1}{l} + 1\right)\right] + 4\,4\,\frac{l}{x_1}\left[1 - e^{-\frac{x_1}{l}}\left(\frac{x_1^2}{2l^2} + \frac{x_1}{l} + 1\right)\right]\right) \qquad (13)$$

$$\sigma_{min} = \frac{3\,P}{t^2}\left(1 - \frac{a_1}{x_1} - 2\,2\left[1 - e^{-\frac{x_1}{l}}\left(\frac{x_1}{l} + 1\right)\right] + 4\,4\,\frac{l}{x_1}\left[1 - e^{-\frac{x_1}{l}}\left(\frac{x_1^2}{2l^2} + \frac{x_1}{l} + 1\right)\right]\right) \qquad (14)$$

TABLE I

x_1 feet	Tensile Stress			
	k = 250 p c i		k = 75 p c i	
	Maximum	Minimum	Maximum	Minimum
1 0	201	131	216	141
1 25	216	141	230	150
1 50	223	145	247	161
1 75	220	144	248	162
2 00	214	139	246	160
2 25	205	133	243	158
2 50	192	126	240	156
3 00	165	107	222	145
3 50	135	88	203	132
4 00	104	68	181	118
4 50	76	50	156	102

strains in the outer surface of the concrete as given in Figure 5, by the modulus of elasticity of the concrete The maximum stresses according to the Westergaard formula are also shown on these graphs

The measured stresses lie between the two limiting stress curves resulting from the hypothesis announced in this paper and are nearer to the curve representing the maximum probable stress, except for the higher values of x_1 in the case of k = 75 In this case, the measured stress values are higher than the calculated stresses and are substantially uniform for several feet in the region of maximum stress This uniform stress situation is verified to some extent by the fact that the four corners of the experimental slab broke off under similar corner The study indicates, however, that the stresses in a slab may be from 0 to 50% greater than those obtained by his analysis of the corner load situation There is

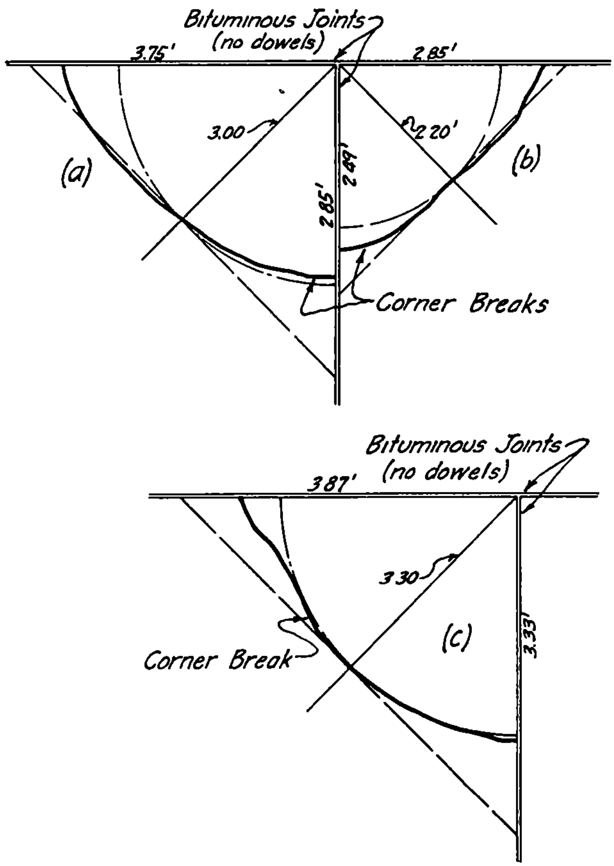


Figure 11. Five-inch Concrete Pavement Reinforced with 6-6-9 wire mesh placed near bottom of slab

loads at distances from the corner which varied from 1 81 to 3 85 ft

Both the measured and the calculated stresses are in substantial agreement with Westergaard's analysis in regard to the distance from the corner at which the maximum stresses occur They also verify his conclusion that the value of the subgrade modulus has relatively little effect on the maximum stress in the slab.

no evidence available to provide a basis for predicting the specific amount of this increase for any particular slab

It is difficult to make a valid comparison between the results obtained in the Bates' Road Tests and the stress formulas offered in this paper, because the modulus of elasticity of the concrete and the modulus of subgrade resistance which prevailed in the Bates' specimens are not known Nevertheless, a general comparison with these experimental data is of value in showing the order of magnitude of the results obtained in the two studies, and the diagram in Figure 19 is offered for this purpose In plotting the curves for the load-thickness relationship resulting from the maximum and minimum probable stress formulas, the value of the modulus of elasticity of the concrete at ultimate has been estimated, or more properly guessed to be about 3,000,000. From the experience gained in connection with the first test slab, where the apparent modulus of subgrade resistance was rapidly reduced from an initial value of roughly 275 p c i to a terminal value of about 40 p c i due to permanent set of the subgrade under a load repeated about 400 to 500 times, it seems logical to suppose that similar phenomena probably occurred at Bates Since those slabs were loaded thousands of times instead of hundreds, a value of 25 p c i for k does not seem far out of line The distance from the corner to the center of gravity of the load, a_1, is known to have been $8\frac{1}{2}$ in , and the reported average value of the modulus of rupture for all sections was 703 p s i

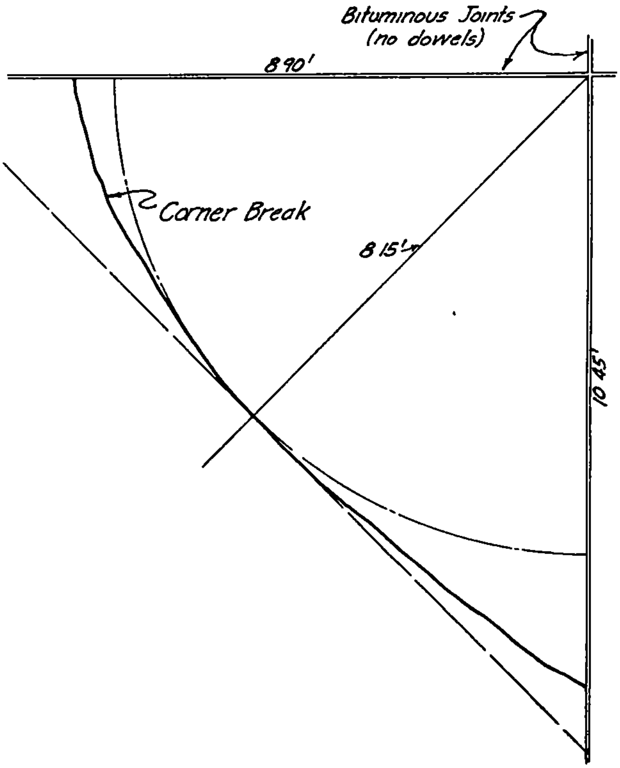


Figure 12 Five-inch Concrete Pavement Reinforced with 6-6-9 wire mesh placed near bottom of slab

In appraising the position and shape of the theoretical curves relative to the ex-

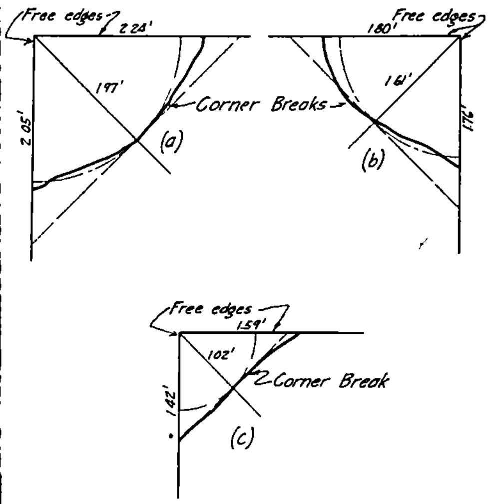


Figure 13 Four-inch Concrete Sidewalk adjacent to a driveway

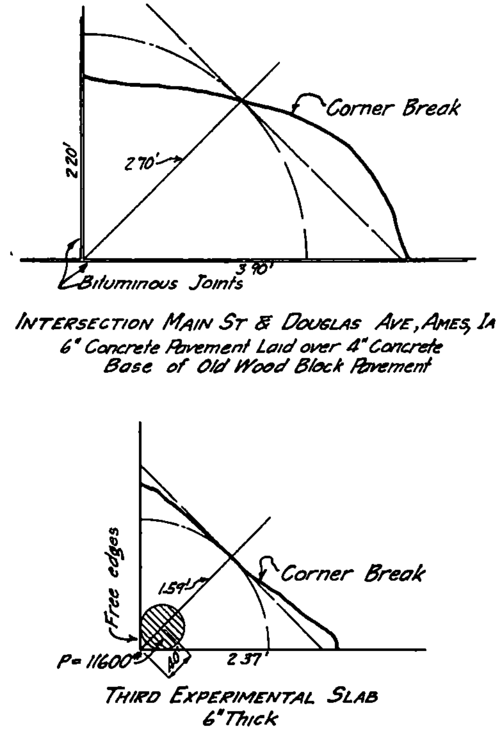


Figure 14

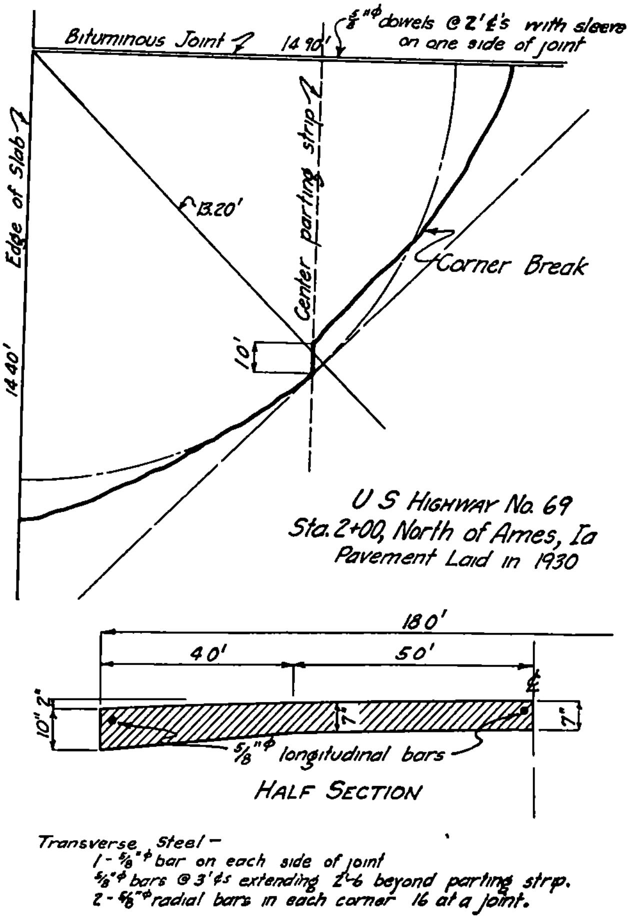


Figure 15

perimental points it must be remembered that the experimental values include the effect of repeated loads producing stresses greater than 50 per cent of the modulus of rupture of the concrete and possibly some impact effect The thickness determined by the Bates experimental data, therefore, may be expected to be greater than those obtained by the proposed formulas by an undetermined amount

Throughout this whole study, the need for adequate fundamental knowledge in regard to the relationship between subgrade pressures and deflections has been apparent, both from the standpoint of research and design, and attempts have been made to study this relationship without satisfactory results Also, there is need for more knowledge regarding the distribution of deflections in the corner region Some information regarding this distribution has been obtained for the third experimental slab which is being observed at the present time. The deflections in the corner region of this slab

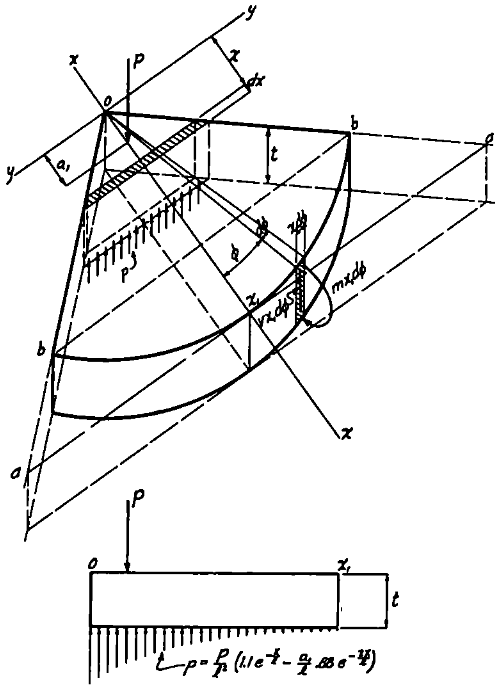


Figure 16

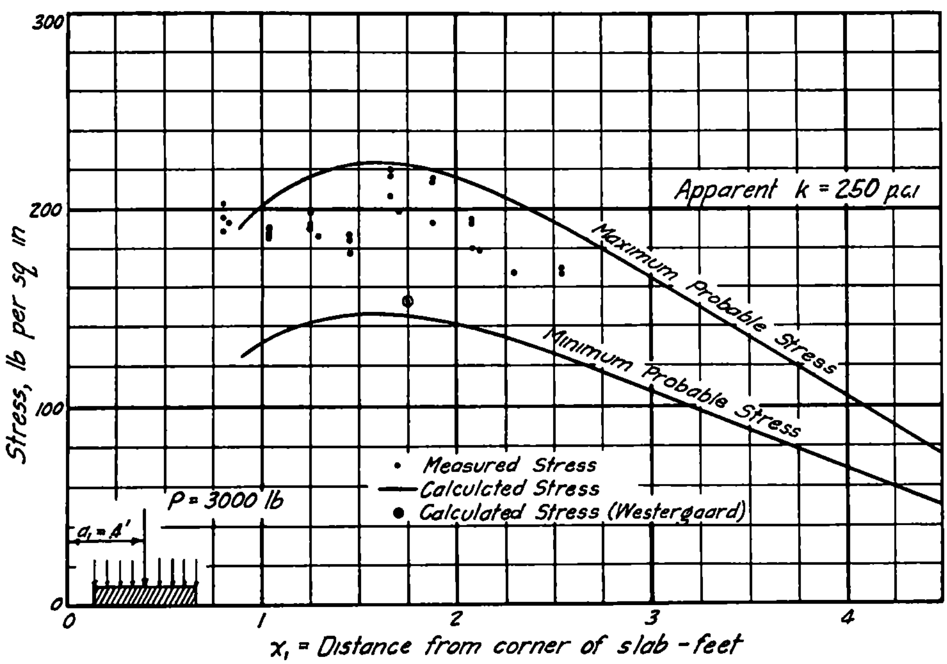


Figure 17

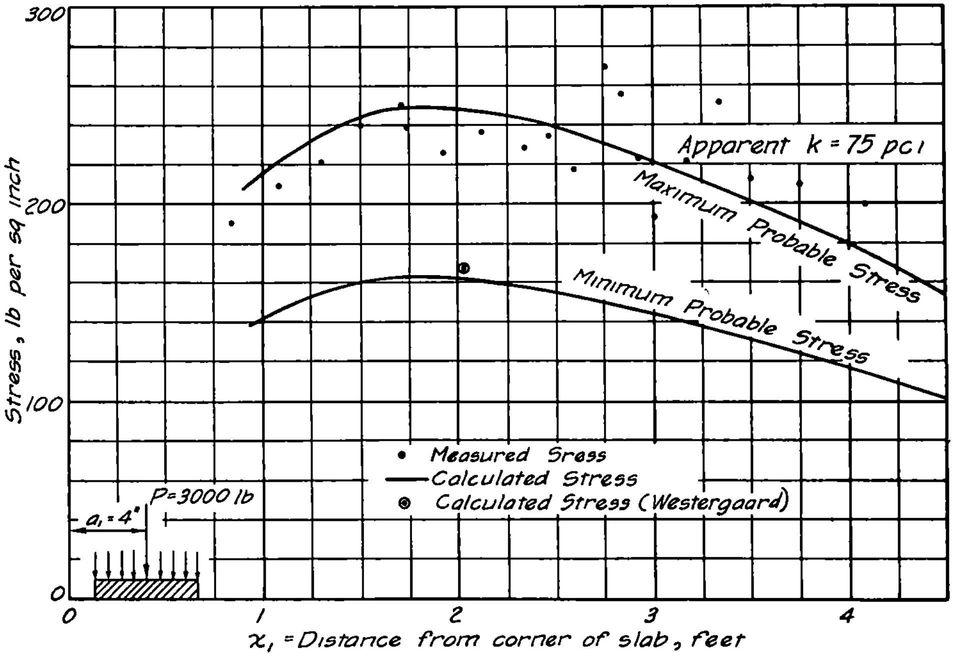


Figure 18

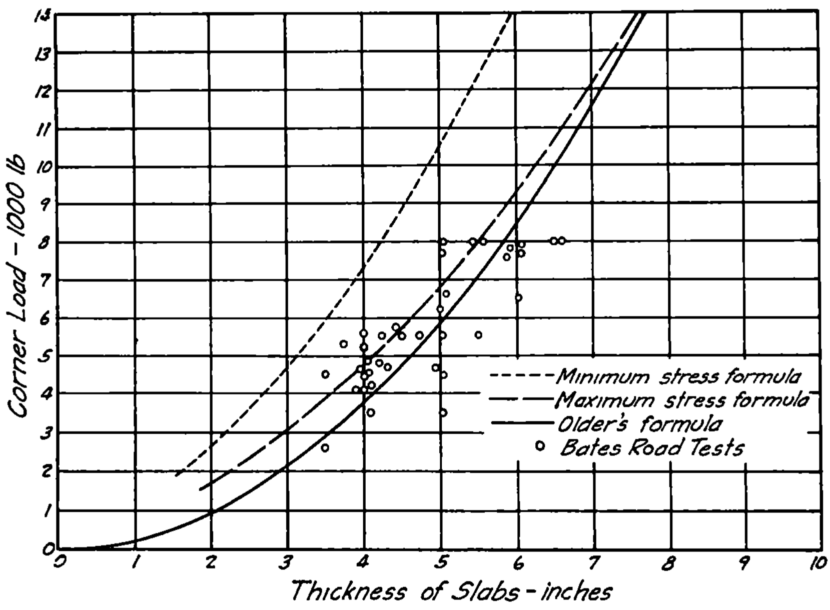


Figure 19

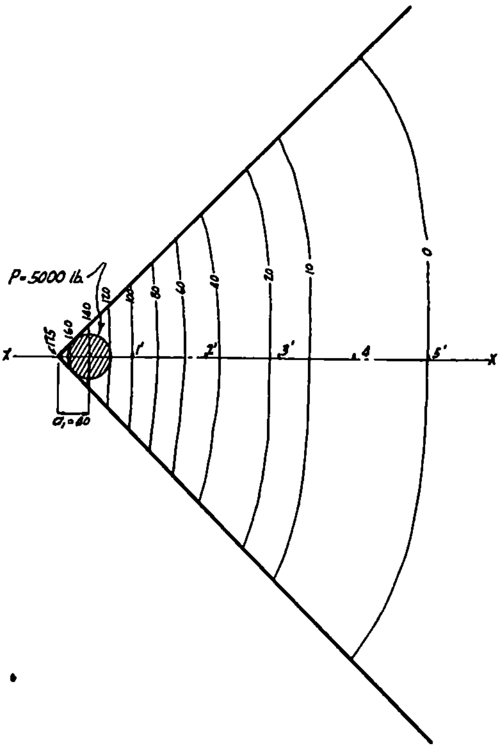


Figure 20. Iso-deflection Diagram Vertical Deflections in 0 0001". Third Experimental Slab

which is similar in dimensions to the first slab, were measured by means of a network of 0 0001-in dials, with the results shown in the iso-deflection diagram in Figure 20 Unfortunately and unexpectedly, the subgrade on which this slab was constructed was unduly stiff, having an apparent modulus in the neighborhood of k = 1000 or 1100 p c i which permitted only very small deflections and makes a study of their distribution difficult and uncertain It should be noted, however, that the deflections along the bisector practically coincide with those obtained by Westergaard's formula for deflection This is also true for points not on the bisector for a distance of about 1½ feet back from the corner Beyond this point the deflections of points not on the bisector deviate slightly from the theoretical, although the deviation is small and not important Whether they might be important or not with a slab on a more nearly normal subgrade, remains to be studied

DISCUSSION ON STRESSES IN CONCRETE PAVEMENT SLABS

MR CLIFFORD OLDER, *Quinlan Older and Consoer* The data presented in Mr. Spangler's paper and his discussion thereof is of much interest

Of particular interest is the matter of the modulus of subgrade resistance, the values of which as derived from the tests, Mr Spangler cautiously calls "apparent modulus of subgrade resistance " He finally emphasizes the need of adequate fundamental knowledge of the relationship between subgrade pressures and deflections, and frankly states that the studies did not produce satisfactory results

It is of interest to note that, after a number of cycles of load applications to the corner of the test slab, the subgrade under the corner was visibly depressed and the apparent modulus of subgrade resistance, as computed by a formula that assumes subgrade resistance to be proportional to slab deflection, decreased from 275 to a minimum of about 40 The unit strains, measured along the bisector, increased substantially as the number of load cycles and the permanent depression of the subgrade increased If, under laboratory conditions, the subgrade modulus may appear to vary 700 percent, what are we to expect under field conditions?

In a pavement slab subject to heavy traffic, various cycles of loading are repeated constantly, not only at corners but along continuous paths In the Bates Test Road permanent depression of the subgrade, caused by the test traffic, was noted not only at corners but continuously along the edge of the pavement This probably caused an unsymmetrical subgrade depression with respect to corners which might distinctly influence the location of the actual subgrade reaction and the locus of maximum stress There is no good reason to assume that similar ruts do not exist under pavement slabs in service

In the case of actual pavement slabs, which warp substantial amounts due to the inevitable alternate warming and cooling of the top surface, the apparent permanent subgrade depression at corners and edges as noted at night may be several times as great as that observed during the day

The deflection of Bates Road corners caused by test loads applied at night was often more than three times as great as that caused by the same loads applied during the day (See Bulletin No 18, Ill Division of Highways, Department of Public Works and Buildings)

The vital importance of warping, which may also be coupled with a permanent local depression of the subgrade, may be illustrated by the following example After reading the elevation of the corner of a 7-in slab with an Ames dial at 3 P M a 6000 lb load was applied and a deflection of 0 02 in noted Upon release of the load the corner recovered its original elevation At midnight the elevation of the unloaded corner was again read and found to be 0 08 in higher than its unloaded 3 P M position The 6000 lb load was again applied and while at this time the deflection was 0 07 in , the elevation of the corner was still 0 01 in higher than its 3 P M unloaded position, and 0 03 in higher than its 3 P M loaded position In other words the deflection under load at night was not sufficient to depress the slab to contact with the subgrade at the corner, and the reaction at that point was evidently zero This is but one of a great many similar observations

In as much as the warping of out of door concrete slabs is probably a universal phenomenon, and permanent subgrade depression may also be present, it is obvious that the use of any formula for stress, that presupposes subgrade reaction to be in proportion to slab deflection, is decidedly illogical in connection with the design of pavements It also seems apparent that if reasonably correct values of the subgrade modulus are necessary in design formulas, the determination of the modulus from slab deflections can not be correctly made by the use of a formula that is based upon the assumption that subgrade resistance is in proportion to slab deflection It would also seem apparent that if the fundamental relations of load, subgrade reaction, and slab stress that apply in service roads are to be confirmed by experiment, the test slabs should be subjected to actual or at least simulated traffic

It is interesting to note the position of the iso-deflection lines, Figure 20, in connection with the location of the radial strain gauge measurements as indicated in Figure 2

Figure 20 indicates that the direction of maximum strains under corner loading is not along radial lines except along the bisector of the angle Near the corner the iso-deflection lines are nearly straight and at right angles to the bisector of the corner angle. In all probability therefore the gauges, except when placed along the bisector, crossed the direction of maximum stress at angles varying from zero to approximately 45 degrees It seems practically certain that except along the bisector, the unit strain computed on the basis of the full gauge length at any given gauge position is less than the maximum in that locality At the edges of the slab and near the corner the difference might be as much as 40 per cent

If the unit strains should be redetermined by using the components of the gauge lengths that are normal to the iso-deflection lines, Figure 20, the iso-strain lines, Figure 8, would all intersect the edges of the slab This would probably represent a reasonably true picture of the actual strains, and the locus of the maximum strain for the 3000 pound load would be nearly parallel to the final fracture under the ultimate load

Mr Spangler's tests clearly illustrate the inconsistent results that must surely follow the assumption that subgrade reaction is everywhere in proportion to slab deflection His findings also indicate that the assumption that corners of pavement slabs must often act as unsupported cantilevers in supporting loads, is entirely reasonable and certainly closer to the truth than the assumption that subgrade reaction is greatest under the corner where the deflection is greatest

He calls attention to the fact that breaks that appear near the corner are likely to occur along straight lines normal to the bisector and the iso-deflection lines of his Fig 20 indicate that this should be so This, together with the demonstrated fact that the slab may be entirely unsupported for some distance from the corner, indicates that of the three assumptions made in the derivation of the simple corner formula $s = 3W/d^2$, only the one in regard to the location of the load is not justified

As to the position of load, the assumption that it is applied at the extreme corner is, in the present day of pneumatic tires, much more of an approximation than it was in the day of solid tires which were capable of imposing the load upon a very small area When, because of the lack of subgrade support for some distance under the corner, the corner acts as a simple cantilever, the bending moment equals the load times the distance from the stressed section to the center of pressure regardless of the area of contact It is then a simple matter to correct the length of the moment arm for the actual position of the center of the wheel load and use the simple beam formula to find the stress at any distance from the corner as far back as the subgrade affords no substantial reaction To determine where the subgrade reaction begins under such adverse conditions as are apt commonly to occur in service pavements is likely to prove to be difficult

In spite of the assumption made in the simple corner formula that the load is applied at the extreme corner, when it is applied to Mr Spangler's test slab, it is found to fit the maximum stresses found experimentally very satisfactorily For a load of 3000 pounds it predicts a stress of 250 pounds per square inch which seems to correspond exactly with the high point of Mr. Spangler's theoretical curve for maximum stress, Figure 18

MR SPANGLER, *Author's Closure*: Mr Older's comments relative to this paper are greatly appreciated, not only because of his eminence in the field of highway engineering, but because they serve to focus attention upon some of the more important factors in this study concerning which many questions are yet unanswered

In regard to his comment relative to the use of a modulus of subgrade reaction and an assumption that subgrade reaction is proportional to slab deflection, there is ample evidence that this assumption is not correct

However, it seems equally true that the opposite extreme, e g , that the corner of a pavement slab acts as an unsupported cantilever beam, is also incorrect Even when the extreme corner of a loaded slab is not in contact with the subgrade, the reaction will, in all probability, be distributed non-uniformly over an area which begins at some undetermined distance back from the corner and the intensity and distribution of the reaction will bear some relationship to slab deflection, even though not directly proportional to it It seems that the true situation lies somewhere between these two extremes

The use of an "apparent modulus of subgrade reaction" is merely an empirical means of evaluating the relationship between slab deflection and subgrade reaction, and a device for bringing an actual unknown situation into the realm of an idealized situation for purposes of analysis

One of the important results of this test is the indication that a large decrease in the value of the apparent modulus (from 250 to 40 lb per sq in) caused by the permanent depression of the subgrade under repeated loading, was accompanied by a relatively small increase in stress, about 20 to 25 percent as shown in Figures 18 and 19 This evidence is supported by the findings of the U S Bureau of Public Roads as reported on page 188 of *Public Roads* for November 1935, which was not available at the time the paper was written

Mr. Older suggests that the stresses reported in the paper, which were obtained by multiplying the strains measured in a radial direction by the modulus of elasticity of the concrete, are probably not the maximum stresses, especially in regions away from the corner bisector This brings out the question of the relationship between radial stresses obtained in this manner and the principal stresses in the corner region In a study of a slab tested later at Ames, which had not been completed when the paper was written, this matter was made a major objective, and very extensive strain measurements on bi-axial and tri-axial rosettes were made, from which the magnitude and direction of the principal stresses over the corner region have been determined The results of this study are now available (February 1936) and they indicate the following general conclusions

The principal stresses along the 45 degree bisector coincide with the radial stresses in direction, due to symmetry about this axis, and are roughly 10 to 12 percent less in magnitude Along the 22 5 and the 67 5 degree elements, the principal stresses bend toward the bisector, that is, tend to become parallel with it The angle of this deviation varies from a maximum of about 17 5 degrees with the radial elements, to a minimum of about 9 5 degrees at 2 5 feet from the corner In magnitude, the principal stresses crossing these elements are equal to or slightly greater than the radial stresses At the edges of the slab, the principal stresses and the radial stresses are coincident, both in direction and magnitude The maximum principal stresses at the edges of the slab are about 68 percent of the maximum principal stress along the bisector

In conclusion, the author wishes to express a high degree of confidence in the results obtained by the use of Mr Older's simple corner stress formula, although disagreeing strongly with the premises upon which it was derived It is felt that the principal result to date of the studies being conducted at Ames has been the partial reconciliation brought about between the results of the Bates Road Tests and an analysis based upon the concepts relative to subgrade reactions, slab deflections, and load position as advanced by Dr Westergaard It is the author's belief that the discrepancy between stresses obtained by Westergaard's corner formula and those indicated by the Bates Tests lies primarily in the inadequacy of the assumption that the locus of maximum stress in the corner region is a straight line normal to the bisector and that the stress is uniformly distributed along this locus