STRESSES UNDER CIRCULAR LOADED AREAS

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SYNOPSIS

A complete solution of the problem of stresses under a uniformly loaded circular area at the surface of a semi-infinite elastically isotropic material was published by A. E. Love in 1929 and by S. D. Carothers in 1924. The general conclusions and numerical computations of the two investigators are in good agreement. The surface of partial maxima principal stress differences was found by Love to extend down to a depth of about 0.7 of the radius of the loaded circle. The greatest principal stress difference is under the perimeter at the surface. Certain approximate methods of computing the vertical and shearing stresses are described. The results are suggested as a rough guide in experimentation involved in the study of the design of flexible highway surfaces.

The main purpose of this paper is to bring to the attention of engineers the fact, that a precise analysis of the complete system of stresses under a uniformly loaded circular area at the surface of a semi-infinite, elastically isotropic material has been in published form for more than ten years. Another purpose of the paper is to indicate approximate methods of checking numerical values of the stresses that have been computed by the precise methods. Still another purpose is to indicate the possibility of using the results of the analyses as a rough guide in experimentation involved in the study of the design of flexible types of highway surfaces. In this latter connection, it is realized generally that the condition of elastic isotropy does not exist in the mass of material under an automobile tire. Hence, it would be a mistake to attempt to apply directly the analytical results and the fact that they can serve only as a rough guide in experimentation needs to be emphasized.

Engineers are confronted with many earth problems for which no theory other than the one of elasticity has been proposed and the treatment of earth problems apart from adequate theory is likely to lead to as many solutions as there are varieties of earth. The experimental study of such problems without reference to any theoretical basis whatsoever is an aimless procedure at the best.

The problem of computing stresses at any point within a semi-infinite, elastically isotropic mass produced by a uniform load over a circular area at the plane boundary has been completely solved by A. E. H. Love (1)[1] and S. D. Carothers (2). The results of these two investigators are in general in good agreement and suggest a means of estimating stresses under wheel loads, since it has been shown by Teller (3) and Buchanan (3) that the pressure distribution of a pneumatic tire on a flexible type of pavement is nearly uniform.

In such problems as the determination of the rate of settlement (by soil consolidation) of a foundation (see Fig. 1A) the vertical stresses at the upper and lower boundaries of a clay layer are computed from formulas derivable from those of Boussinesq by assuming that the entire mass of earth below the foundation is elastically isotropic. That is, the complications that may be involved, owing to the fact that the underground is comprised of alternating strata of dissimilar materials and is, therefore, not elastically isotropic, are ignored completely. Despite the questionable procedure of assuming isotropy in this case, the analy-

[1] Numbers in parentheses refer to list of references at end.

sis of settlements by consolidation is one of the best established theoretical methods of soil mechanics and the present writer believes that in this instance the assumption of isotropy is as reasonable as any other assumption and is to be preferred because it is in the interest of simplicity.

There no doubt is some refraction of the stress trajectories at the boundary of two dissimilar earth materials, that is,

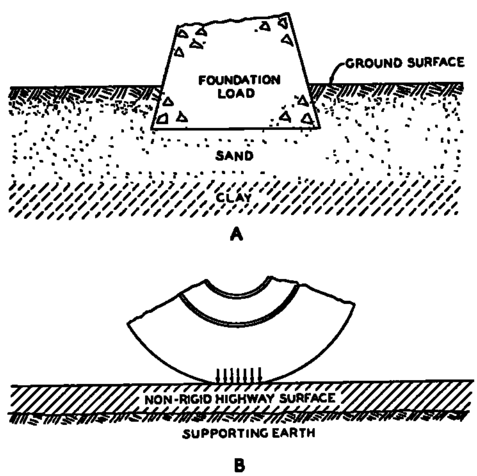

Figure 1

A—Foundation Resting on Sand Underlaid by Clay.

B—Wheel Load Resting on Nonrigid Highway Surface Supported by Earth.

this boundary very likely acts as a plane of discontinuity. But any assumption as to the amount of friction between two layers of dissimilar materials such as clay and sand (Fig. 1A) or as to what extent discontinuity of stresses may exist is likely to be no more valid than the assumption of isotropy in the entire earth mass below the foundation.

For the same reason, the entire mass of materials below a wheel load (Fig. 1B) may be considered as being elastically isotropic.

On the basis of such an assumption, no precise solution can be expected. Nevertheless, the theoretical development meets a definite requirement in that it shows, qualitatively at least, what happens under wheel loads and, therefore, may serve as a guide in planning methods of experimentation. All that is sought is a suggested trace of the transmission of stresses by the wheel load since rigorous results or exact formulas are out of the question.

A. C. Benkelman (4) has presented in considerable detail the present status of our knowledge concerning the design of flexible pavements. It is indeed surprising to note that the conclusions reached by Carothers and Love have not been used in the technical publications discussed in Benkelman's report. However, with practically no exception, the various authors mentioned in his report have based their experimental procedures on theoretical considerations involving numerous simplifying assumptions which are similar in many respects to the classical theories of Love, Carothers and Hencky (5).

The simplest three-dimensional problem considered in the theory of elasticity is the one involving axial symmetry. Hence, the problem of a wheel load on a pavement is simplified by the assumption that the area of contact between a rubber tire and pavement is circular although tests indicate that the contact area is elliptic. Thus an "equivalent circular area" with uniform pressure has been used generally by investigators. The equivalent circle is one having an area equal to the elliptic area of contact and its radius is the square root of the quantity, area of contact divided by π.

The subject of pressure distribution over a circular contact area in connection with foundation design has been discussed at length by Cummings (6), Krynine (7) and various others. The simplest problem is the one of uniform pressure over the contact area. In foundation problems, the actual pressure distribution depends on the relative rigidities of the structure (or loading member) and the earth mass, and in the light of present knowledge no one knows how to express these relative rigidities in quantitative terms. Another difficulty is that any formula expressing the pressure distribution over the contact area can be only approximately true when the deformations are within the elastic limits of the materials and cannot be applicable in any sense as the deformations increase more and more and become characteristic of those attending plastic yield. If an attempt is made to include all of these departures from simplicity in a theoretical development of the subject, any result would be so hopelessly complicated as to be of very small use to practical engineers.

THE GREATEST SHEAR IS NEAR THE PERIMETER OF A UNIFORMLY LOADED CIRCULAR AREA

The outstanding conclusion of Love and Carothers was that the greatest value of the principal stress difference, S, is very close to the perimeter of the uniformly loaded circular area. This difference is twice the maximum shearing stress at any point, that is,

$$S = 2\, s_{max} \quad \ldots\ldots\ldots\ldots 1$$

Table 1 is taken from Love's article and shows values of $\frac{S}{p}$, p being the unit contact pressure over the circular area, corresponding to various positions of the point Q, Figure 2. The ratio, $\frac{r_1}{r_2}$, of the two radial lines, Figure 2, and the magnitudes of the quantities, angle B, $\frac{z}{a}$ and $\frac{r}{a}$, a being the radius of the circle, z being the depth to the point, and B the angle between r_1 and a, determine the position of any point Q.

As the point Q moves in such a manner that the ratio, $\frac{r_1}{r_2}$, approaches zero as a limit, then according to Love, the value of S depends on the angle B, or in other words, S at the point A depends on the direction of approach of point Q to point A. Love has shown that for B=71°, the limiting value of S as $\frac{r_1}{r_2}$ approaches zero is 0.723p (Table 1). The $\frac{S}{p}$ values of Table 1 were computed by Love for

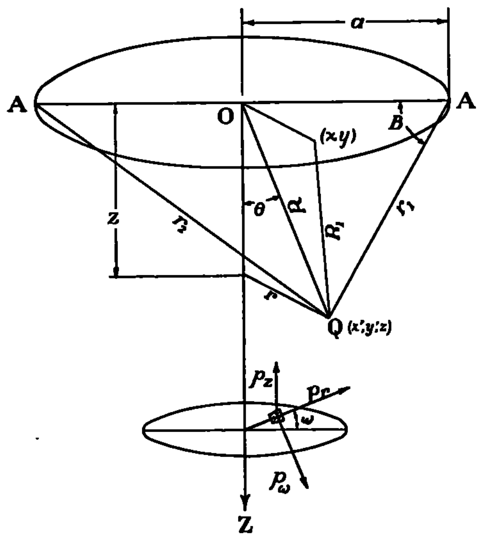

Figure 2. Problem of the Principal Stress Difference, S, under a uniformly Loaded Circular Area.

$\mu = \frac{1}{4}$ and are "partial maxima stress differences," a term that will be considered later in the paper.

From Table 1, the value for s_g, the greatest value of s_{max} at any point Q in the supporting earth is $\frac{0.723p}{2} = 0.36p = \frac{S}{2}$. This is an important fact for the reason that it shows that the greatest shearing stress is at the surface where, due to lack of confinement, there is the least resistance to yield under stress. Love has given in Table 1 more data than are necessary for locating the point Q, Figure 2.

In texts dealing with the theory of elasticity, it is shown that when the point Q (Fig. 2) is restricted to move only along the axis of symmetry, OZ, the greatest value of s_{max} occurs at a considerable depth, z, from the surface. Thus Timoshenko (8) shows that for $\mu=0.3$, the greatest value of s_{max} at any point on OZ is 0.33p and at a depth equal to $\frac{2a}{3}$. The earth at this point is confined in all directions.

TABLE 1

VALUES OF $\frac{S}{p}$ FOR DIFFERENT POSITIONS OF POINT Q IN FIGURE 2

Values given by A. E. H. Love for $\mu = \frac{1}{4}$[1]

	$\frac{r_1}{r_2}$ tends to zero	$\frac{r_1}{r_2}=\sin 5°$	$\frac{r_1}{r_2}=\sin 15°$	$\frac{r_1}{r_2}=\sin 30°$	$\frac{r_1}{r_2}=\sin 50°$	$\frac{r_1}{r_2}$ tends to unity
$\frac{S}{p}$......	0.723	0.714	0.704	0.695	0.690	0.689
B.....	71°	67°	56°	44°	37°	32°
$\frac{r}{a}$......	1	0.934	0.742	0.446	0.184	0
$\frac{z}{a}$......	0	0.156	0.383	0.536	0.615	0.620

[1] The values, $\frac{1}{4}$ and 0.45 for μ, appearing in this paper are values selected by Love and Carothers respectively, in their computations.

Love's solution was obtained by the application of potential theory (9) and involved various elliptic integrals.

Carother's procedure is only very briefly described and therefore requires some discussion. He considers the logarithmic potential (10) of a uniform distribution of matter over a circular area of radius a, expressed by the relation,

$$\psi = \frac{p}{2\pi}\int\int \log(z+R_1)\,dx\,dy \;.....2$$

where dx and dy refer to the coordinates of any point x, y, on the surface and R_1 is the distance of this point to the point Q, Figure 2.

By differentiating equation 2 under the integral sign,

$$\frac{\partial 2\psi}{\partial z^2} = \frac{p}{2\pi}\int\int \frac{z}{R_1^3}\,dx\,dy = \frac{pw}{2\pi} \;....3$$

where w is the solid angle (11) subtended at Q by the circular area.

By use of the equations of equilibrium and the compatibility equations (12) all of the stresses at any point, Q, may be expressed in terms of w, the solid angle subtended at point Q. Thus it is found, for example that

$$s_{rz} = z\frac{\partial}{\partial r}\frac{\partial^2\psi}{\partial z^2} = \frac{p}{2\pi}z\frac{\partial w}{\partial r} \;.....4$$

$$\text{and } p_z = p/2\pi\left(z\frac{\partial w}{\partial z} - w\right) \;........5$$

p_z being the pressure at Q that is normal to the horizontal plane and s_{rz} the shearing stress.

Similarly, p_r and p_w (Fig. 2) may be expressed as functions of the solid angle, w.

For a uniformly loaded circular area, the maximum shearing stress is obtained from the expression,

$$s^2_{max} = \frac{(p_r - p_z)^2}{4} + s^2_{rz} \;.......6$$

Love's very comprehensive tables give values for p_r, p_z and s_{rz}, for $\mu=\frac{1}{4}$, at a great many points. Values for $\frac{S}{p}$ in addition to those given in Table 1, may be computed by means of equation 6 and the other tables of Love.

The stresses at any point, Q, were computed by Carothers by expanding the various functions of w in zonal harmonics. Some of his computed values for $\frac{S}{p}$ are given in Table 2. These values were computed by taking $\mu=0.45$. It is seen from this table that the greatest value of s_{max} is near the perimeter of the loaded circular area where $R=a$ and θ has values ranging from 80° to 90° (see Fig. 2). Love and Carothers are in good agreement as to the region of greatest shear.

POSSIBLE MEANS OF CHECKING THE VALUES OF LOVE AND CAROTHERS

The method of Love, although enormously complicated, is presented in great detail and it is possible to check his numerical values, using his method. Carothers, on the other hand, gives no clue as to what particular functions he expanded in zonal harmonics in making his numerical computations. It would seem desirable then to find some means of checking the values of both Carothers and Love without using exactly the same method of either of these two investigators. This is an exceedingly difficult undertaking for the reason that Love's solution is complete, all of the stresses, p_r, p_z and s_{rz}, being found throughout the entire region within a radial distance, a, of the loaded area. On the basis solely of his numerical values and without regard to how they were obtained, the same is true of Carothers' solution.

For the case of a uniform pressure, p, on a circular area it is easy to show (see Timoshenko (8), page 337) that for any

TABLE 2

VALUES OF $\frac{S}{p}$ FOR DIFFERENT POSITIONS OF POINT Q, FIGURE 2, ACCORDING TO CAROTHERS, FOR $\mu=0.45$

θ	$\frac{S}{p}$ for $R=a$	$\frac{S}{p}$ for $R=\frac{2a}{3}$
0°	0.54	0.60
30°	0.55	0.60
45°	0.57	0.53
60°	0.58	0.50
75°	0.62	0.40
80°	0.63	0 30
85°	0.63	0.18
90°	0.63	0.05

point on the axis of symmetry and for $\mu=\frac{1}{4}$,

$$\frac{S}{p}=\frac{1}{4}+\frac{1}{4}\left[\frac{5a^2z-z^3}{(a^2+z^2)^{\frac{3}{2}}}\right] \quad \ldots\ldots 7$$

This expression may be expanded by the binomial theorem to give the infinite series,

$$\frac{S}{p}=\frac{1}{4}+\frac{5}{4}\left[\frac{z}{a}-\frac{1\times3}{2}\frac{z^3}{a^3}+\frac{1\times3\times5}{2\times4}\frac{z^5}{a^5}-\frac{1\times3\times5\times7}{2\times4\times6}\frac{z^7}{a^7}+,-\right]$$
$$-\frac{1}{4}\left[\frac{z^3}{a^3}-\frac{1\times3}{2}\frac{z^5}{a^5}+\frac{1\times3\times5}{2\times4}\frac{z^7}{a^7}+,-,\right] \quad \ldots\ldots 8$$

if a is greater than z, and

$$\frac{S}{p}=\frac{1}{4}+\frac{5}{4}\left[\frac{a^2}{z^2}-\frac{1\times3}{2}\frac{a^4}{z^4}+\frac{1\times3\times5}{2\times4}\frac{a^6}{z^6}-\frac{1\times3\times5\times7}{2\times4\times6}\frac{a^8}{z^8}+\right]$$
$$-\frac{1}{4}\left[1-\frac{1\times3}{2}\frac{a^2}{z^2}+\frac{1\times3\times5}{2\times4}\frac{a^4}{z^4}-,+,\right] \quad \ldots\ldots 9$$

if z is greater than a.

From a mathematical standpoint, it is not correct to expand a function in zonal harmonics that does not satisfy Laplace's equation. In potential theory, a Newtonian potential function, known to be harmonic (satisfies Laplace's equation) may first be computed on an axis of symmetry and thereafter it may be computed at any point not on the axis, by expanding in zonal harmonics or Legendrian polynomials.

In general, the expression for a stress is a tensor (13) and is not therefore harmonic. However, the expression for a stress may consist of several parts, one or more of which may be harmonic, and it is legitimate to evaluate these by expanding in zonal harmonics. This, no doubt, was Carothers' procedure although he does not indicate it.

In the present consideration, the circle is not one of complete symmetry as Love has shown and in the absence of any general expression for $\frac{S}{p}$, known to be harmonic or otherwise, there is no sound mathematical basis for the following procedure which is to *assume* that the general expression for $\frac{S}{p}$ is harmonic and to pass from equations 7 and 8 to series of Legendrian polynomials. It is also assumed that the circle is symmetrical and this is not quite true.

If now the loaded circular area is considered as analogous to an electrically charged disk in potential theory, then one could substitute R for z in equations 8 and 9 and introduce Legendrian coefficients (14), thereby expanding equation 7 in zonal harmonics. When this is done, equation 8 becomes

$$\frac{S}{p}=\tfrac{1}{4}+\tfrac{5}{4}\left[\frac{R}{a}P_1(\cos\theta)-\frac{1\times3}{2}\frac{R^3}{a^3}P_3(\cos\theta)+\frac{1\times3\times5}{2\times4}\frac{R^5}{a^5}P_5(\cos\theta)-\frac{1\times3\times5\times7}{2\times4\times6}\frac{R^7}{a^7}P_7(\cos\theta)\right]-\tfrac{1}{4}\left[\frac{R^3}{a^3}P_3(\cos\theta)-\frac{1\times3}{2}\frac{R^5}{a^5}P_5(\cos\theta)+\frac{1\times3\times5}{2\times4}\frac{R^7}{a^7}P_7(\cos\theta)\right]\ldots\ldots10,$$

if we neglect all powers of $\frac{R}{a}$ beyond the seventh, and equation 9 becomes

$$\frac{S}{p}=\tfrac{1}{4}+\tfrac{5}{4}\left[\frac{a^2}{R^2}P_1(\cos\theta)-\frac{1\times3}{2}\frac{a^4}{R^4}P_3(\cos\theta)+\frac{1\times3\times5}{2\times4}\frac{a^6}{R^6}P_5(\cos\theta)\right]-\tfrac{1}{4}\left[1-\frac{1\times3}{2}\frac{a^2}{R^2}P_1(\cos\theta)-\frac{1\times3\times5}{2\times4}\frac{a^4}{R^4}(\cos\theta)+\frac{1\times3\times5\times7}{2\times4\times6}\frac{a^6}{R^6}P_5(\cos\theta)\right]\ldots\ldots11,$$

neglecting all powers of $\frac{R}{a}$ beyond the sixth.

The terms, $P_1(\cos\theta)$, $P_3(\cos\theta)$, etc., are the Legendrian coefficients and numerical values for these coefficients, corresponding to different values of θ, Figure 2, may be found in various mathematical treatises. For R less than a and $\theta=0$, equation 10 becomes equation 8 and for R greater than a and $\theta=0$, equation 11 reduces to 9.

It should be emphasized now that aside from the geometric similarity, that is, a circle and an axis of symmetry in both cases, the problem of potential at a point due to a charged disk and the problem of shearing stresses beneath a loaded circular area have nothing in common.

In Table 1, for $\frac{r}{a}=0.184$, $B=37°$ and $\frac{r_1}{r_2}=\sin 50°$, it is found that $\theta=16°$, 40′ and $R=0.6416\,a$. The coefficients, $P_1(\cos\theta)$, $P_3(\cos\theta)$ corresponding to $\theta=16°$, 40′ are

$$P_1(\cos\theta)=0.9579$$
$$P_3(\cos\theta)=0.7609$$
$$P_5(\cos\theta)=0.4573$$
$$P_7(\cos\theta)=0.1207$$

Substituting these values in equation 10,

$$\frac{S}{p}=\tfrac{1}{4}+\tfrac{5}{4}(0.6141-0.3015+0.0930-0.0118)-\tfrac{1}{4}(0.2010-0.0744+0.0101)=0.709,$$

which compares favorably with Love's value in Table 1 which is 0.690.

The series in this case converges rapidly. By using Love's numerical values for p_r, p_z and s_{rz} ($\mu=\frac{1}{4}$) and equation 6 one finds that for $\frac{R}{a}=0.56$ and $\theta=45°$ that $\frac{S}{p}=0.66$. However, by equation 10 the corresponding value for $\frac{S}{p}$ is 0.75 which is 14 percent higher than Love's value. Again, for $\frac{R}{a}=0.19$ and $\theta=45°$, from Love's tables and equation 6, $\frac{S}{p}=0.42$ according to Love, which is exactly the same as the value obtained from equation 10. Jürgenson (15) has computed various values for $\frac{S}{p}$, using Carothers' tables for p_r, p_z, and s_{rz} and finds for example, that for $R=2a$ and $\theta=45°$, $\frac{S}{p}=0.25$. The corresponding value found by an expression similar to equation 11 is 0.27. Love's value for $\frac{S}{p}$ for $\frac{r}{a}=0.446$ and $\frac{z}{a}=0.536$ is 0.695 (Table 1), whereas equation 10 gives a value of 0.716 which is close to Love's values. For θ equal to 90° and $\frac{R}{a}$ less than one, the value of $\frac{S}{p}$ is 0.25 by equation 10, which is Love's value in this region. Equations 10 and 11 do not fit the boundary condition for $\theta=90°$ and R greater than a.

In all other cases it may be said that for θ greater than 45° and for $\frac{R}{a}$ or $\frac{a}{R}$ greater than $\frac{2}{3}$, the values for $\frac{S}{p}$ computed from equations 10 and 11 deviate by at least 15 percent (in many cases much more) from Love's numerical values. For all other positions of the point Q (Fig. 2) the agreement is generally within 15 percent. Carothers states that for values of $\frac{R}{a}$ or $\frac{a}{R}$ approaching unity, his functions expanded in zonal harmonics were "unsatisfactory" and he gives no clue as to an alternative procedure in this case. In the above computations the same limitations are realized.

Equations 10 and 11 have very definite limitations although they tend to give values that are in approximate agreement with those of Love and Carothers at various points.

Functions of solid angles are for the most part of interest only to the mathematician. A practicing engineer has more confidence in numerical values if he can check them by a simple graphical method.

The following simple device is suggested by the author for this purpose. Its use is very limited. The method is as follows:

It is desired, for example, to know the value of $\frac{S}{p}$ at the point where $\theta=45°$ and $R=\frac{2a}{3}$, the value of μ being taken as 0.45 (Carothers' assumed value). In Figure 3, AB is the diameter of the loaded circular area on the horizontal surface and OZ is its axis of symmetry. Draw the radial lines AQ, QB and OQ=R. Draw the line MN through O making an angle θ with AB. The projection of AB on MN is A'B', the minor axis of an ellipse. This ellipse is in the plane that is passed through MN perpendicularly to the plane of the paper. Its major axis is equal to AB which passes through the point O and is perpendicular to MN.

The length A'O is $a\cos\theta=\frac{1}{2}$ the minor axis and $a=AO$ is $\frac{1}{2}$ the major axis. Then the area of the ellipse formed by projecting the circle in the horizontal plane on the plane MN is, $\pi a\,(a\cos\theta)=\pi a^2\cos\theta$.

Let b=the radius of the equivalent circle, that is, the radius of the circle having an area of $\pi a^2 \cos\theta$. Then $\pi b^2 = \pi a^2 \cos\theta$, or $b = a\sqrt{\cos\theta}$. Now replace the ellipse with the circle of radius $= b = a\sqrt{\cos\theta}$. A uniform pressure, p, perpendicular to the plane MN and distributed over the circle of radius $a\sqrt{\cos\theta}$ will produce approximately the same stress, s_{max}, at Q as is caused by the same

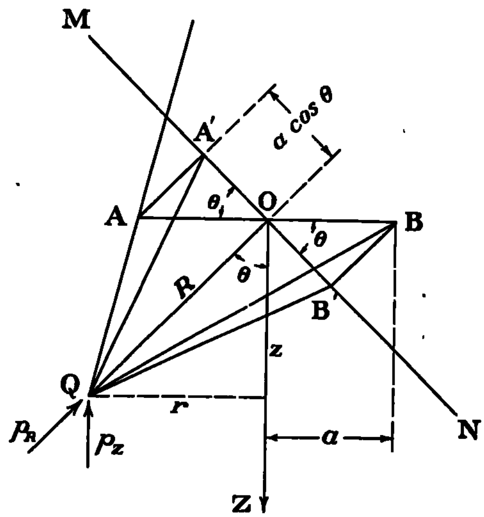

Figure 3. The Projection of the Loaded Circular Area, AB in the Horizontal plane on the plane MN is an ellipse with minor axis A'B' with major axis AB.

pressure, p, distributed over the circular area, AB, in the horizontal plane, provided θ is less than 45°. For $\theta = 0$, it is observed that p_R becomes p_z.

For the general case, the value of s_{max} when the point Q is on the axis of symmetry is obtained from the expression,

$$s_{max} = \frac{p}{2}\left[\frac{1-2\mu}{2} + (1+\mu)\frac{R}{(b^2+R^2)^{\frac{1}{2}}} - \frac{1}{2}\frac{R^3}{(b^2+R^2)^{\frac{3}{2}}}\right] \quad \ldots\ldots\ldots\ldots 12$$

where $b = a\sqrt{\cos\theta}$ = radius of the equivalent circle. For $R = \frac{2}{3}a$, $\theta = 45°$, $\cos\theta = \frac{1}{\sqrt{2}}$ and $\mu = 0.45$,

$$s_{max} \text{ at point } Q = \frac{p}{2}\left[0.05 + \frac{2\times 1.45}{3\left(\frac{1}{\sqrt{2}}+\frac{4}{9}\right)^{\frac{1}{2}}} - \frac{\frac{3}{2}\times\frac{8}{27}}{\left(\frac{1}{\sqrt{2}}+\frac{4}{9}\right)^{\frac{3}{2}}}\right] = 0.296p$$

Then $\frac{S}{p} = \frac{2s_{max}}{p} = 0.592$. Carothers obtains the value, $\frac{S}{p} = 0.53$ for this point (Table 2). The divergence from Carothers' value in this case is likely indicative of the error involved in transforming the elliptic area into one that is circular.

In addition to approximate methods, such as the foregoing, there are other means of checking precise numerical values, such as those published by Love and Carothers. One of such means is the photoelastic method. However, at present there has been but little progress in the analyses of three-dimensional problems of stress distribution by photoelastic devices. R. Weller (16) has suggested that it is possible to use the polarization caused by the scattering of light within a "cloudy" or opaque model in place of the usual analyzer in photoelastic investigations and that this method of polarization enables one to make analyses of three-dimensional stress systems very conveniently. An older method was to cool the model under load from an elevated to room temperature and then analyze it subsequently after slicing it into plates. This is an enormously complicated procedure.

D. P. Krynine (17) has described his device, called a stereogoniometer, which offers much promise in such studies and which has the advantages of simplicity and low cost. This device makes use of the principle of projecting areas from a plane to a spherical surface rather than from one plane to another plane as was done in the preceding example. By this procedure, the stress, p_z, is found in terms of the area of the projection of the loaded surface on the sphere and the volume of the space bounded by this projected area, its projection in turn on the horizontal plane including the load and the perpendicular lines joining the perimeters of the two projected areas. The solid angle at a point below the loaded area is equal to the projected area on the sphere divided by the square of the radius of the sphere.

It is possible to compute the solid angle w, without this device, as follows: On the axis of symmetry, the expression for the solid angle, w is

$$w=2\pi\left(1-\frac{z}{R}\right)\ldots\ldots\ldots 13$$

where $R=(a^2+z^2)^{\frac{1}{2}}$. The general expression for w (see equation 3) is harmonic and it is mathematically legitimate to expand it in zonal harmonics. By so doing, one obtains the two expressions,

$$w=2\pi-2\pi\left[\frac{R}{a}P_1(\cos\theta)-\tfrac{1}{2}\left(\frac{R}{a}\right)^3P_3(\cos\theta)+\tfrac{3}{8}\left(\frac{R}{a}\right)^5P_5(\cos\theta)-\tfrac{15}{48}\left(\frac{R}{a}\right)^7P_7(\cos\theta)+,-,\text{etc.}\right]\ldots\ldots 14$$

for R less than a, and

$$w=2\pi-2\pi\left[1-\tfrac{1}{2}\left(\frac{a}{R}\right)^2P_1(\cos\theta)+\tfrac{3}{8}\left(\frac{a}{R}\right)^4P_3(\cos\theta)-\tfrac{15}{48}\left(\frac{a}{R}\right)^6P_5(\cos\theta)+\tfrac{105}{384}\left(\frac{a}{R}\right)^8P_7(\cos\theta)+,-\text{etc.}\right]\ldots\ldots 15$$

for R greater than a.

For $\frac{R}{a}=\frac{2}{3}$ and $\theta=45°$, one obtains $w=1.07\pi$ from equation 14. Then substituting this value in the expression which is valid only for $\mu=\frac{1}{2}$,

$$p_z=\frac{3p}{2\pi}w\cos^2\theta\ldots\ldots\ldots 16$$

it is found that $p_z=0.803p$. The value for p_z at $R=\frac{2}{3}a$ and $\theta=45°$ is 0.858p when $\mu=0.45$, according to Carothers.

Similarly, for $R=2a$ and $\theta=45°$, $w=0.195\pi$ from equation 15 and on substitution in equation 16, $p_z=0.146p$. The corresponding value obtained by Carothers for $\mu=0.45$ is 0.158p. The differential of equation 16 was proposed by D. P. Krynine (17). The stereogoniometer is a device that performs an exact mechanical integration of the differential of equation 16, θ and w being both variable in the integration.

Equation 16 does not satisfy the boundary conditions, $\theta=0$ and $\theta=90°$. For a uniformly loaded circular area it gives results which are approximately correct for θ greater than 30° and less than 60°, R being greater than zero. On the axis of symmetry,

$$p_z=p\left[1-\frac{z^3}{(a^2+z^2)^{\frac{3}{2}}}\right]\ldots\ldots 17$$

If θ is taken as the average angle between the vertical direction and the radial distance R drawn from any element of loaded surface to the point Q, considering all of the elements of the loaded area, the computations by equation 16 become exact. The analytical procedure of obtaining this average θ is a very difficult one. For R greater than $\frac{3}{2}$a, equation 16 together with equations 14, 15 and 17 give values for p_z that do not diverge more than 15 percent (in most cases much less) from Carothers' values when θ does not exceed 60°.

The purpose of the foregoing procedure is to indicate that in substance, at least, the mechanical method used by Krynine is correct. Its outstanding advantage is that in its use the evaluation of the stress, p_z, can be accomplished for any contour of uniformly loaded area. It is not necessary that there be axial symmetry.

The precise analytical method of obtaining p_z at a point under a uniformly loaded circular area is indicated by equation 5 which is Carothers expression for this stress. Since

$$w = \iint \frac{z}{R_1^3}\, dx\, dy,$$

Then

$$\frac{\partial w}{\partial z} = \iint \left(\frac{1}{R_1^3} - \frac{3z^2}{R_1^5} \right) dx\, dy \quad \ldots\ldots 18$$

and

$$z\frac{\partial w}{\partial z} = \iint \frac{z}{R_1^3}\, dx\, dy - 3\iint \frac{z^2}{R_1^2}\left(\frac{z}{R_1^3} \right) dx\, dy \quad \ldots\ldots 19$$

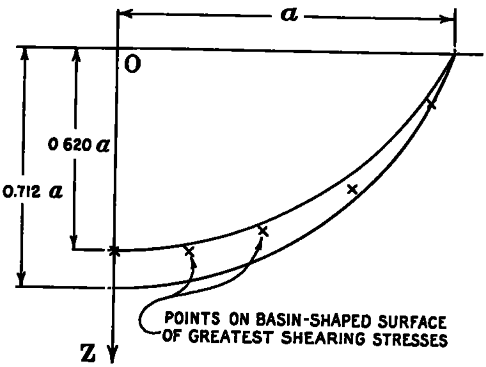

Figure 4. Points at which the Stressed Material under a uniformly Loaded Circular Area Would be Most Likely to Fail.

It is possible to evaluate these two integrals at any point Q. The first integral is w, known to be harmonic, and it may be computed at any desired point by means of equations 14 and 15. The second integral may be simplified by expanding $\frac{z}{R_1^3}$ in Legendrian polynomials since $\nabla^2\left(\frac{z}{R_1^3}\right)=0$. With w and $z\frac{\partial w}{\partial z}$ thus evaluated, the stress, p_z, at any point, Q, is computed from equation 5.

For a uniform pressure produced by a wheel load on the nonrigid surface, the distribution of vertical pressure, p_z, over the area of subgrade bounded by the radial line R is far from uniform as Hawthorne (18) assumed in his analysis.

Love has shown that for any given value of μ, there is a certain value of R, expressed as some multiple of a, for which s_{max} has a greatest value. Thus for $\theta=45°$, the greatest value of s_{max} on the radial line is at any point where R=0.73a. Similarly for $\theta=90°$, the greatest value of s_{max} is at points where R=a, that is, at points just under the perimeter. If the points of "partial maxima stress difference" on all of the radial lines for all values of θ are connected, the locus of such points is found to be a "basin-shaped surface of revolution about the axis of the circle." It passes through the circle and lies between two segments of spheres, which have their centers on the axis of symmetry and pass through the circle (see Fig. 4). These two spheres cut the axis of symmetry at depths equal to 0.620a and at 0.712a when $\mu=\frac{1}{4}$. According to Love, it is reasonable to conclude that the foundation under a round pillar would be most likely to give way at points on such a basin-shaped surface and that it would be nearly as likely to give way at one point of this surface as at any other. The values given in Table 1 are for points on the basin-shaped surface.

A practical consideration is the following: If the supporting subgrade is of questionable supporting power, such as medium or soft clay, then considering all possible values of μ, and the uncertainty of the effect of contact area where pavement and subgrade meet, it would be on the side of safety to have the thickness of the more resistant flexible pavement at least equal to a, the radius of the equivalent circle. This precaution would tend to confine the dangerous surface to the surfacing material. But it is again necessary to emphasize the fact that since the flexible pavement and the subgrade are very different materials, all such conclusions must be considered as only generally indicative and not strictly true in a quantitative sense.

The greatest values of s_{max} may be computed on the axis of symmetry from equation 12 for different values of μ. Table 3 contains such values and the

TABLE 3

MAXIMUM VALUES OF $\frac{s_{MAX}}{p}$ ON THE AXIS OF SYMMETRY FOR DIFFERENT VALUES OF μ

μ	$\frac{s_{max}}{p}$	$\frac{z}{a}$ at point of s_{max}
0.25	0 34	0.62
0.30	0.33	0.64
0.40	0.31	0.67
0.45	0.30	0.69
0.50	0.29	0.71

points at which they are found are defined in terms of $\frac{z}{a}$. It is interesting to note from this table that the shearing stress does not vary greatly as μ varies from 0.25 to 0.50.

Benkelman (4) has rightfully emphasized our lack of information concerning conditions at the boundary plane of contact between the flexible surface and the supporting medium. It is possible that this information may be obtained by intelligent and careful experimentation, making use of adequate theory in all such experimental procedures.

CONCLUSIONS

On the basis of Love's complete solution of the problem of stresses under a uniformly loaded circular area, it is indicated that the greatest shearing stresses may be confined within the flexible pavement and not reach the subgrade if the thickness of the flexible pavement is no less than the radius of a circle having an area equal to the plane of contact between pneumatic tire and flexible pavement.

The conclusions reached by Carothers are in good agreement with those reached independently by Love, but the method used by Carothers requires a more complete presentation.

In the case of a uniform load on a circular area the variation in shearing stress, as Poisson's ratio varies from 0.25 to 0.50, is relatively small.

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