STRESS DISTRIBUTION IN A LOADED SOIL WITH SOME RIGID BOUNDARIES

By Gerald Pickett

Associate Professor of Applied Mechanics, Kansas State College

The stress distribution due to a concentrated load on a semi-infinite elastic body may be found by the Boussinesq equation. The stress distribution due to some other loading may be obtained by integration of the Boussinesq equation. A collection of the resulting formulas for several special cases may be found in the Proceedings of the International Conference of Soil Mechanics, Vol. II, Page 157, 1936. A chart has been prepared by Newmark from which one may obtain the vertical pressure by graphical integration (Illinois Circular No. 24).

This assumes a body with uniform elastic properties. Any change in elastic properties such as increasing Young's Modulus with depth of soil or a discontinuity such as a rock layer, sand or muck from natural causes, or artificial discontinuities such as piers, pilings, bridge abutments or retaining walls will cause the actual stresses to differ from the values obtained by the Boussinesq equation. So far as is known, very little has been done with the problem of variable elastic constants as this would complicate the equations of elasticity. Various investigators have considered the problem of discontinuities. A horizontal rock surface underlying a clay layer has received the most attention. We shall divide this problem into two cases as follows:

Case (a), The rigid rock is frictionless.

Case (b), There is sufficient friction to prevent slipping.

Case (a) has been solved by several writers.[1] We shall refer the reader to these works for the theoretical treatment. There is still need for curves or charts to make these equations usable for the engineer. On the other hand, the writer is of the opinion that this case has limited applications in the field of soil mechanics.

Case (b) appears to be more nearly in accord with actual conditions at the rock surface. The remainder of this paper will be concerned with this case. Its solution is much more difficult than Case (a). Carothers[1] who has a correct solution for Case (a) has an incorrect solution to Case (b) presented at the same Mathematical Congress in 1924. Timoshenko objected to Carothers' solution of Case (b) by oral discussion at the time, but the error was not generally known until very recently.[2]

The first correct solution for this case for plane strain was given by Marguerre.[3] His theory is rather complete but his equations are not in a form that can be readily used.

Biot[4] has solved the problem correctly for not only plane strain but also for the

[1] Filon solved the two dimensional case for a live load. Phil. Transactions of the Royal Society Series A, Vol. 201, 1903, Page 107.

Carothers solved the problem for a strip load, Proceedings of International Math. Congress, Vol. 2, Toronto, Canada.

Timoshenko gives a more general mathematical treatment based on Filon's work, Theory of Elasticity, Page 45.

Biot solved the axial symmetrical three-dimensional problem, Physics, Dec. 1935.

[2] Jurgenson apparently unaware of the error, assumes Carothers' solution to be correct and derives formulas for other loadings which he gives in his paper in the July, 1934 Journal of the Boston Society of Engineers, as well as on page 194, Vol. 2, Proceedings of the International Conference of Soil Mechanics, 1936.

[3] Marguerre, "Druckverteilung durch eine elastische Schicht auf starrer rauher Unterlage." Ing. Archiv 2 (1931).

[4] Biot, "Effect of Certain Discontinuities on the Pressure Distribution in a Loaded Soil." Physics, Dec. 1935.

three-dimensional case of axial symmetry. Since he restricts his paper to an incompressible soil, and he is only concerned with vertical pressure at the rock surface it was felt that there was a need for the complete stress analysis which follows:

Stresses are specified at the ground surface, and displacements are specified at the rock surface. It will therefore be necessary to have expressions for both displacements and stresses. We shall start with equations for displacements and find stresses by differentiation. The equations of equilibrium in rectangular coordinates in terms of the displacements u, v, w are:[5]

$$\left.\begin{aligned} &G\Delta u + (\lambda + G)\frac{\partial \delta}{\partial x} = 0, \qquad G\Delta v + (\lambda + G)\frac{\partial \delta}{\partial y} = 0 \\ &G\Delta w + (\lambda + G)\frac{\partial \delta}{\partial z} = 0 \end{aligned}\right\} \quad (1)$$

Where Δ is the Laplace operator, $\delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ is the volume expansion, and λ and G are the elastic constants of Lamé, related by $\lambda = \frac{2\mu}{1 - 2\mu} G$ where μ is Poisson's ratio.

PLANE STRAIN

For plane strain we shall assume $v = 0$ and let

$$u = \sin kx[Ae^{kz} + Be^{-kz} + zCe^{kz} + zDe^{-kz}] \quad (2)$$

Then it may be shown by substitution that the equilibrium equations are satisfied by:

$$w = -\cos kx\left[Ae^{kz} - Be^{-kz} + \frac{kz - 3 + 4\mu}{k}Ce^{kz} - \frac{kz + 3 - 4\mu}{k}De^{-kz}\right] \quad (3)$$

Since[6]

$$\left.\begin{aligned} S_{zz} &= \lambda\delta + 2G\frac{\partial w}{\partial z} \\ S_{xx} &= \lambda\delta + 2G\frac{\partial u}{\partial x} \\ S_{xz} &= G\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] \end{aligned}\right\} \quad (4)$$

by differentiation we obtain:

$$\left.\begin{aligned} S_{zz} &= 2G\cos kx[(2 - 2\mu - kz)Ce^{kz} - (2 - 2\mu + kz)De^{-kz} - k(Ae^{kz} + Be^{-kz})] \\ S_{xx} &= 2G\cos kx[k(Ae^{kz} + Be^{-kz}) + kz(Ce^{kz} + De^{-kz}) + 2\mu(Ce^{kz} - De^{-kz})] \\ S_{xz} &= 2G\sin kx[k(Ae^{kz} - Be^{-kz}) + kz(Ce^{kz} - De^{-kz}) - (1 - 2\mu)(Ce^{kz} + De^{-kz})] \end{aligned}\right\} \quad (5)$$

Taking the origin of coordinates at the rock surface as shown in Figure 1, the boundary conditions are as follows:

$$u = w = 0 \quad \text{at} \quad z = 0, \qquad S_{xz} = 0 \quad \text{at} \quad z = -h$$

[5] Timoshenko, Theory of Elasticity, Page 200.
[6] Timoshenko, Theory of Elasticity, pages 10 and 11.

Substitution of these three boundary conditions in the above equations gives:

$$B = -A$$

$$C = \frac{-\cosh kh + \dfrac{1 - 2\mu - kh}{3 - 4\mu} e^{kh}}{(1 - 2\mu) \sinh kh - kh \cosh kh} kA$$

$$D = \frac{\cosh kh - \dfrac{1 - 2\mu + kh}{3 - 4\mu} e^{-kh}}{(1 - 2\mu) \sinh kh - kh \cosh kh} kA$$

The equations for stresses now become:

$$\left.\begin{aligned}
S_{zz} = {} & \frac{4GkA \cos kx}{kh \cosh kh - (1 - 2\mu) \sinh kh}\Bigg[2(1 - \mu) \cosh kh \cosh kz \\
& - kz \cosh kh \sinh kz - \frac{(2 - 2\mu)(1 - 2\mu) + kz\,kh}{3 - 4\mu} \cosh k(h + z) \\
& + \frac{(1 - 2\mu)kz + 2kh(1 - \mu)}{3 - 4\mu} \sinh k(h + z) - (kh \cosh kh \\
& - (1 - 2\mu) \sinh kh) \sinh kz\Bigg] \\
S_{xx} = {} & \frac{4GkA \cos kx}{kh \cosh kh - (1 - 2\mu) \sinh kh}\Bigg[\frac{kh\,kz - 2\mu(1 - 2\mu)}{3 - 4\mu} \cosh k(h + z) \\
& + \left(\frac{2\mu kh - kz(1 - 2\mu)}{3 - 4\mu}\right) \sinh k(h + z) + kz \cosh kh \sinh kz) \\
& + 2\mu \cosh kh \cosh kz + (kh \cosh kh - (1 - 2\mu) \sinh kh) \sinh kz\Bigg] \\
S_{xz} = {} & \frac{4GkA \sin kx}{kh \cosh kh - (1 - 2\mu) \sinh kh}\Bigg[\frac{(1 - 2\mu)^2 + kz\,kh}{3 - 4\mu} \sinh k(h + z) \\
& - \frac{(1 - 2\mu)k(h + z)}{3 - 4\mu} \cosh k(h + z) + kz \cosh kh \cosh kz \\
& - (1 - 2\mu) \cosh kh \sinh kz + (kh \cosh kh - (1 - 2\mu) \sinh kh) \cosh kz\Bigg]
\end{aligned}\right\} \quad (6)$$

If we let

$$q = 4GkA \frac{\left[2(1 - \mu) \cosh^2 \alpha + (1 - 2\mu) \sinh^2 \alpha + \dfrac{\alpha^2 - 2(1 - \mu)(1 - 2\mu)}{3 - 4\mu}\right]}{\alpha \cosh \alpha - (1 - 2\mu) \sinh \alpha}$$

Where $\alpha = kh$, and substitute $z = -h$ in the above equations for stress we find the normal load S_{zz} at the ground surface to be $S_{zz} = q \cos kx$. By means of Fourier analysis we may reqresent any loading as a superposition of loads like the above. A concentrated load P at $x = 0$ is given by

$$S_{zz} = \frac{P}{\pi} \int_0^\infty \cos kx\, dk.$$

This means that we substitute $\frac{P}{\pi}\,dk$ for q and integrate from zero to infinity in each of the stress equations.

We now have:

$$
\left.
\begin{aligned}
S_{zz} = \frac{P}{\pi h}\int_0^\infty \Bigg[& 2(1-\mu)\cosh\alpha\cosh\frac{\alpha z}{h} - \frac{\alpha z}{h}\cosh\alpha\sinh\frac{\alpha z}{h} \\
& - \frac{2(1-\mu)(1-2\mu) + \frac{\alpha^2 z}{h}}{3-4\mu}\cosh\alpha\left(1+\frac{z}{h}\right) \\
& + \frac{(1-2\mu)\alpha\frac{z}{h} + 2\alpha(1-\mu)}{3-4\mu}\sinh\alpha\left(1+\frac{z}{h}\right) \\
& + (\alpha\cosh\alpha - (1-2\mu)\sinh\alpha)\sinh\frac{\alpha z}{h}\Bigg]\frac{\cos\frac{\alpha x}{h}}{M}\,d\alpha \\
S_{xx} = \frac{P}{\pi h}\int_0^\infty \Bigg[& \frac{\alpha^2\frac{z}{h} - 2\mu(1-2\mu)}{3-4\mu}\cosh\alpha\left(1+\frac{z}{h}\right) \\
& + \left(\frac{2\mu\alpha - (1-2\mu)}{3-4\mu}\alpha\frac{z}{h}\right)\sinh\alpha\left(1+\frac{z}{h}\right) + \alpha\frac{z}{h}\cosh\alpha\sinh\alpha\frac{z}{h} \\
& + 2\mu\cosh\alpha\cosh\alpha\frac{z}{h} + (\alpha\cosh\alpha - (1-2\mu)\sinh\alpha)\sinh\alpha\frac{z}{h}\Bigg]\frac{\cos\alpha\frac{x}{h}}{M}\,d\alpha \\
S_{xz} = \frac{P}{\pi h}\int_0^\infty \Bigg[& \frac{(1-2\mu)^2 + \alpha^2\frac{z}{h}}{3-4\mu}\sinh\alpha\left(1+\frac{z}{h}\right) \\
& - \frac{(1-2\mu)\alpha\left(1+\frac{z}{h}\right)}{3-4\mu}\cosh\alpha\left(1+\frac{z}{h}\right) \\
& + \alpha\frac{z}{h}\cosh\alpha\cosh\alpha\frac{z}{h} - (1-2\mu)\cosh\alpha\sinh\alpha\frac{z}{h} \\
& + (\alpha\cosh\alpha - (1-2\mu)\sinh\alpha)\cosh\alpha\frac{z}{h}\Bigg]\frac{\sin\frac{\alpha x}{h}}{M}\,d\alpha \\
\text{where} \qquad & \\
M = 2(1-\mu)\cosh^2\alpha & + (1-2\mu)\sinh^2\alpha + \frac{\alpha^2 - 2(1-\mu)(1-2\mu)}{3-4\mu}.
\end{aligned}
\right\} \quad (7)
$$

If we let $z = -h$ we find, of course, that $S_{xz} = 0$ for all values of x; and $S_{zz} = 0$ for all values of x except $x = 0$, where S_{zz} is infinite. The total value of this concentrated load is P The horizontal stress is not zero but is given by.

$$S_{xx} = \frac{P}{\pi h}\int_0^\infty \left[2\mu \cosh^2\alpha + (1-2\mu)\sinh^2\alpha - \frac{\alpha^2 + 2\mu(1-2\mu)}{3-4\mu}\right]\frac{\cos\frac{\alpha x}{h}}{M}\,d\alpha \qquad (8)$$

If we let $z = 0$ we find the following stresses at the rock surface.

$$\left.\begin{aligned} S_{zz} &= \frac{P}{\pi h}\int_0^\infty \left[\frac{4(1-\mu)^2}{3-4\mu}\cosh\alpha + \frac{2(1-\mu)}{3-4\mu}\alpha\sinh\alpha\right]\frac{\cos\frac{\alpha x}{h}}{M}\,d\alpha \\ S_{xx} &= \frac{P}{\pi h}\int_0^\infty \frac{2\mu}{3-4\mu}\left[2(1-\mu)\cosh\alpha + \alpha\sinh\alpha\right]\frac{\cos\frac{\alpha x}{h}}{M}\,d\alpha \\ S_{xz} &= \frac{P}{\pi h}\int_0^\infty \left[\frac{2(1-\mu)}{3-4\mu}\alpha\cosh\alpha - \frac{2(1-\mu)(1-2\mu)}{3-4\mu}\sinh\alpha\right]\frac{\sin\frac{\alpha x}{h}}{M}\,d\alpha \end{aligned}\right\} \qquad (9)$$

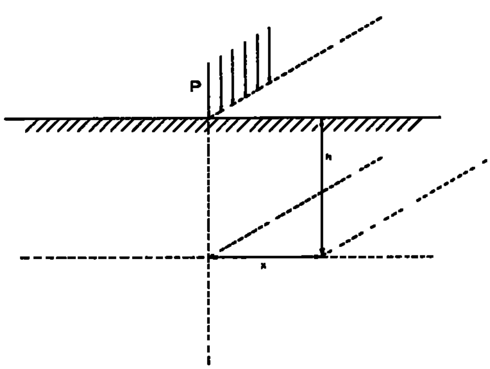

Figure 1

AXIAL SYMMETRY

Consider now the three dimensional case of axial symmetry Using the usual cylindrical coordinates r, θ, z, the problem becomes very similar to the two dimensional case, since displacements and stresses depend only on the two coordinates r and z In fact, it is proper in many respects to treat this case as two dimensional The differential equations and their solutions are similar though not so well known, because Bessel instead of circular functions are involved

Again letting $v = 0$ the two necessary equilibrium equations in terms of the displacements u and w are

$$\left.\begin{aligned} G\left(\Delta u - \frac{u}{r^2}\right) + (\lambda + G)\frac{\partial\delta}{\partial r} &= 0 \\ G\Delta w + (\lambda + G)\frac{\partial\delta}{\partial z} &= 0 \end{aligned}\right\} \qquad (1a)$$

where δ now becomes $\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}$

$$u = J_1(kr)\,[Ae^{kz} + Be^{-kz} + zce^{kz} + zDe^{-kz}] \qquad (2a)$$

and

$$w = -J_0(kr)\left[Ae^{kz} - Be^{-kz} + \frac{kz - 3 + 4\mu}{k}Ce^{kz} - \frac{kz + 3 - 4\mu}{k}De^{-kz}\right] \quad (3a)$$

satisfy equations (1a). The stresses in terms of displacements are as follows:

$$\left.\begin{aligned} S_{zz} &= \lambda\delta + 2G\frac{\partial w}{\partial z} \\ S_{rr} &= \lambda\delta + 2G\frac{\partial u}{\partial r} \\ S_{\theta\theta} &= \lambda\delta + 2G\frac{u}{r} \\ S_{rz} &= G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right). \end{aligned}\right\} (4a)$$

By differentiation we obtain:

$$S_{zz} = 2GJ_0(kr)[(2 - 2\mu - kz)Ce^{kz} - (2 - 2\mu + kz)De^{-kz} - k(Ae^{kz} + Be^{-kz})]$$

$$S_{rr} = 2G\left[J_0(kr) - \frac{J_1(kr)}{kr}\right]k[Ae^{kz} + Be^{-kz} + zCe^{kz} + zDe^{-kz}] + 4G\mu J_0(kr)[Ce^{kz} - De^{-kz}]$$

$$S_{\theta\theta} = 2G\frac{J_1(kr)}{r}[Ae^{kz} + Be^{-kz} + zCe^{kz} + zDe^{-kz}] + 4G\mu J_0(kr)[Ce^{kz} - De^{-kz}]$$

$$S_{rz} = 2GJ_1(kr)[k(Ae^{kz} - Be^{-kz}) + kz(Ce^{kz} - De^{-kz}) - (1 - 2\mu)(Ce^{kz} + De^{-kz})].$$

For the boundary conditions which we have assumed the integration constants are related in exactly the same way as for plane strain. We shall omit a portion of the tedious mathematical work and proceed to the final equation for special cases.

SPECIAL CASES

We shall evaluate the stresses at the ground and rock surfaces for certain values of of Poisson's ratio for both Plane Strain and Axial Symmetry as follows:

Plane Strain.

$\mu = 0, z = -h$

$$S_{xx} = \frac{P}{\pi h}\int_0^\infty \frac{\sinh^2\alpha - \frac{\alpha^2}{3}}{2\cosh^2\alpha + \sinh^2\alpha + \frac{\alpha^2}{3} - \frac{2}{3}}\cos\alpha\frac{x}{h}\,d\alpha$$

$\mu = 0 \quad z = 0$

$$S_{zz} = \frac{P}{\pi h}\int_0^\infty \frac{\frac{4}{3}\cosh\alpha + \frac{2}{3}\alpha\sinh\alpha}{2\cosh^2\alpha + \sinh^2\alpha + \frac{\alpha^2}{3} - \frac{2}{3}}\cos\alpha\frac{x}{h}\,d\alpha$$

$$S_{xz} = \frac{P}{\pi h}\int_0^\infty \frac{\frac{2}{3}\alpha\cosh\alpha - \frac{2}{3}\sinh\alpha}{2\cosh^2\alpha + \sinh^2\alpha + \frac{\alpha^2}{3} - \frac{2}{3}} \sin\alpha\frac{x}{h}\,d\alpha$$

$\mu = \frac{1}{4}, z = -h$

$$S_{xx} = \frac{P}{\pi h}\int_0^\infty \frac{\cosh^2\alpha + \sinh^2\alpha - \alpha^2 - \frac{1}{4}}{3\cosh^2\alpha + \sinh^2\alpha + \alpha^2 - \frac{3}{4}} \cos\frac{\alpha x}{h}\,d\alpha$$

$\mu = \frac{1}{4}, z = 0$

$$S_{zz} = \frac{P}{\pi h}\int_0^\infty \frac{\frac{9}{4}\cosh\alpha + \frac{3}{2}\alpha\sinh\alpha}{3\cosh^2\alpha + \sinh^2\alpha + \alpha^2 - \frac{3}{4}} \cos\alpha\frac{x}{h}\,d\alpha$$

$$S_{xx} = \frac{P}{\pi h}\int_0^\infty \frac{\frac{3}{4}\cosh\alpha + \alpha\sinh\alpha}{3\cosh^2\alpha + \sinh^2\alpha + \alpha^2 - \frac{3}{4}} \cos\alpha\frac{x}{h}\,d\alpha$$

$$S_{xz} = \frac{P}{\pi h}\int_0^\infty \frac{\frac{3}{2}\alpha\cosh\alpha - \frac{3}{4}\sinh\alpha}{3\cosh^2\alpha + \sinh^2\alpha + \alpha^2 - \frac{3}{4}} \sin\alpha\frac{x}{h}\,d\alpha$$

$\mu = \frac{1}{2}; z = -h$

$$S_{xx} = \frac{P}{\pi h}\int_0^\infty \frac{\cosh^2\alpha - \alpha^2}{\cosh^2\alpha + \alpha^2} \cos\frac{\alpha x}{h}\,d\alpha$$

$\mu = \frac{1}{2}; z = 0$

$$S_{zz} = S_{xx} = \frac{P}{\pi h}\int_0^\infty \frac{\cosh\alpha + \alpha\sinh\alpha}{\cosh^2\alpha + \alpha^2} \cos\frac{\alpha x}{h}\,d\alpha$$

$$S_{xz} = \frac{P}{\pi h}\int_0^\infty \frac{\alpha\cosh\alpha}{\cosh^2\alpha + \alpha^2} \sin\frac{\alpha x}{h}\,d\alpha$$

Axial Symmetry.

$\mu = 0; z = -h$

$$S_{rr} = \frac{P}{2\pi h^2}\int_0^\infty \frac{\sinh^2\alpha - \frac{\alpha^2}{3}}{2\cosh^2\alpha + \sinh^2\alpha + \frac{\alpha^2}{3} - \frac{2}{3}} \left[\alpha J_0\left(\frac{\alpha r}{h}\right) - \frac{h}{r} J_1\left(\frac{\alpha r}{h}\right)\right] d\alpha$$

$$S_{\theta\theta} = \frac{P}{2\pi h^2}\int_0^\infty \frac{\sinh^2\alpha - \frac{\alpha^2}{3}}{2\cosh^2\alpha + \sinh^2\alpha + \frac{\alpha^2}{3} - \frac{2}{3}} \frac{h}{r} J_1\left(\frac{\alpha r}{h}\right) d\alpha$$

$\mu = 0; z = 0$

$$S_{zz} = \frac{P}{2\pi h^2}\int_0^\infty \frac{\frac{4}{3}\cosh\alpha + \frac{2}{3}\alpha\sinh\alpha}{2\cosh^2\alpha + \sinh^2\alpha + \frac{\alpha^2}{3} - \frac{2}{3}} \alpha J_0\left(\frac{\alpha r}{h}\right) d\alpha$$

$$S_{rz} = \frac{P}{2\pi h^2} \int_0^\infty \frac{\frac{2}{3}\alpha \cosh\alpha - \frac{2}{3}\sinh\alpha}{2\cosh^2\alpha + \sinh^2\alpha + \frac{\alpha^2}{3} - \frac{2}{3}} \alpha J_1\left(\frac{\alpha r}{h}\right) d\alpha$$

$\mu = \frac{1}{4}; z = -h$

$$S_{rr} = \frac{P}{2\pi h^2} \int_0^\infty \left[\frac{(\sinh^2\alpha - \alpha^2)\left(\alpha J_0 \frac{\alpha r}{h} - \frac{h}{r} J_1 \frac{\alpha r}{h}\right) + \left(\cosh^2\alpha - \frac{1}{4}\right)\alpha J_0 \frac{\alpha r}{h}}{3\cosh^2\alpha + \sinh^2\alpha + \alpha^2 - \frac{3}{4}} \right] d\alpha$$

$$S_{\theta\theta} = \frac{P}{2\pi h^2} \int_0^\infty \left[\frac{(\sinh^2\alpha - \alpha^2)\frac{h}{r} J_1 \frac{\alpha r}{h} + \left(\cosh^2\alpha - \frac{1}{4}\right)\alpha J_0\left(\frac{\alpha r}{h}\right)}{3\cosh^2\alpha + \sinh^2\alpha + \alpha^2 - \frac{3}{4}} \right] d\alpha$$

$\mu = \frac{1}{4}; z = 0$

$$S_{zz} = \frac{P}{2\pi h^2} \int_0^\infty \frac{\frac{9}{4}\cosh\alpha + \frac{3}{2}\alpha\sinh\alpha}{3\cosh^2\alpha + \sinh^2\alpha + \alpha^2 - \frac{3}{4}} \alpha J_0\left(\frac{\alpha r}{h}\right) d\alpha$$

$$S_{rr} = S_{\theta\theta} = \frac{P}{2\pi h^2} \int_0^\infty \frac{\frac{3}{4}\cosh\alpha + \alpha\sinh\alpha}{3\cosh^2\alpha + \sinh^2\alpha + \alpha^2 - \frac{3}{4}} \alpha J_0\left(\alpha \frac{r}{h}\right) d\alpha$$

$$S_{rz} = \frac{P}{2\pi h^2} \int_0^\infty \frac{\frac{3}{2}\alpha\cosh\alpha - \frac{3}{4}\sinh\alpha}{3\cosh^2\alpha + \sinh^2\alpha + \alpha^2 - \frac{3}{4}} \alpha J_1\left(\alpha \frac{r}{h}\right) d\alpha$$

$\mu = \frac{1}{2}; z = -h$

$$S_{rr} = \frac{P}{2\pi h^2} \int_0^\infty \frac{(\cosh^2\alpha - \alpha^2)\,\alpha J_0\left(\alpha\frac{r}{h}\right) + \alpha^2 \frac{h}{r} J_1\left(\alpha\frac{r}{h}\right)}{\cosh^2\alpha + \alpha^2} d\alpha$$

$$S_{\theta\theta} = \frac{P}{2\pi h^2} \int_0^\infty \frac{\alpha\cosh^2\alpha J_0\left(\alpha\frac{r}{h}\right) - \alpha^2\frac{h}{r} J_1\left(\frac{\alpha r}{h}\right)}{\cosh^2\alpha + \alpha^2} d\alpha$$

$\mu = \frac{1}{2}; z = 0$

$$S_{rr} = S_{\theta\theta} = S_{zz} = \frac{P}{2\pi h^2} \int_0^\infty \frac{\cosh\alpha + \alpha\sinh\alpha}{\cosh^2\alpha + \alpha^2} \alpha J_0\left(\frac{\alpha r}{h}\right) d\alpha$$

$$S_{rz} = \frac{P}{2\pi h^2} \int_0^\infty \frac{\alpha^2\cosh\alpha}{\cosh^2\alpha + \alpha^2} J_1\left(\frac{\alpha r}{h}\right) d\alpha$$

It is believed that the above equations for stresses are mathematically exact, and certain qualitative conclusions may be drawn from them The method used to evaluate the integrals so that quantitative information may be obtained is given in the Appendix The results are summarized in the following.

Figure 2 shows the distribution of the vertical stress and horizontal shear in the case of a concentrated point load for various values of μ It is to be noted that for $\mu = 0\ 5$, the rock tends to prevent spreading of the load, producing an intensity 56 percent greater than the Boussinesq, when $r = 0$ Whereas for

$\mu = 0$, the maximum intensity is only about 60 percent of the Boussinesq.

It is of interest to note that a value of μ somewhere between 0.25 and 0.50 would give a distribution approximately the same as the Boussinesq.

Figure 3 shows a comparison of formulas for a strip loading of 2a in width, and of intensity P on an elastic material supported by a rough, rigid rock at a distance h = 2a below the loaded surface.

Curves C and D show maximum and horizontal shearing stress respectively, as obtained from integration of the Boussinesq equation, and necessarily neglects the discontinuity of the rock surface. Curve A shows the shearing stress obtained by the Carothers formula which fails to meet one boundary requirement at the rock surface. According to this formula the horizontal shear is the maximum shear at this surface. These curves are independent of μ. Curve B is the theoretically correct curve for $\mu = 0.5$. It is of interest to note that the Carothers formula has the proper shape but gives values about 20 per cent too high for $\mu = 0.5$. The Carothers formula has had considerable use in recent years, since Jurgenson[2] extended it to triangular loading, and it may readily be extended to various types, as shown in Figure 4. The author is not convinced that it is safe to assume that $\mu = 0.5$ for soil. But, if

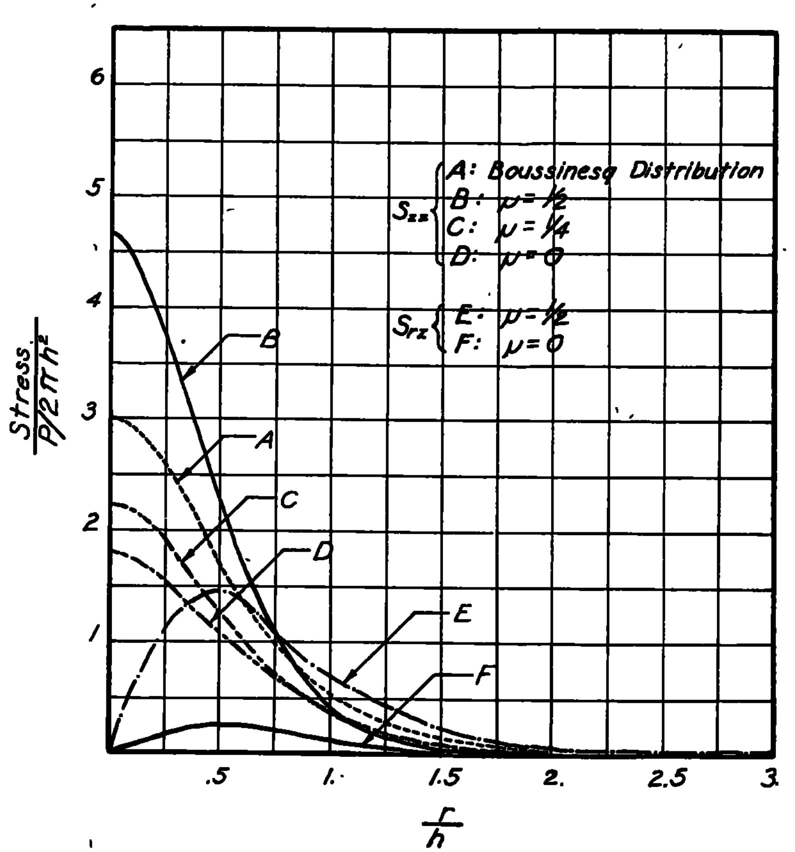

Figure 2. Pressure Distribution at a Rough Rock Surface in the Three Dimensional Problem

such could be assumed, the comparatively simple formulas of Jurgenson, based on the Carothers' assumption, would be very useful.

It was the author's intention to prepare Newmark charts from which the stresses due to any distribution of normal loading could be obtained by graphical integration. A few such charts were prepared but are omitted from the paper for the following reasons: (a) The existing Newmark chart (Illinois Circular No. 24) is sufficiently accurate for vertical stress for all values of μ from 0.25 to 0.50. (b) No method of constructing such a chart for shearing stresses was discovered. (c) It was not considered advisable to construct such charts for horizontal stresses at the present time.

In conclusion, the necessary equations have been derived for the exact determination of the stresses, vertical, horizontal or shearing, at any point in a homogeneous, isotropic, elastic solid of known Poissons' ratio, with thickness h bounded on one side by a rough, rigid rock, and carrying any known normal load on the other side.

The Appendix shows how the equations are evaluated to give numerical answers for special cases.

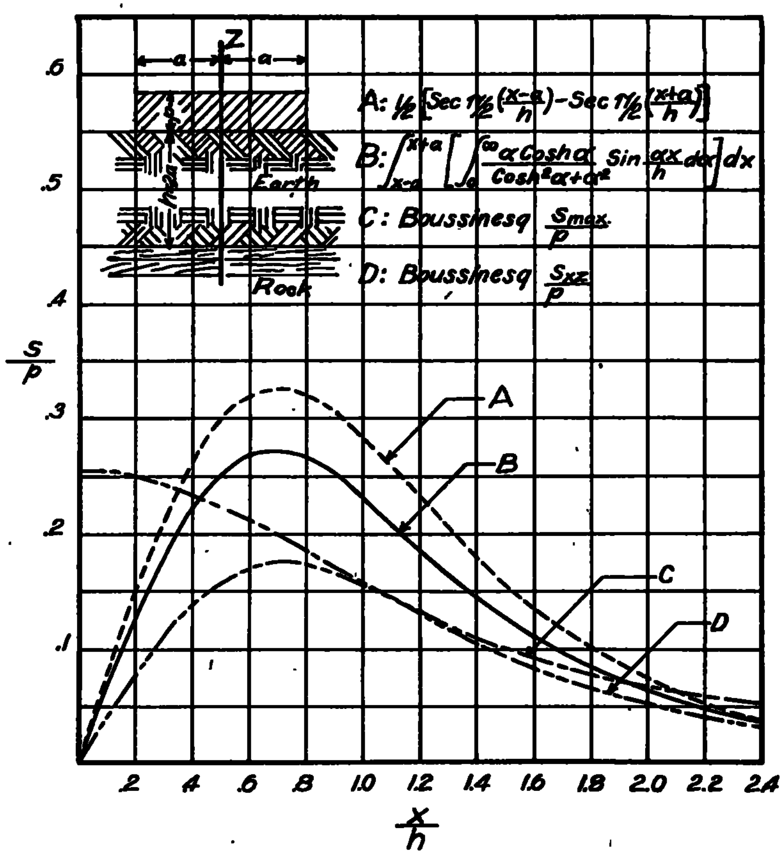

Figure 3. Shearing Stress at Depth h Below a Uniform Loaded Strip

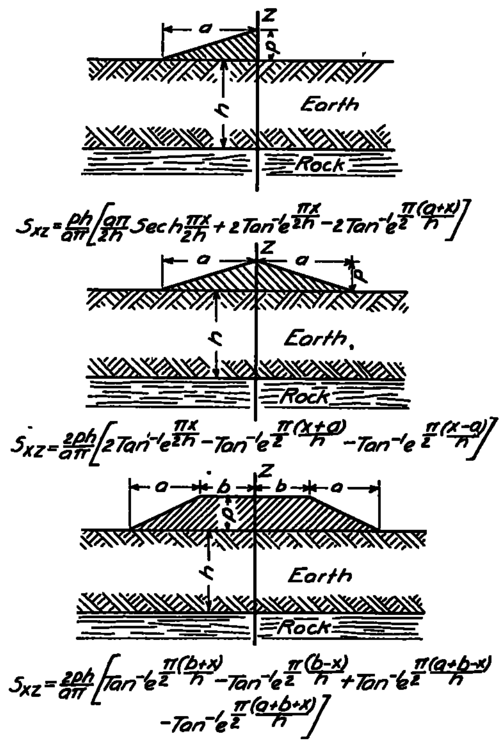

Figure 4

APPENDIX

Since exact values of the integrals involved are not known, we shall use an approximate method which may be made to approach the exact value as nearly as desired. For example, to evaluate the integrals involved for the determination of stresses at the rock surfaces for $\mu = 0.5$, $\mu = 0.25$ and $\mu = 0$, the following substitutions may be made:

$$\frac{\cosh \alpha}{\cosh^2 \alpha + \alpha^2} = 2e^{-\alpha} - e^{-2\alpha} - 0.345(\alpha^2 e^{-2\alpha} + 100\alpha^5 e^{-5.5\alpha})$$

$$\frac{\sinh \alpha}{\cosh^2 \alpha + \alpha^2} = 2e^{-\alpha} - 2(1+\alpha)\, e^{-2.5\alpha} - 5\alpha^4 e^{-4\alpha}$$

$$\frac{\cosh \alpha}{3\cosh^2 \alpha + \sinh^2 \alpha + \alpha^2 - \frac{3}{4}} = \frac{1}{2} e^{-\alpha} - \frac{1+8\alpha}{18} e^{-17\alpha} - \frac{\alpha^2}{2} e^{-3\alpha} + 7\alpha^3 e^{-6\alpha}$$

$$\frac{\sinh \alpha}{3\cosh^2 \alpha + \sinh^2 \alpha + \alpha^2 - \frac{3}{4}} = \frac{1}{2} e^{-\alpha} - \frac{9+19\alpha}{18} e^{-4\alpha} - \frac{\alpha^2}{2} e^{-3\alpha} + \frac{7\alpha^3}{4} e^{-6\alpha}$$

$$\frac{\cosh \alpha}{2\cosh^2 \alpha + \sinh^2 \alpha + \frac{\alpha^2}{3} - \frac{2}{3}} = \frac{2}{3} e^{-\alpha} + \frac{1+10\alpha}{12} e^{-2\alpha} - 7\alpha^3 e^{-4\alpha}$$

$$\frac{\sinh \alpha}{2\cosh^2 \alpha + \sinh^2 \alpha + \frac{\alpha^2}{3} - \frac{2}{3}} = \frac{2}{3} e^{-\alpha} - \frac{8+15\alpha}{12} e^{-4\alpha} - 7\alpha^5 e^{-5.5\alpha}$$

It can be seen by inspection that the above equations become identities as α approaches zero or infinity, and it can be shown by numerical check that in no case does the substituted value differ from the correct value by as much as one per cent

After making the above substitutions in the equations for stress at the rock surface we find integrals whose solutions are known The general forms are as follows.

$$\int_0^{\infty} e^{-a\alpha} \sin\left(\alpha \frac{x}{h}\right) d\alpha = \frac{\frac{x}{h}}{a^2 + \left(\frac{x}{h}\right)^2}$$

$$\int_0^{\infty} \alpha^n e^{-a\alpha} \sin\left(\alpha \frac{x}{h}\right) d\alpha = \frac{\partial^n}{\partial a^n}\left[\frac{\frac{x}{h}}{a^2 + \left(\frac{x}{h}\right)^2}\right](-1)^n$$

$$\int_0^{\infty} e^{-a\alpha} \cos\left(\alpha \frac{x}{h}\right) d\alpha = \frac{a}{a^2 + \left(\frac{x}{h}\right)^2}$$

$$\int_0^{\infty} \alpha^n e^{-a\alpha} \cos\left(\alpha \frac{x}{h}\right) d\alpha = \frac{\partial^n}{\partial a^n}\left[\frac{a}{a^2 + \left(\frac{x}{h}\right)^2}\right](-1)^n$$

$$\int_0^{\infty} e^{-a\alpha} J_1\left(\alpha \frac{r}{h}\right) d\alpha = \frac{h}{r} - \frac{ah}{r\sqrt{a^2 + \left(\frac{r}{h}\right)^2}}$$

$$\int_0^{\infty} \alpha^n e^{-a\alpha} J_1\left(\alpha \frac{r}{h}\right) d\alpha = -\frac{\partial^n}{\partial a^n}\left[\frac{ah}{r\sqrt{a^2 + \left(\frac{r}{h}\right)^2}}\right](-1)^n$$

$$\int_0^{\infty} e^{-a\alpha} J_0\left(\alpha \frac{r}{h}\right) d\alpha = \left[a^2 + \left(\frac{r}{h}\right)^2\right]^{-\frac{1}{2}}$$

$$\int_0^{\infty} \alpha^n e^{-a\alpha} J_0\left(\alpha \frac{r}{h}\right) d\alpha = \frac{\partial^n}{\partial a^n}\left[a^2 + \left(\frac{r}{h}\right)^2\right]^{-\frac{1}{2}}(-1)^n$$

Two examples will be worked out in more detail

Example I The case of Rectangular loading illustrated in Figure 3 Assuming μ = 0 50 The horizontal shear, which is the maximum shear, at the rock surface due to a line load is given by

$$S_{xz} = \frac{P}{\pi h}\int_0^{\infty} \frac{\alpha \cosh \alpha}{\cosh^2 \alpha + \alpha^2} \sin \frac{\alpha x}{h} d\alpha$$

Making the above substitution this becomes.

$$S_{xz} = \frac{P}{\pi h}\int_0^{\infty} [2\alpha e^{-\alpha} - \alpha e^{-2\alpha} - 345\alpha^3 e^{-2\alpha} - 34\ 5\alpha^6 e^{-5\ 5\alpha}] \sin\left(\frac{\alpha x}{h}\right) d\alpha$$

which upon integration becomes.

$$S_{xs} = \frac{P}{\pi h}\left[\frac{2\frac{x}{h}}{\left[1+\left(\frac{x}{h}\right)^2\right]^2} - \frac{\frac{x}{2h}}{2}\,\frac{\left[2.035 + .965\left(\frac{x}{2h}\right)^2 + \left(\frac{x}{2h}\right)^4\right]}{\left[1+\left(\frac{x}{2h}\right)^2\right]^4} - \frac{34.5 \times 720\,\frac{x}{h}}{(5.5)^8}\,\frac{\left[7 - 35\left(\frac{x}{5.5h}\right)^2 + 21\left(\frac{x}{5.5h}\right)^4 - \left(\frac{x}{5.5h}\right)^6\right]}{\left[1+\left(\frac{x}{5.5h}\right)^2\right]^7}\right]$$

Since the above is only for a line load, we must now multiply by dx and integrate between the limits (x + a) and (x − a). This integration gives:

$$S_{xs} = \frac{P}{\pi}\left[\frac{2}{1+\left(\frac{x}{h}\right)^2} - \frac{3.517 + 3.965\left(\frac{x}{2h}\right)^2 + 3\left(\frac{x}{2h}\right)^4}{6\left[1+\left(\frac{x}{2h}\right)^2\right]^3} - \frac{34.5 \times 720\left[1 - 15\left(\frac{x}{5.5h}\right)^2 + 15\left(\frac{x}{5.5h}\right)^4 - \left(\frac{x}{5.5h}\right)^6\right]}{(5.5)^7 \times 6\left[1+\left(\frac{x}{5.5h}\right)^2\right]^6}\right]_{x+a}^{x-a}$$

This is plotted as curve B in Fig. 3.

Example II. To find the vertical stress at the rock surface beneath the center of a uniformly loaded circular area of radius r.

Assuming in this case $\mu = 0$, the necessary integral is:

$$S_{ss} = 2\pi \int_0^r \left[\frac{P}{2\pi h^2}\int_0^\infty \frac{\frac{4}{3}\cosh\alpha + \frac{2}{3}\alpha\sinh\alpha}{2\cosh^2\alpha + \sinh^2\alpha - \frac{2}{3}}\,\alpha J_0\left(\frac{\alpha r}{h}\right)d\alpha\right] r\,dr$$

The part within the brackets gives the vertical stress at any distance r due to a concentrated load P. We shall find it first.

Making the substitutions as previously, the bracketed expression becomes:

$$\frac{P}{2\pi h^2}\int_0^\infty \left[\frac{8}{9}\alpha e^{-\alpha} + \frac{\alpha + 10\alpha^2}{9}e^{-2\alpha} - \frac{28}{3}\alpha^4 e^{-4\alpha} + \frac{4}{9}\alpha^2 e^{-\alpha} - \frac{8\alpha^2 + 15\alpha^3}{18}e^{-4\alpha} - \frac{14}{3}\alpha^7 e^{-5.5\alpha}\right] J_0\left(\frac{\alpha^r}{h}\right) d\alpha$$

Upon integration we obtain:

$$\frac{P}{2\pi h^2}\left[\frac{16 + 4\left(\frac{r}{h}\right)^2}{9\left[1+\left(\frac{r}{h}\right)^2\right]^{\frac{5}{2}}} + \frac{11 - 4\left(\frac{r}{2h}\right)^2}{36\left[1+\left(\frac{r}{2h}\right)^2\right]^{\frac{5}{2}}}\right.$$

$$- \frac{\left[1162 - 2973\left(\frac{r}{4h}\right)^2 + 234\left(\frac{r}{4h}\right)^4 - 32\left(\frac{r}{4h}\right)^6\right]}{4608\left[1 + \left(\frac{r}{4h}\right)^2\right]^{\frac{9}{2}}}$$

$$- \frac{0.0002508\left[112 - 936\left(\frac{r}{5.5h}\right)^2 + 1470\left(\frac{r}{5.5h}\right)^4 - 245\left(\frac{r}{5.5h}\right)^6\right]}{\left[1 + \left(\frac{r}{5.5h}\right)^2\right]^{\frac{15}{2}}}\Bigg]$$

The above is shown as Curve D in Figure 2 The other curves in this figure were obtained in like manner After making the next integration, considerable labor is involved in reducing it to the following

$$S_{rz} = P\left[\frac{4}{9}\left[2 - \frac{2 + \left(\frac{r}{h}\right)^2}{\left[1 + \left(\frac{r}{h}\right)^2\right]^{\frac{3}{2}}}\right] + \frac{1}{9}\left[1 - \frac{1 - \left(\frac{r}{2h}\right)^2}{\left[1 + \left(\frac{r}{2h}\right)^2\right]^{\frac{3}{2}}}\right]\right.$$

$$\left. - \left(\frac{r}{4h}\right)^2\left[\frac{61{,}005 - 1785\left(\frac{r}{4h}\right)^2 + 3360\left(\frac{r}{4h}\right)^4}{30{,}240\left[1 + \left(\frac{r}{4h}\right)^2\right]^{\frac{7}{2}}}\right]\right]$$

The terms containing $\left(\frac{r}{5.5h}\right)$ have been omitted because of negligible numerical value The above equation is represented by Curve C in Figure 5

Newmark charts may readily be constructed from the curves of Figure 5, in fact, Curve B gives the original Newmark chart

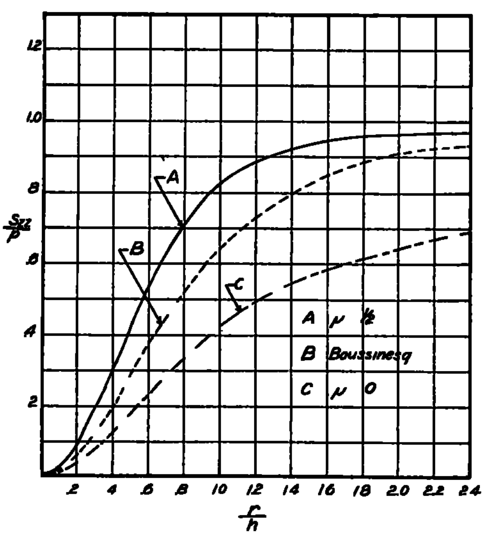

Figure 5 Vertical Stress at Rock Surface Beneath a Uniformly Loaded Circle of Radius r Newmark charts may be constructed from these curves