SOIL DISPLACEMENT UNDER A CIRCULAR LOADED AREA

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SYNOPSIS

A procedure for evaluating the supporting characteristics of the subgrade under flexible types of pavements is indicated This procedure is to use laboratory stress-deformation curves of the subgrade soil in conjunction with rational theoretical analyses This method of approach does not include the making of penetration and loading tests directly on the subgrade, a procedure that has various disadvantages A correlation between measured deflections of the subgrade under wheel load and the deflections computed from laboratory test data and theory is an indicated experimental procedure that should extend our knowledge in this field.

The system of stresses at any point within a semi-infinite, elastically isotropic body produced by a uniform load over a circular area at the surface has been determined by A E H. Love[1] and S. D Carothers[2] Formulas for the vertical displacement, V, at any point of the elastic body due to the surface load are given in various texts[3] dealing with the theory of elasticity. The use of such formulas involves knowledge concerning two elastic constants, Poisson's ratio, μ, and the modulus of elasticity, E. Test data as published by Terzaghi[4] and others show that deformations of soils under load are not characteristic of elastic materials Hence, there is difficulty in applying the formulas based on the assumption of elastic properties.

[1] The Stress Produced in a Semi-Infinite Solid by Pressure on Part of the Boundary by A E H Love Philosophical Transactions of the Royal Society, series A, vol 228, 1929

[2] Test Loads on Foundations as Affected by Scale of Tested Area by S D Carothers Proceedings, International Mathematical Congress, Toronto, pp 527–549, 1924

[3] See for example equations 203 and 204, page 335, of Theory of Elasticity by S Timoshenko McGraw-Hill Book Company, first edition, 1934

[4] Determination of Consistency of Soils by Means of Penetration Tests by Charles Terzaghi Public Roads, vol 7, no 12, February, 1927

The purpose of this paper is to indicate a method of computing vertical displacement in soil due to a uniform load over a circular area by the use of triaxial compression test data The vertical displacement or settlement so computed is that caused solely by lateral yield of the soil It is assumed that this type of settlement, S_L, is completed prior to the realization of any settlement due to consolidation of the supporting soil. In the computations, a modulus of deformation, C, is used in the formulas instead of the modulus of elasticity, E, and since it is assumed that the settlement, S_L, occurs at constant volume μ is necessarily equal to $\frac{1}{2}$

The modulus of deformation, C, is herein defined as the ratio of stress to deformation without regard to the nature of the deformation whether it be elastic or plastic deformation or both.

The method of analysis used in this paper for the case of a uniform load over a circular area may be applied, with certain modifications, for the case of a parabolic or conical distribution of load over the circular area

THE MODULUS, C, IS DETERMINABLE FROM STABILOMETER TEST DATA

In stabilometer tests, cylindrical soil samples encased in rubber sleeves are compressed to failure by applying a vertical load with or without lateral pressure Lateral pressure is applied by air or fluid and is constant during an individual test. The decrease in length, Δ h, of the sample with increasing vertical load may be obtained by means of a micrometer dial attached to the moving plunger or by an automatic recording device. The initial length of the sample is designated as h.

A more complete description of plotting the stabilometer test data in connec-

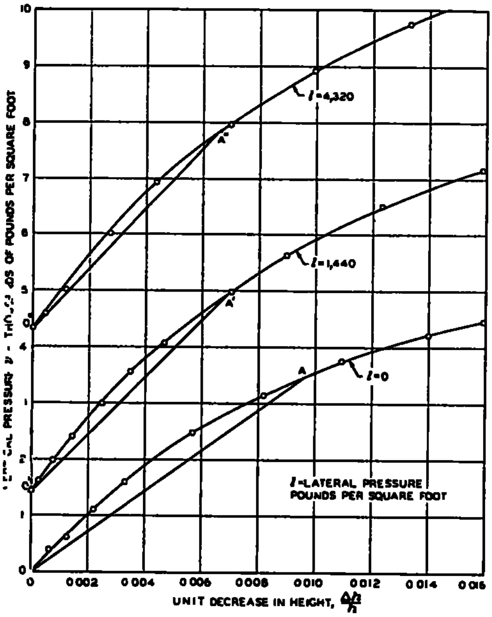

Figure 1. Load-compression test results

tion with computing C has been published elsewhere[5]. In Figure 1, the portions of the three curves for which the vertical pressure, v, is less than the lateral pressure, l, have been omitted and the coordinate axes have been moved to the right so that the point where $l = v$ falls on the $\frac{\Delta h}{h} = 0$ axis.

In Figure 1, the slope of any of the secant lines is taken as the modulus of deformation, C, within the range of loading indicated. The more nearly linear the v versus $\frac{\Delta h}{h}$ relationship, the more nearly the secant modulus becomes a tangent modulus The greater the curvature of the v versus $\frac{\Delta h}{h}$ relationship, the greater is the divergence of the secant line from the curve. Usually this divergence increases with increasing load.

DERIVATION OF EXPRESSIONS FOR S_L

From the theory of elasticity, the vertical strain ϵ_z, at any point on the axis of loading in the stressed earth below the uniformly loaded circular area of radius, a, is

$$\epsilon_z = \frac{\partial V}{\partial z} = \frac{1}{E}(p_z - 2\mu p_r) \ldots . (1)$$

where z is the depth of the point, V is the vertical displacement at the point and p_z and p_r are normal stresses in the vertical and radial directions, respectively, acting at the point.

If instead of E, the modulus of deformation, C, is used and taken as a constant, equation (1) becomes

$$\frac{\partial V}{\partial z} = \frac{1}{C}(p_z - 2\mu p_r) \qquad (2)$$

The expressions for the normal stresses, p_z and p_r, are

$$p_z = p\left[1 - \frac{z^3}{(a^2 + z^2)^{3/2}}\right] \qquad (3)$$

and

$$p_r = \frac{p}{2}\left[1 + 2\mu - \frac{2(1+\mu)z}{(a^2 + z^2)^{1/2}} + \frac{z^3}{(a^2 + z^2)^{3/2}}\right] \qquad (4)$$

where p is the unit surface load. The origin of coordinates is taken at the center of the circular area of earth surface. By

[5] The Settlement of Embankments by L A Palmer and E. S. Barber. Public Roads, vol 21, no 9 November, 1940

substituting equations (3) and (4) in (2) and integrating between the limits, z and ∞, one obtains

$$\text{maximum } V = \frac{p}{C}\left[(2 - 2\mu^2)(a^2 + z^2)^{1/2} - \frac{(1+\mu)z^2}{(a^2+z^2)^{1/2}} + (\mu + 2\mu^2 - 1)z\right] \quad (5)^6$$

By taking $\mu = \frac{1}{2}$,

$$S_L = \frac{3pa^2}{2C(a^2+z^2)^{1/2}}. \quad (6)$$

or in general,

$$\text{max. } V = \frac{pa}{C}F \quad (7)$$

where

$$F = (2 - 2\mu^2)\sqrt{1 + \left(\frac{z}{a}\right)^2} - \frac{(1+\mu)\left(\frac{z}{a}\right)^2}{\sqrt{1+\left(\frac{z}{a}\right)^2}} + (\mu + 2\mu^2 - 1)\frac{z}{a}$$

F may be called the "settlement factor." In Figure 2, F is plotted against values of z/a for values of 0, 0.2, 0.3, 0.4, and 0.5 assigned to μ. It is observed that the value of μ has but very little influence on the value of F if z/a exceeds unity.

In equations (5) and (6), z can have any value. For $\mu = \frac{1}{2}$ and z = 0, equation (6) becomes

$$S_L = \frac{3pa}{2C} \quad (8)$$

which is the total settlement due to lateral yield from the load down to infinite depth, and it also is the downward displacement of a soil particle at the center of the circular surface contact area. Equation (8) gives S_L along the center line. At points removed from the center line, the settlement is less than this value. According to Timoshenko[3] the average settlement under the uniformly loaded circular area, which is the average deflection of all points over the circular contact area, is 85 percent of the maximum deflection at the center. Timoshenko[3] also shows that the average settlement under a uniformly loaded circular area is practically the same as that under a uniformly loaded square area. He shows further that the average settlement under a uniformly loaded rectangular area having sides of

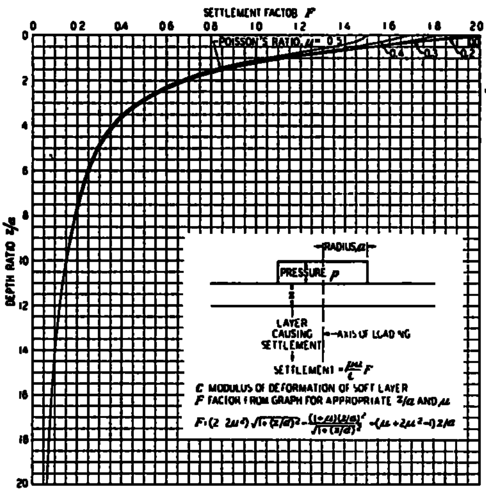

Figure 2. Settlement under center of uniform circular load

ratio 2:1 is about four percent less than that realized in the case of the circle or square. Thus the effect of shape of loaded area on settlement is not so important as might at first be considered.

The depth, z, Figure 2, may be considered as the thickness of an incompressible yet flexible layer, infinite in lateral extent, below which there is yielding soil of infinite depth. Such a condition may be realized for practical purposes if a bed of stabilized soil-aggregate is spread over a clay soil and a load is applied to the stabilized surface. The settlement in

[6] Equation (5) is very similar to equation (3) on page 32 of an article, "Physical Properties of Earth," by John H. Griffith, Bulletin 101, Iowa Engineering Experiment Station, Iowa State College.

[3] Loc. cit., see pages 338 and 339

this case would be due entirely to yielding of the clay if the thickness, z, of the soil-aggregate bed is assumed to remain constant under load and to have no flexural strength. Then, as shown in Figure 2, for given values of p and C increasing the ratio, z/a, either by increasing z or decreasing a, reduces the settlement factor, F, and hence the value of S_L.

This reasoning is based on the simplifying assumption that the bed of material of thickness z and the more yielding material below it comprise a single, homogeneous and isotropic mass of material insofar as the system of stresses is concerned. As pointed out previously,[7] a very similar assumption has been made

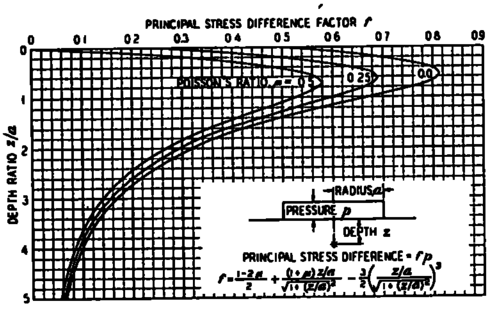

Figure 3. Principal stress difference under center of uniform circular load

and used to very good advantage in estimating the settlements of structures supported by one or more layers of compressible soil with one or more intervening layers of sand.

SHEARING STRESSES UNDER A CIRCULAR LOADED AREA

For a uniform load on a circular area at the surface, the maximum shearing stress at each point on the axis of loading is

$$s_{max} = \tfrac{1}{2}(p_z - p_r) \qquad (9)$$

where the difference, $p_z - p_r$, is the principal stress difference.

[7] "Stresses Under Circular Loaded Areas" by L A Palmer Proceedings of the Highway Research Board, vol. 19, 1939

From equations (3) and (4),

$$p_z - p_r = p\left[\frac{1-2\mu}{2} + (1+\mu)\cdot\frac{z}{(a^2+z^2)^{1/2}} - \frac{3z^3}{2(a^2+z^2)^{3/2}}\right] \qquad (10)$$
$$= pf$$

where

$$f = \frac{1-2\mu}{2} + \frac{(1+\mu)\frac{z}{a}}{\sqrt{1+\left(\frac{z}{a}\right)^2}} - \frac{3}{2}\left(\frac{\frac{z}{a}}{\sqrt{1+\left(\frac{z}{a}\right)^2}}\right)^3$$

Values for the principal stress differences at different depths on the axis of loading and for different values of μ are shown in Figure 3 It may be noted from this figure that for $\mu = \frac{1}{2}$ the maximum principal stress difference is equal to 0.58p and occurs at the depth, z = 0 71a.

It is necessary to bear in mind that $p_z - p_r$ is the difference between the vertical and lateral pressures. Obviously there could be no lateral yielding of the supporting soil if p_z and p_r were equal in magnitude and of the same sign at all points.

In Figure 1, the points A, A′ and A″ were selected so as to give $v - l$ the same value. In general, the modulus, C, obtained from the slopes of secant lines, such as OA, O′A′, etc , vary somewhat with the magnitude of the lateral pressure, l, maintained constant during a single test. Usually the value of C is lowest for the curve, $l = 0$. The procedure in this paper is to use an average value of C obtained from two or more curves of the type shown in Figure 1 for a definite value of $v - l$, applicable to the particular problem. The value of $v - l$ which determines the average C to be used is the maximum principal stress difference at any point within the yielding soil mass and on the axis of loading Obviously this procedure is on the side of safety.

NUMERICAL EXAMPLES

The use of these principles may be illustrated by a numerical example.

Assume the existence of a clay layer extending from the earth surface to an indefinite depth Assume there is a uniform load at the surface of 3 tons per sq. ft distributed over a circular area having a radius of 10 ft Taking μ as $\frac{1}{2}$, the greatest principal stress difference on the axis of symmetry is at a depth of 0 71a = 0 71 × 10 = 7.1 ft. Soil samples taken at this depth have the stress deformation characteristics shown in Figure 1. The greatest principal stress difference = 0.58p = 0 58 × 6,000 = 3,480 lb. per sq ft

On the curve, l = 0, Figure 1, for $v - l$ = 3,480, v = 3,480 lb per sq ft. This is at point A on this curve The secant line, OA is drawn

The slope of OA is $\frac{3{,}480}{.0096}$ or 362,000 lb per sq ft., the value of C from this curve For the curve obtained with l = 1,440 lb. per sq ft in the triaxial test, the point A′ is determined by adding 1,440 to 3,480 to obtain a value, v = 4,920 lb. per sq. ft The corresponding percentage deformation is 0.68 Then C from this curve is $\frac{3{,}480}{.0068}$ or 512,000 lb. per sq. ft. Similarly, from the secant line O″A″ for the curve, l = 4,320 lb per sq ft, C corresponding to v = 7,800 lb. per sq. ft. on the curve, is computed as $\frac{3{,}480}{0065}$ or 535,000 lb per sq ft The average modulus is then $\frac{1}{3}$ (362,000 + 512,000 + 535,000) or 470,000 lb per sq ft

In this problem, the depth of a layer or zone wherein compression due to yielding does not occur is zero and hence z/a is zero. From Figure 2, for z/a = 0 and $\mu = \frac{1}{2}$, F = 1.5. Then

$$S_L = \frac{pa}{C} F = \frac{6{,}000 \times 10}{470{,}000} \times 1.5$$

$$= 0.19 \text{ ft., or about } 2\,3 \text{ in.}$$

Suppose now that there is a bed of compact sand 5 ft thick at the surface and that the clay is below this Neglecting the lateral displacement or vertical compaction of the sand, with z/a = $\frac{5}{10}$ = 0 5, the displacement factor, Figure 2 is seen to be now reduced from 1 5 to 1 33 and the value of S_L is reduced to 2 3 × $\frac{1\,33}{1.5}$ = 2 0 in. Similarly, if the bed of sand is 10 ft thick, z/a becomes 1 and F, Figure 2, is 1 06.

With a bed of compact sand 10 ft. thick, the greatest principal stress difference is in the sand and not in the clay. The greatest principal stress difference in the clay is then 10 ft down from the ground surface on the axis of symmetry, and from Figure 3, its value at the point, z/a = 1, is 0 53p For the soil data plotted in Figure 1, the moduli of deformation corresponding to various $v - l$ values are shown in Figure 4 for the three curves, l = 0, l = 1,440, and l = 4,320 lb per sq ft An average for these three curves is also shown For p = 6,000 lb per sq. ft and a maximum principal stress difference of 0 53p = 0 53 × 6,000 = 3,180 lb per sq ft, the corresponding C from the average curve, Figure 4 is 490,000 lb per sq ft Taking F = 1.06 corresponding to z/a = 1 and $\mu = \frac{1}{2}$,

$$S_L = \frac{6{,}000 \times 10}{490{,}000} \times 1.06$$

$$= 0.13 \text{ ft. or } 1.6 \text{ in.}$$

This is approximately 70 percent of the settlement without the sand, assuming no lateral displacement or compaction of the sand

Actually, the sand would undergo some

lateral displacement or compaction or both but to a considerably lesser extent than a soft clay. Greatest settlements due to lateral yield are to be expected in the case of plastic soils in which pore pressures of considerable magnitude are developed by loading.

With reference again to Figure 4, consider a wheel load producing 10,000 lb. per sq. ft. pressure on the surface of the clay, assuming a balloon tire and an equivalent radius of 6.5 in. for the contact area. Here the greatest principal

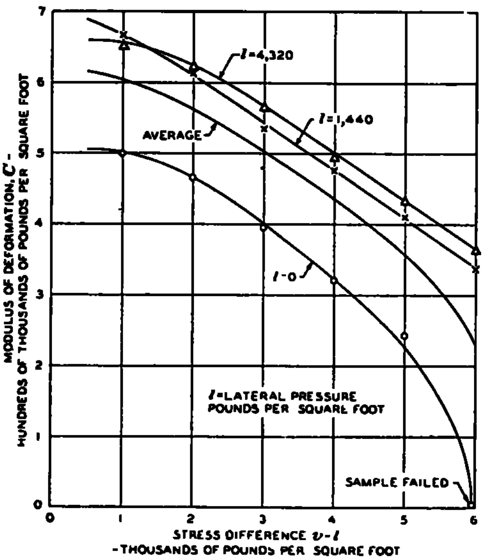

Figure 4. Variation of modulus of deformation with stress

stress difference on the axis of loading is 0.58 × 10,000 or 5,800 lb. per sq. ft. corresponding to an average C value, Figure 4, of 270,000 lb. per sq. ft. For z/a = 0, F = 1.5 and

$$S_L = \frac{10{,}000 \times 6.5}{270{,}000} \times 1.5 = 0.36 \text{ in.}$$

Suppose now that the wheel rests on a flexible type of pavement 6 5 in. thick and that the lateral movement and compaction of pavement material under the wheel load is negligible. Under the pavement is the same clay, extending to an indefinite depth. The maximum principal stress difference in the clay is 0 53p for z/a = 1 and $\mu = \frac{1}{2}$ and 0.53 × 10,000 = 5,300 lb. per sq. ft. The corresponding C value, average curve, figure 4, is 330,000 lb. per sq. ft. and F = 1.06, Figure 2. Then

$$S_L = \frac{10{,}000 \times 6.5}{330{,}000} \times 1.06 = 0.21 \text{ in.}$$

By making similar computations for the same soil and for various values of z, the

TABLE 1

THE EFFECT OF PAVEMENT THICKNESS ON THE ESTIMATED SETTLEMENT, S_L, DUE TO YIELDING OF THE SUBGRADE

Unit load = 10,000 lb. per sq ft, equivalent radius of wheel load = 6.5 in. and $\mu = \frac{1}{2}$

Thickness of pavement, z	$\frac{z}{a}$	Maximum principal stress difference in subgrade, $v - l$	Modulus of deformation, C	Deflection factor, F	Settlement, S_L
inches		*lb. per sq. ft.*	*lb. per sq. ft*		*inches*
0	0	5,800	270,000	1 50	0.36
3 25	0.50	5,800	270,000	1 33	0 32
6 50	1 00	5,300	330,000	1 06	0.21
8 00	1 23	4,600	390,000	0.93	0 16
10 00	1 54	3,750	455,000	0 80	0 11
12 00	1 85	2,950	505,000	0.71	0 09
15 00	2 31	2,200	550,000	0 59	0 07
18 00	2 77	1,600	580,000	0.51	0 06

settlements of Table 1 are obtained. It is observed in Table 1 that when the pavement thickness is of magnitude such that z/a is equal to or greater than two, any further increase in pavement thickness effects only relatively small decreases in S_L. It is also noted that the modulus, C, increases rapidly as the principal stress difference decreases, a fact that is indicative of the curvalinear relationship between v and $\frac{\Delta h}{h}$.

The effect of moisture content and compaction on the modulus, C, of a typical clay soil is shown in Table 2 The settlement, S_L, caused by lateral displacement of this clay when subjected to a uniform load over a circular area is also shown. For the computations shown in Table 2, μ is taken as $\frac{1}{2}$ and the surface load is assumed to be applied directly to the clay, namely, $z/a = 0$

The effect of moisture content and density is strikingly illustrated by the computed values of Table 2. For the condition of ultimate failure in the stabilometer test, the value of C is taken as zero and with reference to the computed values of C, Table 2, it is seen that only for the condition of 100 percent maximum density is this soil able to deform in place without failure under a principal stress difference of 6,000 lb. per sq. ft. which corresponds to a unit surface load of 10,344 lb. per sq. ft. For this particular table, a decrease of 4.8 percent in moisture content, 24.5 to 19 7 percent, is attended by a relatively small gain in dry density, from 103 to 105 lb. per cu. ft. However, the increase in C corresponding to this moisture difference is quite large and is indicative of benefits derivable from having a well compacted and stable subgrade.

TABLE 2

THE EFFECT OF MOISTURE CONTENT AND COMPACTION ON THE MODULUS, C, OF A CLAY SOIL AND ON THE SETTLEMENT, S_L, OF THIS SOIL WHEN UNDER A UNIFORMLY LOADED CIRCULAR AREA OF RADIUS 6 5 IN., $\mu = \frac{1}{2}$

Soil moisture content	Dry density		Modulus, C, corresponding to principal stress difference, $v - l$				Settlement, S_L, corresponding to unit load, $p = \frac{v - L}{0\ 58}$, $\frac{z}{a} = 0$			
			$v - l =$ 500	$v - l =$ 1000	$v - l =$ 3000	$v - l =$ 6000	p = 862	p = 1724	p = 5172	p = 10344
percent of dry weight	*lb. per cu ft*	*percent of maximum density*	*lb per sq. ft*	*lb. per sq ft*	*lb per sq. ft.*	*lb per sq ft.*	*in.*	*in.*	*in.*	*in.*
35 2	86	82	50,000	zero*	zero*	zero*	0 17	failure	failure	failure
30 5	92	88	170,000	80,000	zero*	zero*	0 05	0 21	failure	failure
24 5	103	98	600,000	440,000	120,000	zero*	0 01	0 04	0 42	failure
19 7	105	100	900,000	790,000	590,000	250,000	0 01	0 02	0 09	0 40

* Samples failed at these principal stress differences

CONCLUSIONS

On the basis of the earlier report[7] and on the computations and data in this paper, it is believed that the following general conclusions are warranted.

1. For a uniform load on a circular area at the earth surface, failure of a cohesive supporting earth is most likely to begin at any point of a basin-shaped surface intersecting the axis of symmetry at a depth equal to about 0.7 of the radius of the loaded circular area and extending to the perimeter. For points on this surface, the principal stress differences have maximum values.

2 Since stresses of a certain magnitude may be insufficient to cause failure of the subgrade and yet sufficient to cause considerable settlement, depending on the properties of the subgrade, a knowledge of the stress-deformation characteristics of the subgrade soil is absolutely necessary.

3. It is believed that a method of estimating deflections of the subgrade on the

[7] Loc. cit.

basis of laboratory test data and without the necessity of loading tests in the field is an objective worth striving for.

4 A method is described for determining the modulus of deformation of cohesive soils by means of the triaxial compression device. In general this is a secant modulus that diminishes in magnitude as the principal stress difference is increased.

5 By substituting the appropriate value for this modulus in integrated expressions similar to those that apply to elastic behavior, it is possible to make some sort of estimate of the deflection of the supporting soil under load.

6 It is shown that the movement under stress diminishes according to determinable relations as the thickness of pavement is increased, assuming that both lateral yield and compaction within the flexible pavement itself is relatively small.

7 Values of settlement computed from theory and stabilometer test data indicate the possibility of very materially reducing deflections under wheel loads by having the subgrade compacted at maximum density.