# SHEARING STRESSES UNDERNEATH A SPREAD FOUNDATION

By D. P. Krynine

*Research Associate in Soil Mechanics, School of Engineering, Yale University*

The only scientific tool at our disposal for determining the stresses underneath a foundation is the mathematical theory of elasticity which deals with homogeneous elastically isotropic bodies. Soils are neither homogeneous nor elastically isotropic; in addition, besides the applied forces and body forces as considered in the theory of elasticity—mechanical action of moisture percolating through a mass must be taken into consideration. A great difficulty in stress computation is the existence of the so-called "disturbed zone" under the structure itself. Physical properties of this zone undergo a certain change during the process of loading; hence elastic constants of the material in the "disturbed zone" also change. Formulas of the theory of elasticity, however, have been developed under the assumption that the elastic constants of the loaded body are the same before, during and after loading. Consequently, the use of elastic formulas is very questionable if stresses are to be determined within the "disturbed zone." Apparently, the deeper the layer, the more correct is the use of elastic formulas and in addition for determining stresses acting at a great depth the distributed load applied at the surface may be replaced by a concentrated force. The latter statement may be proved by the Saint Venant principle. According to this principle, stresses remote from the boundary where a system of forces is applied, differ but very little from those due to the resultant of this system. The principle in question is valid for determining stresses in an earth mass at a great depth.

The most important stresses in foundation engineering are the vertical pressure and the maximum shearing stress. The horizontal pressure is of importance in particular cases, for instance, in designing a retaining wall. From the two most important stresses mentioned above one may assume the rôle of "controlling stress." For instance, in the case of a heavy bridge pier founded at a considerable depth, there is practically no shear danger; but owing to the presence of a still deeper soft layer, if any, there may be settlement due to consolidation of that layer. Hence the controlling stress in this case is the vertical pressure used to compute settlements which are due to consolidation. In the case of a shallow foundation, the shear danger is present, especially if the structure is simply placed on the earth surface, as an earth dam or a highway embankment. In such a case the controlling stress is the maximum shear.

Since stresses cannot be computed exactly owing to the difference between an elastically isotropic body and the actual earth mass, it is not worth while for a foundation engineer to compute them very accurately using complicated formulas. In the case when the shear stress is the stress controlling the situation, all that is needed, is a rough estimation of the maximum shearing stress and its checking against the shearing strength (shearing value) of the given material. In this paper a simplified method is given for determining the maximum shearing stress at a point of a semi-infinite elastically isotropic body loaded at its surface by a uniformly distributed unit load acting on an arbitrary area.

BOUSSINESQ FORMULAS

According to Boussinesq,[1] if a concentrated load, P, is applied at a point, A, of the boundary of an elastically isotropic body the stresses at a point, O, of that body are (Fig. 1):

$$\left.\begin{aligned}
\sigma_x &= \frac{3P}{2\pi}\left[\frac{x^2z}{R^5} + \frac{m-2}{3m}\left(\frac{1}{R(R+z)} - \frac{(2R+z)\cdot x^2}{(R+z)^2\cdot R^3} + \frac{z}{R^3}\right)\right]\\
\sigma_y &= \frac{3P}{2\pi}\left[\frac{y^2z}{R^5} + \frac{m-2}{3m}\left(\frac{1}{R(R+z)} - \frac{(2R+z)\cdot y^2}{(R+z)^2\cdot R^3} + \frac{z}{R^3}\right)\right]\\
\sigma_z &= \frac{3P}{2\pi}\cdot\frac{z^3}{R^5}\\
\tau_{zy} &= \frac{3P}{2\pi}\cdot\frac{yz^2}{R^5}\\
\tau_{zx} &= \frac{3P}{2\pi}\cdot\frac{xz^2}{R^5}\\
\tau_{xy} &= \frac{3P}{2\pi}\left[\frac{x\cdot y\cdot z}{R^5} - \frac{m-2}{3m}\cdot\frac{(2R+z)x\cdot y}{(R+z)^2\cdot R^3}\right]
\end{aligned}\right\}\quad(1)$$

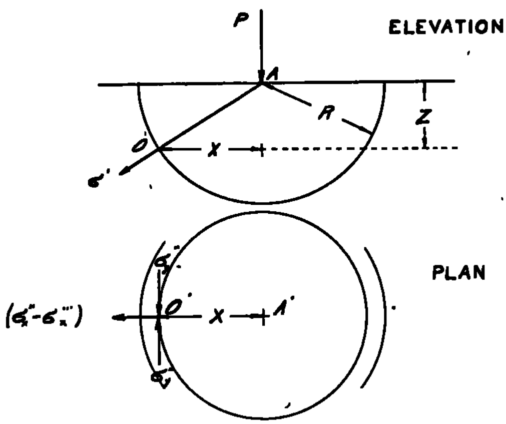


**Figure 1. Boussinesq Stress Distribution**

In formulas (1) the designations: $\sigma_x$; $\sigma_y$; $\sigma_z$ are normal stress components parallel to the axes OX, OY; OZ; respectively; $\tau_{zy}$; $\tau_{zx}$; $\tau_{xy}$ are tangential stress components (shears), the second index meaning the direction of the axis to which the given shear is parallel, and the first is the direction of the normal to the plane of action of the given shear. The value of the radius vector R equals $\sqrt{x^2 + y^2 + z^2}$ The positive sign (+) in formulas (1) means compression so far as normal stresses are concerned.

To simplify these formulas, let us imagine a vertical plane passing through point, O, and containing the vertical line of action of the load, P. Furthermore, let us take this plane for the X-Z plane. In such a case all the ordinates, y, in formulas (1) vanish, since $y = 0$. Formulas (1) may then be rewritten thus:

[1] J. Boussinesq, Application des Potentiels etc., Paris, 1885.

$$\left.\begin{aligned}
\sigma_x &= \frac{3P}{2\pi}\cdot\left[\frac{x^2z}{R^5} + \frac{m-2}{3m}\left(\frac{1}{R(R+z)} - \frac{(2R+z)\cdot x^2}{(R+z)^2\cdot R^3} + \frac{z}{R^3}\right)\right]\\
\sigma_y &= \frac{3P}{2\pi}\cdot\frac{m-2}{3m}\cdot\left(\frac{1}{R(R+z)} + \frac{z}{R^3}\right)\\
\sigma_z &= \frac{3P}{2\pi}\cdot\frac{z^3}{R^5}\\
\tau_{zx} &= \frac{3P}{2\pi}\cdot\frac{xz^2}{R^5}
\end{aligned}\right\}\quad(2)$$

MAXIMUM SHEAR IN THE RADIAL STRESS DISTRIBUTION

If Poisson's ratio of the given earth material were equal to ½, or in other words, if its reciprocal were m = 2, Formulas (2) would be:

$$\left.\begin{aligned} \sigma'_x &= \frac{3P}{2\pi}\cdot\frac{x^2 z}{R^5} \\ \sigma'_y &= 0 \\ \sigma'_z &= \frac{3P}{2\pi}\cdot\frac{z^3}{R^5} \\ \tau'_{zx} &= \frac{3P}{2\pi}\cdot\frac{xz^2}{R^5} \end{aligned}\right\} \quad (3)$$

Formulas (3) describe the components of a stress, $\sigma'$:

$$\sigma' = \frac{3P}{2\pi R^2}\cdot\frac{z}{R} \quad (4)$$

Stress, $\sigma'$, acting at point, O, radiates from point, A, that of application of the given load, P. The stress distribution described by formulas (3) is the spherical radial stress distribution; stress, $\sigma'$, is the major principal stress, the two other principal stresses being equal to zero. The value of the maximum shear in this case is:

$$\max\tau_1 = \frac{\sigma'}{2} \quad (5)$$

PRINCIPAL STRESSES IN THE RADIAL STRESS DISTRIBUTION VISUALIZED AS VOLUMES

In Figure 2 point O is that where stresses are to be determined, and MN is a hypothetical foundation of arbitrary shape, loaded with a uniformly distributed load, p. Consider an element of the area MN and trace straight lines joining all points of its perimeter with point O, thus forming an elementary solid angle, $d\omega$. Trace a sphere passing through that element of the foundation, the center of the sphere being at point O. The solid angle, $d\omega$, cuts out at the surface of the sphere an area element, $R^2 \cdot d\omega$, where R is the radius vector or the distance from point O to the element in question. The area of the element of the foundation is then $\frac{R^2\cdot d\omega}{\text{Cos }\Theta}$ where $\Theta$ is the variable angle formed by the radius vector with the vertical. Remembering that $\text{Cos }\Theta = \frac{z}{R}$, this area is $\frac{R^3\cdot d\omega}{z}$ and the value of the load acting on it is $\frac{p\cdot R^3\cdot d\omega}{z}$ The value of the stress at point O caused by the load acting on that element may be determined using Equation (4):

$$d\sigma' = \frac{3p\cdot R^3\cdot d\omega}{2\pi\cdot z}\cdot\frac{z}{R^3} = \frac{3p\cdot d\omega}{2\pi} \quad (6)$$

ELEVATION

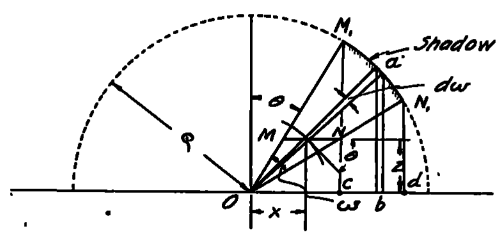


**Figure 2**

If the element of the spherical surface intersecting the given element of the foundation, is loaded with the unit load, p, acting normally to that surface, the stress at point O would be the same as determined by Formula (6). Actually, in this case z = R, from which, applying Formula (4):

$$d\sigma' = \frac{3p\cdot R^2\cdot d\omega}{2\pi}\cdot\frac{R}{R^3} = \frac{3p\cdot d\omega}{2\pi} \quad (7)$$

Formula (7) does not contain the value of R; hence it is valid for the surface element of any sphere having its center at O and which can be measured by the elementary solid angle, $d\omega$. Particularly, this is the case of the sphere of the radius,

$\rho$, as shown in Figure 2. The discussion which follows refers to the horizontal plane, but may be extended to the case of any other plane passing through point O. The elementary vertical pressure, $d\sigma_z'$, or, in other words, pressure normal to the given horizontal plane, may be computed from equation (7) thus:

$$d\sigma_z' = d\sigma' \cdot \text{Cos}^2\,\theta = \frac{3\text{p}\cdot d\omega}{2\pi}\cdot \text{Cos}^2\,\theta \quad (8)$$

horizontal projection of that element (point b in Fig. 2); $\rho$.Cos Θ is the vertical projection of the radius vector; (line ab in Fig. 2) $\frac{2}{3}\pi\rho^3$ is the volume of a hemisphere of the radius $\rho$ (designation A will be used). Hence the product [$\rho^2$.d$\omega$.Cos Θ].[$\rho$.Cos Θ] is the volume of an elementary body bounded: at the top by the given surface element, $\rho^2$.d$\omega$; at the bottom by the projection of that element $\rho^2$.d$\omega$.Cos Θ; and at the sides by

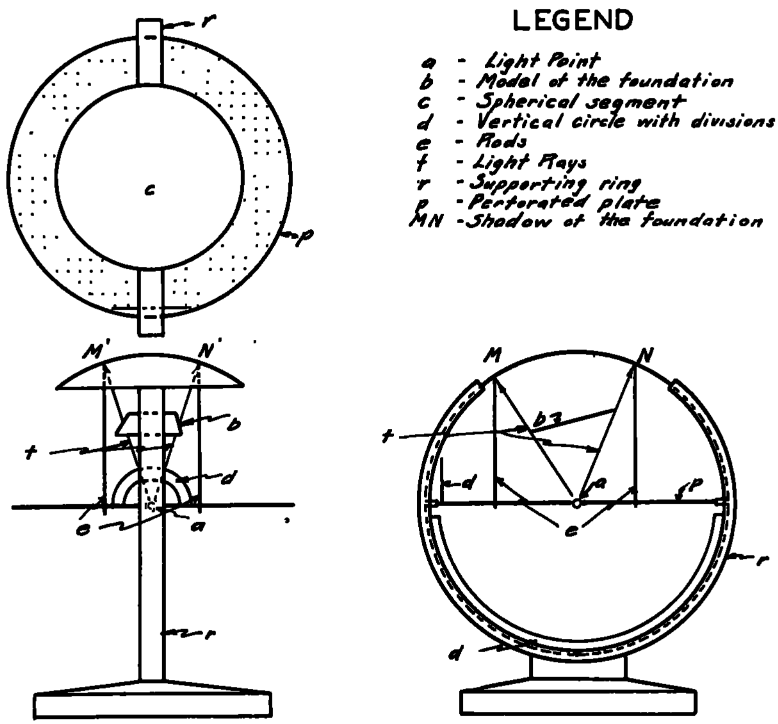


**Figure 3. Device for Determining Principal Stresses**

Multiply and divide expression (8) by $\rho^3$; thus:

$$d\sigma_z' = \frac{3\text{p}\cdot d\omega}{2\pi}\cdot \text{Cos}^2\,\theta \cdot \frac{\rho^3}{\rho^3}$$

$$= \text{p}\cdot\frac{[\rho^2\cdot d\omega\cdot \text{Cos}\,\theta]\cdot[\rho\cdot \text{Cos}\,\theta]}{\frac{2}{3}\pi\rho^3} \quad (8(\text{a}))$$

The interpretation of Formula (8(a)) is as follows: $\rho^2$.d$\omega$ is an element of the surface of the hemisphere of the radius, $\rho$ (point a in Fig. 2); $\rho^2$.d$\omega$.Cos Θ is the

the normals dropped from all points of the perimeter of the given surface element, $\rho^2$.d$\omega$, to the horizontal plane. Extending this statement to the whole angle $\omega$, a body designated in Fig. 2 with the letters $M_1cdN_1$ is to be considered. Let V be its volume; then:

$$\sigma_z' = \text{p}\cdot\frac{\text{V}}{\text{A}} \quad (9)$$

Suppose a point light is placed at O; then M′N′ will be the hypothetical shadow of the foundation at the interior surface of the sphere. Hence: the normal pressure on a plane caused by the given foundation is proportional to the volume of a body bounded by the shadow referred to; by the projection of that shadow on the given plane and by the perpendiculars dropped from all points of the perimeter of the shadow in question on that plane. A position of this plane which corresponds to the maximum volume of that body corresponds to the major principal stress.

### APPARATUS FOR MEASURING NORMAL STRESSES SUGGESTED

The device for determining normal stresses (Fig. 3) consists of a light source, a, and a spherical segment c, on which the shadow of the model of the foundation b, is obtained. Instead of being placed horizontally, the model of the foundation may be placed obliquely keeping its relative position with respect to the point O. The scale of the model obviously has an influence on the result. A perforated plane, p, may revolve in all vertical directions, and its movements, are recorded on the rings r and d. The perimeter of the shadow is reached by special thin rods passing through the holes of the plate, p. As soon as a rod reaches the surface of the experimental segment, it is fixed in its position. The body thus formed by the rods is filled then with shot or with sand and this material is weighed, from which the volume of the corresponding body is found.

### STEREOGONIOMETER

The apparatus suggested is similar to the "Stereogoniometer" constructed according to a general idea of the writer in the Sloane Physics Laboratory of Yale University (Fig. 4). The "stereogoniometer" may be used for measuring solid angles and for determining the sum of principal stresses at a point within a semi-infinite elastically isotropic body loaded at its boundary with a uniformly distributed load. The sum of principal stresses in question is proportional to the shadow of the loaded area at the surface of a sphere having the given point for center. An automobile tail light bulb (three candle power) represents the point, where stresses are determined. The apparatus

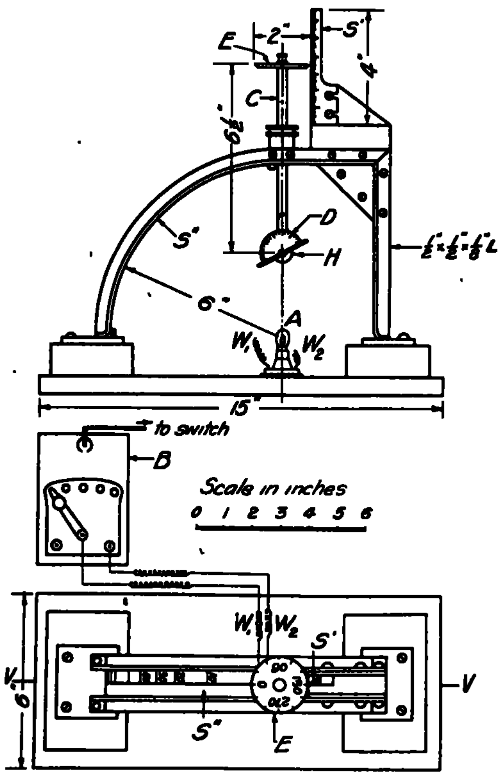


**Figure 4. Stereogoniometer**

is characterized by the vertical line, AA, passing through the light source, and a vertical plane, VV, practically constituting the plane of symmetry of the apparatus. There is a rotating shaft, C, on the vertical line, AA, which carries the model holder, H, provided with a vertical 180 deg. disk, D, and a 360 deg. horizontal disk, E. The height of the holder, H, may be read on the vertical scale, S′. A quadrant bent from stiff aluminum angles ½ in. by ½ in. by ⅛ in. carries a transparent celluloid scale, S″, which is graduated following the law (1 − Cos Φ), where Φ is the angle formed with the horizontal by the radius vector joining a given division with the light point. The disk E, is turned by constant angles (usually by 30 deg.) and in this way, arcs of great circles of the shadow are successively brought for measurement on the scale, S″. The model of the foundation is simply cut out of cardboard. The solution furnished by the stereogoniometer is an approximation quite satisfactory for all practical purposes.

### INFLUENCE OF POISSON'S RATIO IN THE CASE OF A CONCENTRATED LOAD

Stress components (2) may be broken into three stress systems: stress components $\sigma_x'$; $\sigma_z'$; $\tau_{xz}$ as given by Formulas (3); and two systems of plane stresses (10) and (11) acting in the horizontal plane, z units deep. (See below.)

The system (10) corresponds to a plane state of stress analogous to hydrostatic pressure since the two principal stresses $\sigma_x''$ and $\sigma_y''$ are equal. The system (11) describes tension, $\sigma_x'''$ being the only principal stress which does not vanish. Both systems (10) and (11), if superimposed, describe the influence of Poisson's ratio on the value of stresses under a concentrated load, P acting at the boundary of a semi-infinite elastically isotropic body.

### INFLUENCE OF POISSON'S RATIO IN THE CASE OF A UNIFORMLY LOADED AREA

The horizontal projection of an area at the boundary of a semi-infinite body is the area itself. Stresses at point O caused by different elements of that loaded area may be computed graphically by subdividing the projection of the area in question A, into circular rings with center at point, O, where stresses are to be determined (Fig. 5). Summation may be made graphically, and it will be shown how in the case of a uniformly loaded area (unit load p) stresses as caused by tension $\sigma_x'''$, may be computed.

Let x = az, where a is a number.

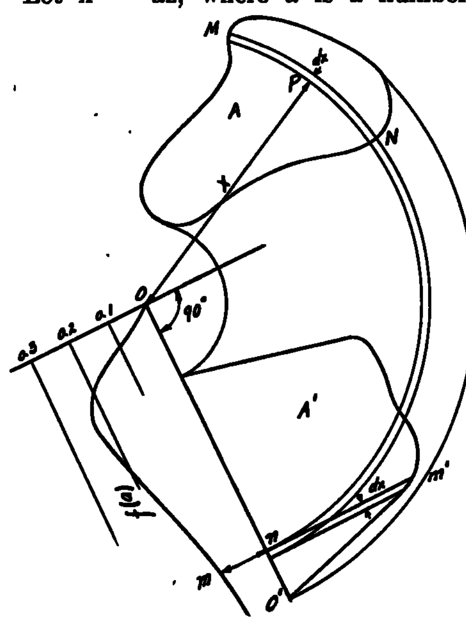


**Figure 5**

$$\left.\begin{aligned} \sigma_x'' &= \frac{3P}{2\pi}\cdot\frac{m-2}{3m}\left[\frac{1}{R(R+z)}+\frac{z}{R^3}\right] \\ \sigma_y'' &= \frac{3P}{2\pi}\cdot\frac{m-2}{3m}\left[\frac{1}{R(R+z)}+\frac{z}{R^3}\right] \end{aligned}\right\} \quad (10)$$

and:

$$\left.\begin{aligned} \sigma_x''' &= -\frac{3P}{2\pi}\cdot\frac{m-2}{3m}\cdot\frac{(2R+z)\cdot x^2}{(R+z)^2\cdot R^3} \\ \sigma_y''' &= 0 \end{aligned}\right\} \quad (11)$$

Since for a given horizontal plane $z$ = const., the expression for $\sigma_x'''$ may be represented thus:

$$\sigma_x''' = -\frac{3P}{2\pi}\cdot\frac{m-2}{3m}\cdot\frac{f(a)}{z^2} \qquad (12)$$

where f(a) is a function of a. Its values for different values of $a = \frac{x}{z}$ are represented in Fig. 6. Using this graph, stresses $\sigma_x'''$ may be readily computed for any value of x at a given horizontal plane. An arbitrary line, OO′, is traced through point O, and curve f(a) is traced according to Fig. 6. The full

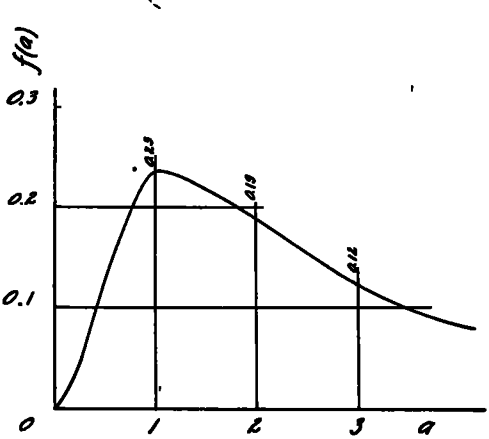


Figure 6

stress $d\sigma_x'''$ at point O, caused by the load acting at the ring [MN].dx is:

$$d\sigma_x''' = -\frac{3p[MN]\cdot dx}{2\pi}\cdot\frac{m-2}{3m}\cdot\frac{[mn]}{z^2}. \qquad (13)$$

or introducing, for the sake of brevity, a designation:

$$c = -\frac{3p}{2\pi}\cdot\frac{m-2}{3m}\cdot\frac{1}{z^2},$$

Equation (13) may be represented thus:

$$d\sigma_x''' = c\cdot[MN]\cdot[mn]\cdot dx \qquad (13(a))$$

Ordinate nm′ in Fig. 5 is supposed to represent at a certain scale the product c.[MN].[mn]. Repeating this operation for all rings of area A, of which one only, namely (MN).dx is shown in Fig. 5 and plotting corresponding ordinates, such as nm′, an area, A′, is obtained, which, taking into consideration the scale of the drawing, Fig. 5, is proportional to the sum of principal stresses at point O, as caused by tensions $\sigma_x'''$.

Method of breaking the sum of principal stresses into the two principal stresses when the stressed condition is

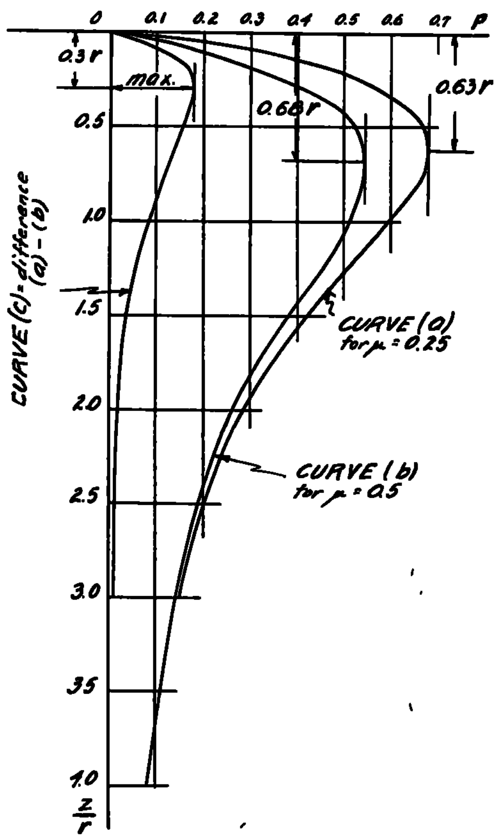


**Figure 7. Stress Difference at the Center of a Circular Loaded Disk**

caused by a uniform load acting on a circular arc has been given by the writer elsewhere.[2] The corresponding half stress difference is the maximum shear, max $\tau_3$, caused by the stress system (11) at point O. Since both principal stresses of system (10) are equal and of the same

[2] *Proceedings*, Am. Soc. C. E., *Transactions*, Vol. 62, p. 1291.

sign, it will be assumed that system (10) does not contribute to the maximum shear stress at point, O. The two maximum shearing stresses max $\tau_1$; and max $\tau_3$ are to be added geometrically to determine approximately the full maximum shear at point O. The correct procedure would be, however, to determine the principal stresses at point, O,

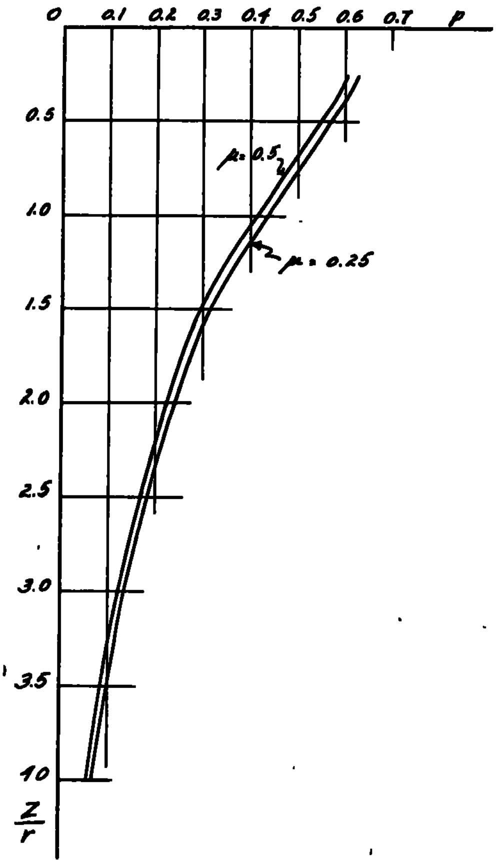


**Figure 8. Stress Difference under the Edge of a Circular Loaded Disk**

from the three systems (3), (10) and (11) acting simultaneously and to compute the corresponding half stress difference neglecting the influence of the middle principal stress.

### CIRCULAR DISK ANALOGY

The preceding discussion shows how troublesome is the determination of the influence of Poisson's ratio on the maximum shearing stress at a point of the mass. It has been decided, therefore, to consider the stress difference under a uniformly loaded circular disk and to generalize conclusions drawn from such consideration. Figures 7 and 8 represent stress differences under the center and the edge of a circular disk of a radius, r. It follows from these curves that the maximum influence of Poisson's ratio (curve (c) in Fig. 7) as found from the comparison of the curves (a) and (b), traced for the values of Poisson's ratio $\mu = 0.25$ and $\mu = 0.5$ in no case is greater than 50% of the stress difference corresponding to the value of $\mu = 0.5$.

As stated in the beginning of this paper, the value of Poisson's ratio is not constant in the case of soils; besides, it is not known how to determine this value in the field. Hence it is not worth while to make complicated computations operating with a quite arbitrary value of Poisson's ratio.

### CONCLUSION

It is being proposed to determine the shearing stress at a given point of the earth mass as caused by a loaded area at the earth surface, from the radially acting system of stresses (4) and to add 50 per cent to take care of the influence of Poisson's ratio. The apparatus shown in Figure 3 is to be constructed and improved.

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