STRESS TRANSMISSION IN EARTHS

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SYNOPSIS

There are many theories for the behavior of earth as an engineering structure A complete solution based on the theory of elasticity satisfies (1) the boundary conditions, (2) the equilibrium equations which give the basic relationships between the rates of change of the stresses, and (3) the compatibility or integrability conditions For the transmission of stress in a semi infinite elastic medium supporting surface loads, the classical solutions of Boussinesq for a normal surface load and that of Cerruti for a tangential surface load satisfy the desired conditions In each of these solutions three of the stresses are found to be independent of the elastic moduli of the medium and three of them depend on Poisson's ratio This latter dependence is a consequence of the fact that the compatibility relations must be satisfied in order to rationalize the system of stresses with the accompanying displacements

J. H Griffith, late professor of civil engineering at Iowa State College, first generalized the stresses of the Boussinesq, by omitting those terms in these stresses which depended upon Poisson's ratio and generalized the radial stress in the form

$$\sigma_R = \frac{-nP}{2\pi R^2} \cos^{n-2}\phi \qquad (n > 2)$$

where R is the radius vector which makes an angle ϕ with the line normal to the surface and through the point of application of the load. He showed that the system of stresses arising from this generalization satisfied only the conditions (1) and (2) but the integrability conditions (3) were not complied with unless some form of variable moduli were employed in the relationships between stress and strain. He did not attempt to furnish the required relationships in order that conditions (3) are satisfied.

The present paper utilizes this generalization of Professor Griffith for the Boussinesq problem and a similar generalization by J Ohde in 1939 for the Cerruti problem to bring out the fact that if a "compression modulus" or perhaps a "foundation modulus" of the Form $E = E_0 z^\lambda$ (where z is the variable depth from the surface) is employed then a complete solution of stress transmission is provided by the generalized system of stresses for any value of n if

$$n-1 = 2 + \lambda = \frac{1}{\nu}$$

where ν is Poisson's ratio. Under these restrictions, conditions (1), (2), and (3) are satisfied. Whether the displacements arising from this complete solution can be correlated with the actual displacements arising in an undisturbed earth remains unanswered

Formulas are presented and the results graphically exhibited for uniform circular loads, parabolic and conical circular loads for n = 3, 4, . . . , 8 Similar formulas for all six stresses induced by normal and tangential surface loads over a rectangular area are given

There are many theories for the behavior of earth as an engineering structure. In the present paper the equilibrium equations of elasticity are employed to present the basic relationships for the transmission of stresses. In this theory a problem is said to be solved when the determined system of stresses satisfy not only the equilibrium equations and boundary conditions but also the "compatibility relationships" or the "integrability conditions." The generalization given in this paper does not assume the homogeneity of the elastic medium. The disparate character of the supporting earth is treated by including a parameter which may be selected by comparing the analysis with field experiment. Although the introduction of this parameter introduces a desirable flexibility in the analysis it also places some restrictions upon the dependence of this parameter and the elastic moduli in order that the compatibility relationships may be satisfied. It is also assumed that the principle of superposition holds so that stress distributions due to distributed surface loads may be obtained by summation methods when the distribution is known for a unit load. Only stresses due to surface loads are considered, and the stresses induced by the weight of the medium are neglected.

In the case of the earth, it is convenient to designate this medium as a semi-infinite elastic medium bounded by a horizontal surface plane which is infinite in extent. The surface loads on this plane may be normal or tangential loads, uniformly or nonuniformly distributed over circular or rectangular areas By combining the results for normal and tangential loads, the distribution of the stresses induced by any directed surface load may be obtained.

KNOWN SOLUTIONS

A formal general solution of the problem for a point load applied on the surface of an isotropic homogeneous medium was given by Boussinesq [1].[1] In his solution the components of the displacement and components of stress were expressed by means of various types of potentials and it was applicable to any form of boundary of the loaded area, and to any law of pressure variation over the area. The stresses induced by a tangential surface force were given by Cerruti [2]. As a special case of these solutions, the two dimensional plane strain and plane stress problems were solved by Michell [3] All these solutions satisfy the proper boundary conditions, the equilibrium conditions and the compatibility relations

During the college school year 1927–28, the late Professor J. H. Griffith [4] of Iowa State College, at a colloquium staff meeting of the mathematics department, first presented his generalized solution of the Boussinesq problem. In 1929 in Engineering and Construction under the caption, "Pressures under Substructures" and subsequently in 1934 in Bulletin 117 of the Iowa State College Engineering Experiment Station, Professor Griffith published his results. In 1933, Dr. O. K. Frohlich [5] in Europe independently arrived at the same result. This generalization of the Boussinesq law of stress propagation is considered as a rectilinear type of stress transmission in that the radial directions emanating from the load point are principal stress directions and that the radial stress σ_R is the only nonvanishing stress. The radial stress σ_R of the Boussinesq problem on a unit horizontal element is the vector sum of the vertical stress σ_z and the tangential stress τ_{rz} on this horizontal element, that is

$$\sigma_R = -\frac{3P}{2\pi R^2}\cos^2\phi = -\frac{3P}{2\pi z^2}\cos^4\phi.$$

This stress on a unit horizontal element is equivalent to the stress

$$\sigma_R = -\frac{3P}{2\pi z^2}\cos^3\phi$$

on the unit element normal to the radial direction making an angle ϕ with the vertical direction (See Fig. 1) The generalization of this stress by Professor Griffith and by Dr. Frohlich is to set

$$\sigma_R = -\frac{nP}{2\pi z^2}\cos^n\phi = -\frac{nP}{2\pi R^2}\cos^{n-2}\phi \quad (1)$$

[1] Figures in brackets refer to list of references at end

where $n > 2$. The case $n = 3$ is that of Boussinesq, producing the corresponding stresses σ_x, σ_y, σ_z, τ_{yz}, τ_{xz}, and τ_{yx} only if Poisson's ratio $\nu = \frac{1}{2}$

The generalized stresses for a point load P normal to the loading surface are. (See Fig. 2)

$$\sigma_z = \sigma_R \cos^2\phi = -\frac{nP}{2\pi z^2}\cos^{n+2}\phi$$

$$\sigma_r = \sigma_R \sin^2\phi = -\frac{nP}{2\pi z^2}\cos^{n}\phi\sin^2\phi$$

$$\tau_{rz} = \sigma_R \sin\phi\cos\phi = -\frac{nP}{2\pi z^2}\cos^{n+1}\phi\sin\phi$$

$$\sigma_x = \sigma_r\cos^2\theta = -\frac{nP}{2\pi z^2}\cos^{n}\phi\sin^2\phi\cos^2\theta$$

$$\sigma_y = \sigma_r\sin^2\theta = -\frac{nP}{2\pi z^2}\cos^{n}\phi\sin^2\phi\sin^2\theta$$

$$\tau_{xz} = \tau_{rz}\cos\theta = -\frac{nP}{2\pi z^2}\cos^{n+1}\phi\sin\phi\cos\theta$$

$$\tau_{yz} = \tau_{rz}\sin\theta = -\frac{nP}{2\pi z^2}\cos^{n+1}\phi\sin\phi\sin\theta$$

$$\begin{aligned}\tau_{xy} &= \sigma_r\sin\theta\cos\theta \\ &= -\frac{nP}{2\pi z^2}\cos^{n}\phi\sin^2\phi\sin\theta\cos\theta. \qquad (2)\end{aligned}$$

The first three of the stresses satisfy the equilibrium equation for an axially symmetric loading, namely

$$\frac{\partial\sigma_z}{\partial z} + \frac{\partial\tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} = 0$$

$$\frac{\partial\sigma_r}{\partial r} + \frac{\partial\tau_{rz}}{\partial z} + \frac{\sigma_r}{r} = 0.$$

The stresses σ_x, σ_y, σ_z, τ_{xz}, τ_{yz}, and τ_{xy} formally satisfy the following differential equations of equilibrium for any value of $n > 2$.

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} = 0$$

$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{yz}}{\partial z} = 0$$

$$\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} = 0. \qquad (3)$$

Although the generalized stresses of Equation (2) satisfy these equilibrium equations, they do not satisfy the compatibility equations for an isotropic medium having constant elastic moduli. The stresses in Eq. (2) satisfy the boundary conditions on the surface plane $z = 0$. At the load point, these stresses become infinite corresponding to a singularity due to a point load. On any plane $z = h$, the total normal force made by the summation of the stress σ_z is statically equivalent to the load $-P$.

GENERALIZATION FOR TANGENTIAL FORCES

Frohlich [5] in his book, "Druckverteilung im Baugrunde," gives a general-

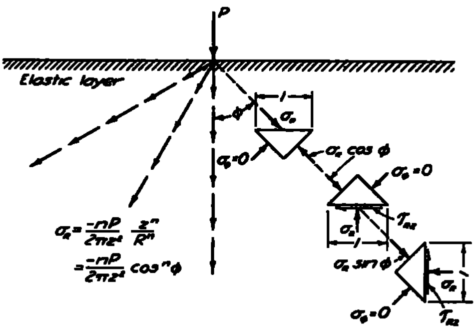

Figure 1

ized formula for a rectilinear propagation of stress induced by a tangential force on the surface of a semi-infinite medium similar to the problem of Cerruti. His generalized stresses are deduced from the stress

$$\sigma_R = -\frac{nP}{2\pi R^2}\sin^{n-2}\phi. \qquad (4)$$

It seems probable that this generalization was obtained from the corresponding generalization for a normal load (Boussinesq case) by replacing ϕ by $\frac{\pi}{2} - \phi$. It is obvious that no axial symmetry is present for a directed tangential load. Again if n is an even integer greater than two, all values of σ_R give compressive stresses throughout the entire medium. While such a state is admissible in the case of a vertical load, it is evident that Eq. (4) is not valid for a tangential load. Ohde [6] gave a correct generalization for a load directed along the x axis by the relationship

$$\sigma_R = -\frac{n(n-2)P}{2\pi R^2} \cdot \cos^{n-3}\phi \sin\phi \cos\theta. \quad (5)$$

The coefficient in (5) has been adjusted to satisfy the static load P acting tangential to the surface and also so that when $n = 3$ the derived stresses agree with those of the classical problem of Cerruti if $\nu = \frac{1}{2}$.

The generalized stresses for a point load P tangential to the surface are

$$\sigma_z = \sigma_R \cos^2\phi = -\frac{n(n-2)P}{2\pi z^2} \cos^{n+1}\phi \sin\phi \cos\theta$$

$$\sigma_r = \sigma_R \sin^2\phi = -\frac{n(n-2)P}{2\pi z^2} \cos^{n-1}\phi \sin^3\phi \cos\theta$$

$$\tau_{rz} = \sigma_R \sin\phi \cos\phi = -\frac{n(n-2)P}{2\pi z^2} \cos^{n}\phi \sin^2\phi \cos\theta$$

$$\sigma_x = \sigma_r \cos^2\theta = -\frac{n(n-2)P}{2\pi z^2} \cos^{n-1}\phi \sin^3\phi \cos^3\theta$$

$$\sigma_y = \sigma_r \sin^2\theta = -\frac{n(n-2)P}{2\pi z^2} \cos^{n-1}\phi \cdot \sin^3\phi \cos\theta \sin^2\theta$$

$$\tau_{xy} = \tau_{rz} \cos\theta = -\frac{n(n-2)P}{2\pi z^2} \cos^{n}\phi \sin^2\phi \cos^2\theta$$

$$\tau_{yz} = \tau_{rz} \sin\theta = -\frac{n(n-2)P}{2\pi z^2} \cos^{n}\phi \cdot \sin^2\phi \sin\theta \cos\theta$$

$$\tau_{xz} = \sigma_r \sin\theta \cos\theta = -\frac{n(n-2)P}{2\pi z^2} \cos^{n-1}\phi \cdot \sin^3\phi \sin\theta \cos^2\theta. \quad (6)$$

The stresses in Eq. (6) formally satisfy Eq. (3) for any value of n greater than 2. They also satisfy the proper loading conditions at the boundary surface, but they do not satisfy the compatibility relations of an elastic medium with constant moduli unless $n = 3$ and $\nu = \frac{1}{2}$. In the case of earths a "modulus of compression" or more appropriately a "foundation modulus" of the form

$$E = E_0 z^\lambda$$

is a conceivable modulus which varies with the depth z measured from the surface of loading. However, such a modulus vanishes when $z = 0$, and may not appear feasible. Nevertheless such a modulus might be applied to an undisturbed earth which initially offers no resistance to initial displacement of the boundary surface. Another form of such a modulus which avoids this difficulty is $E = E_0(z_0 + z)^\lambda$. Ohde [6] has shown that the stresses given in Eqs. (2) and (6) will satisfy a set of compatibility equations when derived from this modulus if

$$n - 1 = \lambda + 2 = 1/\nu, \quad (7)$$

and $z_0 = 0$. Thus when $n = 3$, $\lambda = 0$, and $\nu = \frac{1}{2}$, the classical compatibility relations hold for in that case Equations (2) and (6) reduce to those of Boussinesq and Cerruti, provided that in the latter $\nu = \frac{1}{2}$. Not only do the restrictions given in Eq. (7) qualify the given stresses to satisfy a consistent elastic theory from which the relative displacements may be obtained but they also insure a minimum strain energy per unit volume.

PLANE STRAIN CASES

The plane strain case of a line load is most readily obtained from the generalized three dimensional case by summing the stresses for an infinite line load. (See Fig. 3.)

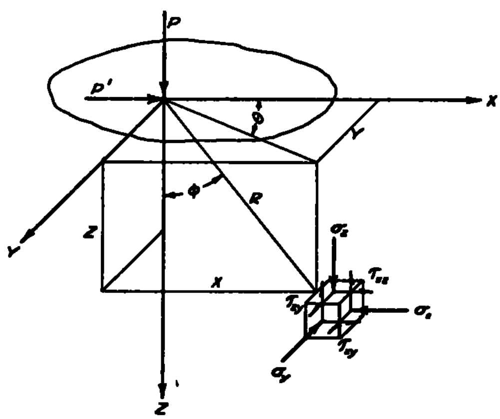

Figure 2

The stresses for a *normal line load* of constant intensity p as obtained from this summation are of the form

$$\sigma_z = -\frac{p}{z} K \cos^{n+1}\phi$$

$$\sigma_x = -\frac{p}{z} K \cos^{n-1}\phi \sin^2\phi$$

$$\tau_{xy} = -\frac{p}{z} K \cos^{n}\phi \sin\phi \qquad (8)$$

where the constant K is determined by statically balancing the load. The values of K are

$$n = 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8$$

$$K = \frac{2}{\pi} \quad \frac{3}{4} \quad \frac{8}{3\pi} \quad \frac{15}{16} \quad \frac{16}{5\pi} \quad \frac{35}{32}. \qquad (9)$$

When $n = 3$, the stresses in Eq. (8) yield the known stresses for the classical case

$$\sigma_z = -\frac{2p}{\pi z} \cos^4\phi$$

$$\sigma_x = -\frac{2p}{\pi z} \cos^2\phi \sin^2\phi$$

$$\tau_{xy} = -\frac{2p}{\pi z} \cos^3\phi \sin\phi. \qquad (10)$$

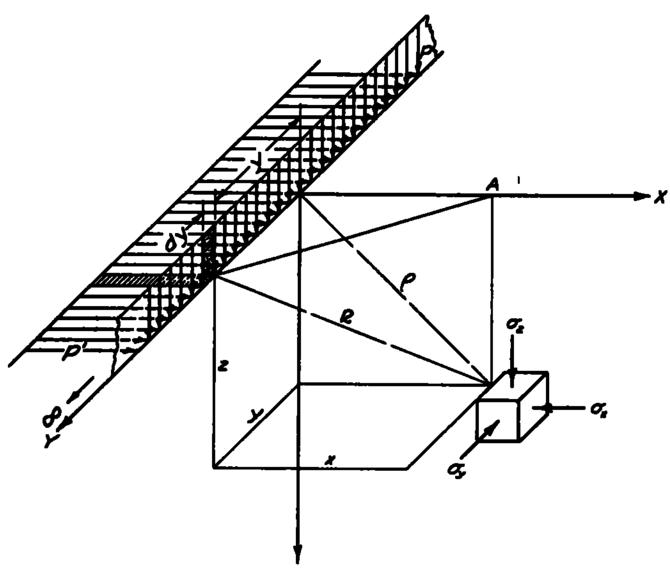

Figure 3

The stresses for the plane strain case of a tangential line load are

$$\sigma_z = -(n-2)K\frac{p}{z}\cos^n\phi\sin\phi$$

$$\sigma_x = -(n-2)K\frac{p}{z}\cos^{n-2}\phi\sin^3\phi$$

$$\tau_{xy} = -(n-2)K\frac{p}{z}\cos^{n-1}\phi\sin^2\phi \qquad (11)$$

with K having the values in (9). Ohde [6] has doubted the validity of Eq. (10) and corresponding ones from Eq. (11) for $n = 3$ for the plane case unless Poisson's ratio $\nu = \frac{1}{2}$. It is not difficult to show that these stresses in the classical plane strain case are independent of ν.

APPLICATIONS

Uniform circular load. The maximum vertical compressive stress on the central axis at any depth z is

$$\sigma_z = -p\left[1 - \left(\frac{z}{\sqrt{z^2+a^2}}\right)^n\right] = -p[1 - \cos^n\phi_0], \qquad (12)$$

where $\tan\phi_0 = a/z$.

Similarly the generalized lateral stress on the central axis is

$$\sigma_x = -\frac{p}{2}\left[\frac{n}{n-2}(1-\cos^{n-2}\phi_0) - (1-\cos^n\phi_0)\right]. \qquad (13)$$

The stress σ_z in Eq. (12) can be graphically determined by a generalization of Newmark [1] charts. The principal shearing stress on the central axis is found by taking half the difference of σ_z and σ_x. The maximum value of this principal stress τ occurs at $z/a = \sqrt{2}/2$ regardless of the value of n. At $z = 0$, $\sigma_x = -p/(n-2)$ and $\sigma_z = -p$. These stresses are shown in Fig 4.

Parabolic distribution over circular area. The maximum vertical and lateral compressive stress on the central axis is (See Fig. 5):

$$\sigma_z = -np_0\left[\frac{a^2+z^2}{na^2}(1-\cos^n\phi_0) - \frac{z^2(1-\cos^{n-2}\phi_0)}{a^2(n-2)}\right]$$

$$\sigma_x = -\frac{np_0}{2a^2}\left[-\frac{(a^2+z^2)}{n}(1-\cos^n\phi_0) + \left(\frac{a^2+2z^2}{n-2}\right)(1-\cos^{n-2}\phi_0) - \frac{z^2}{n-4}(1-\cos^{n-4}\phi_0)\right]. \qquad (14)$$

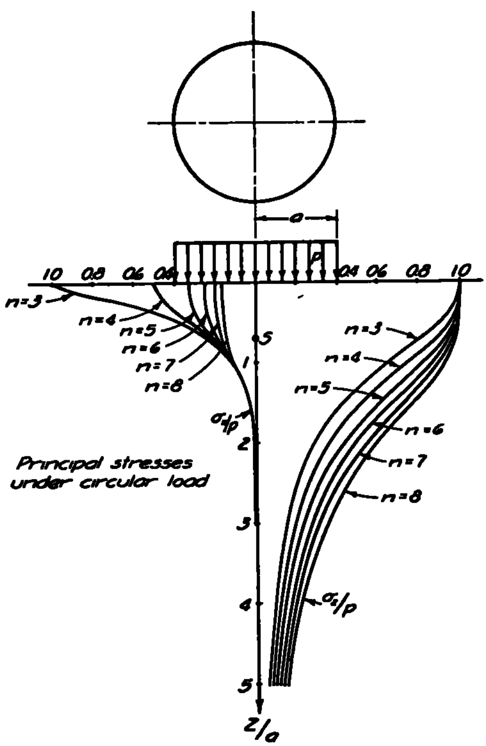

Figure 4

Conical distribution over circular area. The central vertical compressive stresses are

$n = 3 \quad \sigma_z = -p_0(1 + \cos\phi_0 - 2\cos^3\phi_0)$

$n = 4 \quad \sigma_z = -p_0(1 - \frac{1}{2}\cos^2\phi_0 - \frac{1}{2}\phi_0\cot\phi_0)$

$$n = 5 \quad \sigma_z = -p_0 \left(1 + \tfrac{2}{3} \cos \phi_0 + \tfrac{1}{3} \cos^3 \phi_0 - 2 \cos^5 \phi_0\right)$$

$$n = 6 \quad \sigma_z = -p_0 \left(1 - \tfrac{3}{8} \phi_0 \cot \phi_0 - \tfrac{3}{8} \cos^2 \phi_0 - \tfrac{1}{4} \cos^4 \phi_0\right)$$

$$n = 7 \quad \sigma_z = -p_0 \left(1 - \tfrac{8}{15} \cos \phi_0 - \tfrac{4}{15} \cos^3 \phi_0 - \tfrac{1}{5} \cos^5 \phi_0\right)$$

$$n = 8 \quad \sigma_z = -p_0 \left(1 - \tfrac{5}{16} \phi_0 \cot \phi_0 - \tfrac{5}{16} \cos^2 \phi_0 - \tfrac{5}{24} \cos^4 \phi_0 - \tfrac{1}{6} \cos^6 \phi_0\right). \quad (15)$$

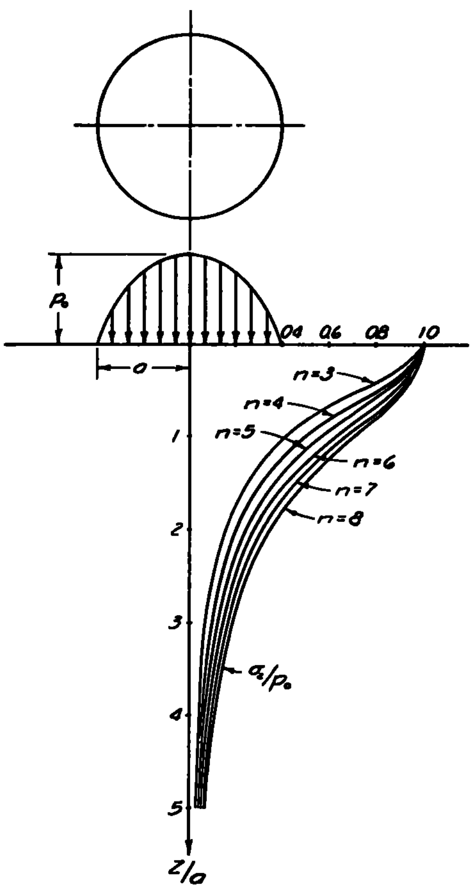

Figure 5

Under the center of the load the stress σ_z is the compressive load $-p$. The stress vanishes at $z = \infty$. In the interest of brevity the central compressive lateral stresses σ_x are omitted

In all of the above axially symmetric cases, the general stresses at any point in the supporting medium are not given by simple functions.

Rectangular Loads of Uniform Distribution

Adopting the notation of Figure 6 the following results are given for the stresses directly under the corner of a rectangle of dimensions a by b. Two types of loads are shown in Figure 6. In one section there is a normal load over a rectangular

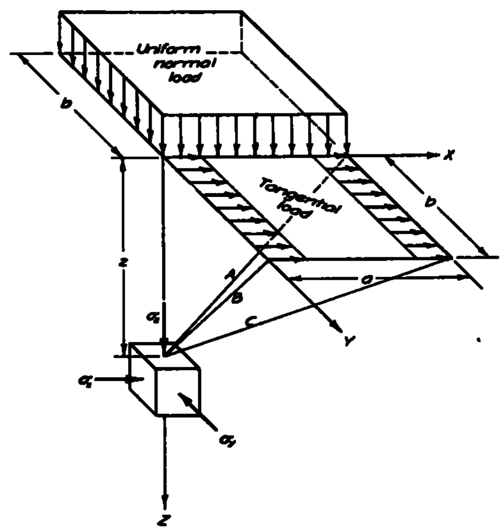

Figure 6

area and in another section there is a tangential load over a rectangular area.

$n = 3$ *Normal load*

$$\sigma_z = -\frac{p}{2\pi}\left[\arctan \frac{ab}{zC} + \frac{abz}{C}\left(\frac{1}{A^2} + \frac{1}{B^2}\right)\right]$$

$$\sigma_x = -\frac{p}{2\pi}\left[\arctan \frac{ab}{zC} - \frac{abz}{A^2C}\right]$$

$$\sigma_y = -\frac{p}{2\pi}\left[\arctan \frac{ab}{zC} - \frac{abz}{B^2C}\right]$$

$$\tau_{xz} = -\frac{p}{2\pi}\left[\frac{b}{B} - \frac{z^2 b}{A^2C}\right]$$

$$\tau_{yz} = -\frac{p}{2\pi}\left[\frac{a}{A} - \frac{z^2 a}{B^2 C}\right]$$

$$\tau_{xy} = -\frac{p}{2\pi}\left[1 + \frac{z}{C} - z\left(\frac{1}{A} - \frac{1}{B}\right)\right]. \quad (16)$$

$n = 3$ *Tangential shear load in x direction*

$$\sigma_z = -\frac{p}{2\pi}\left[\frac{b}{B} - \frac{z^2 b}{A^2 C}\right]$$

$$\sigma_x = -\frac{p}{\pi}\left[\log\frac{A(b+B)}{z(b+C)} - \frac{a^2 b}{2A^2 C}\right]$$

$$\sigma_y = -\frac{p}{2\pi}\left[\log\frac{A(b+B)}{z(b+C)} - b\left(\frac{1}{B} - \frac{1}{C}\right)\right]$$

$$\tau_{xz} = -\frac{p}{2\pi}\left[\arctan\frac{ab}{zC} - \frac{abz}{A^2 C}\right]$$

$$\tau_{yz} = -\frac{p}{2\pi}\left[1 + \frac{z}{C} - z\left(\frac{1}{A} + \frac{1}{B}\right)\right]$$

$$\tau_{xy} = -\frac{p}{2\pi}\left[\log\frac{(A+a)(C-a)}{zb} + a\left(\frac{1}{C} - \frac{1}{A}\right)\right]. \quad (17)$$

$n = 4$ *Normal load*

$$\sigma_z = -\frac{p}{2\pi}\left[\frac{a}{A}\left(1 + \frac{z^2}{2A^2}\right)\arctan\frac{b}{A} + \frac{b}{B}\left(1 + \frac{z^2}{2B^2}\right)\arctan\frac{a}{B} + \frac{abz^2}{2C^2}\left(\frac{1}{A^2} + \frac{1}{B^2}\right)\right]$$

$$\sigma_x = -\frac{p}{4\pi}\left[\frac{a^3}{A^3}\arctan\frac{b}{A} + \frac{b}{B}\arctan\frac{a}{B} - \frac{abz^2}{A^2 C^2}\right]$$

$$\tau_{xz} = -\frac{p}{4\pi}\left[\arctan\frac{b}{z} - \frac{z^3}{A^3}\arctan\frac{b}{A} + bz\left(\frac{1}{B^2} - \frac{z^2}{A^2 C^2}\right)\right]$$

$$\tau_{xy} = -\frac{p}{4\pi}\left[\frac{(a^2+b^2)^2}{z^2 C^2} - \frac{1}{z^2}\left(\frac{a^4}{A^2} + \frac{b^4}{B^2}\right)\right]. \quad (18)$$

$n = 4$ *Tangential load*

$$\sigma_x = -\frac{p}{\pi}\left[\arctan\frac{b}{z} - \frac{z}{A}\left(1 + \frac{a^2}{2A^2}\right)\cdot\arctan\frac{b}{A} - \frac{a^2 bz}{2A^2 C^2}\right]$$

$$\sigma_y = -\frac{p}{2\pi}\left[\arctan\frac{b}{z} - \frac{z}{A}\arctan\frac{b}{A} - bz\left(\frac{1}{B^2} - \frac{1}{C^2}\right)\right]$$

$$\tau_{xy} = -\frac{p}{2\pi}\left[\arctan\frac{a}{z} - \frac{z}{B}\arctan\frac{a}{B} + \frac{a}{z}\left(\frac{a^2}{A^2} - \frac{a^2+b^2}{C^2}\right)\right]. \quad (19)$$

$n = 5$ *Normal load*

$$\sigma_z = -\frac{p}{2\pi}\left[\arctan\frac{ab}{zC} + \frac{abz}{C}\left(\frac{1}{A^2} + \frac{1}{B^2}\right) + \frac{abz^3}{C}\left(\frac{1}{A^4} + \frac{1}{B^4}\right) - \frac{abz^3}{3C^3}\left(\frac{b^2}{A^4} + \frac{a^2}{B^4}\right)\right]$$

$$\sigma_x = -\frac{p}{6\pi}\left[\arctan\frac{ab}{zC} - \frac{abz}{A^2 C} + \frac{abz}{C}\left(\frac{3a^2}{A^4} - \frac{b^2}{A^2 C^2} + \frac{a^2}{B^2 C^2}\right) + \frac{ab^3 z^3}{C^3 A^3}\right]$$

$$\tau_{xz} = -\frac{p}{3\pi}\left[\frac{b}{B} - \frac{bz^4}{CA^4} + \frac{bz^2}{2}\left(\frac{1}{B^3} - \frac{1}{C^3}\right) + \frac{bz^2 a^2}{2C^3 A^2}\right]$$

$$\tau_{xy} = -\frac{p}{6\pi}\left[1 - z^3\left(\frac{1}{A^3} - \frac{1}{B^3} - \frac{1}{C^3}\right)\right]. \quad (20)$$

$n = 5$ *Tangential load*

$$\sigma_x = -\frac{p}{\pi}\left[\frac{b}{B} - \frac{b}{C}\left(1 + \frac{a^2}{2C^2}\right) + \frac{a^4 b}{A^4 C}\left(1 + \frac{A^2}{2C^2}\right)\right],$$

$$\sigma_y = -\frac{p}{2\pi}\left[\frac{b^3 a^2}{A^2 C^3} + b^3\left(\frac{1}{B^3} - \frac{1}{C^3}\right)\right]. \quad (21)$$

$n = 6$ *Normal load*

$$\sigma_z = -\frac{p}{2\pi}\left[\frac{a}{A}\left(1 + \frac{z^2}{2A^2} + \frac{3z^4}{8A^4}\right)\arctan\frac{b}{A} + \frac{b}{B}\left(1 + \frac{z^2}{2B^2} + \frac{3z^4}{8B^4}\right)\arctan\frac{a}{B} + \frac{abz^2}{2C^2}\left\{\left(\frac{1}{A^2} + \frac{1}{B^2}\right) + \frac{z^2}{2C^2}\left(\frac{1}{A^2} + \frac{1}{B^2}\right) + \frac{3z^2}{4}\left(\frac{1}{A^4} + \frac{1}{B^4}\right)\right\}\right]. \quad (22)$$

$n = 7$

$$\sigma_z = [\sigma_z]_{n=5} - \frac{abz^5 p}{C 2\pi}\left[\left(\frac{1}{A^6} + \frac{1}{B^6}\right) - \frac{2}{3C^2}\left(\frac{b^2}{A^6} + \frac{a^2}{B^6}\right) + \frac{1}{5C^4}\left(\frac{b^4}{A^6} + \frac{a^4}{B^6}\right)\right]. \quad (23)$$

$n = 8$

$$\sigma_z = [\sigma_z]_{n=6} - \frac{p}{2\pi}\left[\frac{5az^6}{16A^7}\arctan\frac{b}{A} + \frac{5bz^6}{16B^7}\arctan\frac{a}{B} + \frac{abz^6}{2C^2} \cdot \left\{\frac{1}{3C^4}\left(\frac{1}{A^2} + \frac{1}{B^2}\right) + \frac{5}{12C^2}\left(\frac{1}{A^4} + \frac{1}{B^4}\right) + \frac{5}{8}\left(\frac{1}{A^6} + \frac{1}{B^6}\right)\right\}\right]. \quad (24)$$

For $n = 3$, all six stresses are given for both the normal and the tangential loads. In the case of the normal loading the stresses σ_x and σ_y as well as τ_{xz} and τ_{yz} are obtained from each other by interchange of the x and y directions, that is, by replacing a, b, A, B by b, a, B, A, respectively. Thus in cases $n = 4$ and $n = 5$ only four of these stresses are included. In the tangential or shear loading all six stresses are given for the case $n = 3$. One notes that the stresses (when $n = 3$)

τ_{xz}, τ_{yz}, and σ_z for a tangential load agree respectively with

σ_x, τ_{xy}, and τ_{xz} of a normal load.

A comparison of these respective stresses in Eqs. (6) and (2) shows that these three stresses for a shearing surface load may always be obtained from the corresponding three stresses for a normal load if the latter are multiplied by the factor $(n - 2)$. This explains the peculiar listing of the stresses for the cases $n = 4$ and $n = 5$. Only the direct stress σ_z due to a normal load is given in the cases $n = 6$, 7, and 8. The complete system for these cases is quite lengthy and is omitted in this abbreviated paper. One observes that the first two terms in σ_z for $n = 5$ coincides with σ_z for $n = 3$. Similarly one may write Eqs. (23) and (24) for σ_z in cases $n = 7$ and $n = 8$ in terms of the earlier values of σ_z for $n = 5$ and $n = 6$, respectively.

The variation of the stress σ_z on the vertical axis under a normal load over a square area is shown in Figure 7 for different values of n These curves are similar to those in Figure 4 for a uniform load over a circular area of radius a but entail more calculation than is needed in Eq. 12 for the latter curves. Figure 8 graphically presents the variation in the central compressive stress for three values of the concentration factor n, when the area of normal loading changes from $b/a = 1$ to $b/a = \infty$ The lateral variation of the vertical stress σ_z at three depths z/a is shown in Figures 9 and 10. In Figure 9, $b/a = 1$ and the stresses for $n = 3$ (right) and $n = 6$ (left) occur on the horizontal elements which are normal to the vertical plane of symmetry through the centroid of the loaded area. In Figure 10 where $b/a = 2$, the lateral

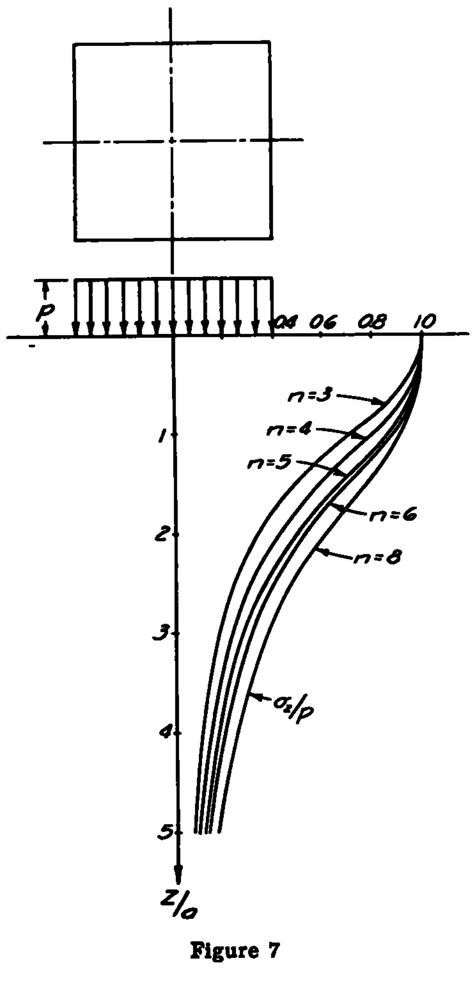

Figure 7

variation of σ_z is shown for the longitudinal and transverse directions. One readily observes that a greater concentration of stress exists about the vertical axis of symmetry when the law of transmission involves the larger values of n. When tables or graphs are given for the stress σ_z under the corner or center of a rectangular loading, it is easier to evaluate the lateral distribution under rectangular loads than under circular loads, since the latter are not generally given by elementary functions.

Figure 11 is a reproduction of Figure 15, p. 250 of the paper, "Wheel Load Stress Distribution Beneath Flexible Type Pavements" by Spangler and Ustrud in this volume. The heavy dots are values calculated for σ_z from Eq. 22 for a rectangular area $b/a = 2$ for a total load of 3000 pounds applied by a tire. The calculated values agree with the experimental values just as well as the empirical curve does.

By appropriately superposing rectangles, the above equations may be employed to completely describe the state of stress at an arbitrary point in the supporting medium. For example, under the center of a rectangle of normal loading of dimensions $2a$ by $2b$ stresses σ_x, σ_y, σ_z are multiplied by four, whereas the stresses τ_{xz}, τ_{yz} and τ_{xy} vanish. If two rectangles are placed so that the dimensions are a by $2b$, and b allowed to become infinite the results produce the known results for line loads. These latter stresses for the line loads may be checked by direct integration of the appropriate equations selected from Eqs. (8) and (11). Even the stress σ_y as obtained from this infinitely long rectangle reduces to the corresponding plane strain equivalent

$$\sigma_y = \nu(\sigma_x + \sigma_y)$$

if for each n, the value used for Poisson's ratio is $\nu = 1/(n - 1)$, again agreeing with the result stated in Eq. (7).

In all of the above cases the stresses produced by the surface loads are derived by the principle of superposition and from a direct generalization of the Boussinesq and Cerruti formulas. They satisfy the equilibrium equations and the compatibility relations which result when the modulus E is given as a function of the depth z, provided that if $z_0 = 0$, then

$$n = \lambda + 3 = \frac{1 + \nu}{\nu}$$

Similarly the lateral stress σ_x and σ_y have surface values of $-p/(n - 2)$ under the loaded portion, except at the edges and corners where they suffer abrupt changes.

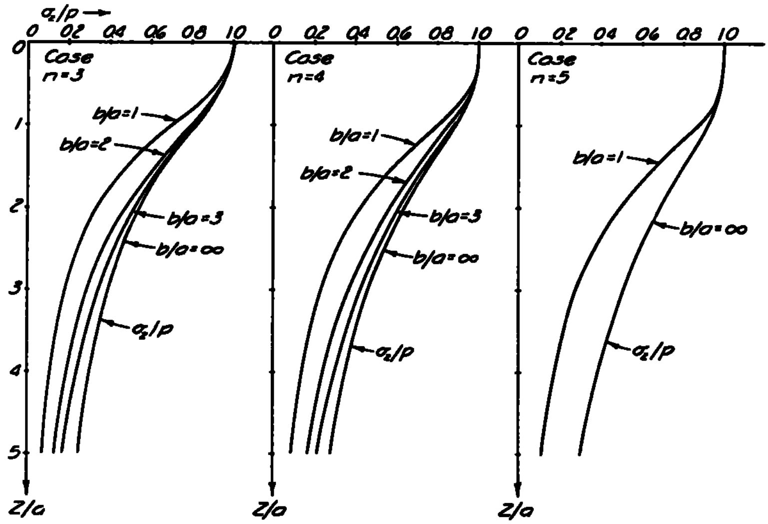

Figure 8

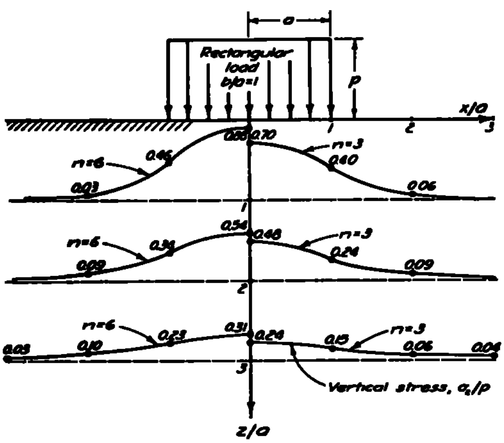

Figure 9

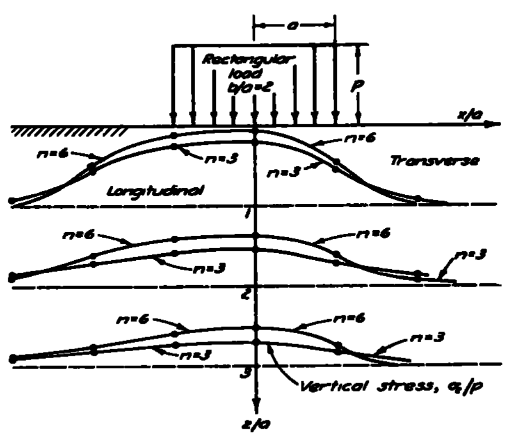

Figure 10

holds. The boundary conditions at the surface $z = 0$ are always satisfied under the loaded surface as well as at surface points exterior to the load. The usual discontinuities arise at the edges and corners of a uniformly loaded area. For example the edge stress is $\sigma_z = -p/2$ and at the corners of the rectangle it is $-p/4$.

In the classical case $n = 3$, Love [7] discusses the indeterminacy of some of the other stresses at the edges of the rectangle The results for the normal load for $n = 3$ may be confirmed by comparing with the results of Love for the rectangular loading if in the latter ν is given the value of $\frac{1}{2}$.

In the classical case, the lateral stress σ_x for a point load P is

$$\sigma_x = -\frac{P}{2\pi}\left[\frac{3x^2 z}{R^5} - (1 - 2\nu) \cdot\left(\frac{z}{R^3} - \frac{1}{R(R+z)} + \frac{(2R+z)x^2}{R^3(R+z)^2}\right)\right].$$

$$\sigma_y = -\frac{2p}{\pi}\left[\arctan\frac{ab}{zC} - \frac{abz}{B^2C} + (1 - 2\nu)\left(\arctan\frac{a}{b} - \arctan\frac{za}{bC} - \arctan\frac{ab}{zC}\right)\right].$$

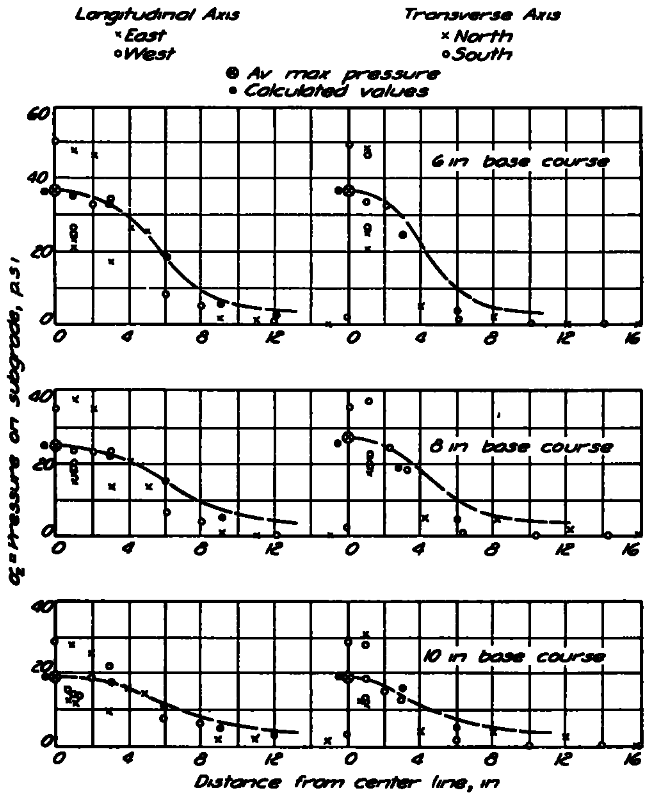

Figure 11

A similar form holds for σ_y. By summation one finds that the lateral stresses under the center of a normal surface load over a region $2a$ by $2b$ are

$$\sigma_x = -\frac{2p}{\pi}\left[\arctan\frac{ab}{zC} - \frac{abz}{A^2C} + (1 - 2\nu)\left(\arctan\frac{b}{a} - \arctan\frac{zb}{aC} - \arctan\frac{ab}{zC}\right)\right]$$

It will be seen that when $\nu = \frac{1}{2}$ these results confirm the generalized stresses of Eq. (16). When $b \to \infty$, these results yield the correct "plane deformation" stresses when ν is arbitrary and it is not required that $\nu = \frac{1}{2}$ as Ohde [6] has contended in order for the two dimensional plane strain problem to be valid.

The result for σ_z in the classical case $n = 3$ as given in the first of Eq. (16) was obtained independently by the author early in 1928. It was suggested to the writer by Dean Marston and the result employed by him in his studies on loaded culverts. This result was transmitted by Dean Marston to the Bureau of Public Roads in 1929 and later published in Bulletin 96 of the Iowa State College Engineering Experiment Station. A similar result for this stress was published in 1935 by N. W. Newmark in Circular No. 24 of the University of Illinois Engineering Experiment Station.

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