STABILITY OF COHESIVE EARTH MASSES IN VERTICAL EMBANKMENTS[1]

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This paper contains a direct mathematical derivation of the failure surface in its simplest form of vertical cohesive embankments synopsis, together with an investigation of the equilibrium of this embankment section as to its effect on maximum stress, critical height, and stability. Model studies were made to provide physical corroboration and to emphasize the significance of the mathematical findings Verification was obtained by means of gelatin models, of the effect of tension on the sliding surface, and of the outline of the failure surface itself

It is said that the conception of initial failure due to tension is of definite importance in safe design practice and in the study of the mechanics of embankment failure. Current design methods which presume to mobilize shear along the entire sliding surface are subject to occasional discrepancies in practice. Many cohesive soils, when moisture content and other conditions are favorable, display a certain quasi-tension or "suction", and may for a time develop shearing resistance over the whole surface. Under other circumstances, however, such as the bank of a recently drained reservoir, a tension crack may precipitate sudden shear failure. The report shows that embankments based on critical heights near 4 S/w are best suited for temporary work and require an element of judgment in their use. Conservative design values in the range of 2 S/w are recommended for permanence and safety.

The problem here investigated is that of the stability and the mechanics of failure of vertical cohesive embankments subject to the vertical gravitational forces of their own weight. This problem has received attention in technical literature from time to time, most treatments known to the writer belonging to one of two general classifications. The solutions such as the Fellenius "Swedish Circle" method, based on assumed sliding surfaces, are characteristic of the first group, and the mathematical derivation of maximum stress by Résal and Frontard represent the second In neither case can the embankment sections arrived at satisfy all the three requirements of static equilibrium when subject to a rigid vector analysis.

The solution presented in this paper consists of a direct mathematical derivation of the failure surface in its simplest form, together with an investigation of the equilibrium of this embankment section as to its effect on maximum stress, critical height, and stability.

In order to treat this specific problem mathematically certain assumptions and idealizations of the cohesive material and its behavior are necessary. The material considered in this analysis is assumed to be purely cohesive and by definition is capable of maintaining a constant difference between principal stresses. The shearing stress is thus independent of normal stress and the material fails according to the maximum shear theory along the path of maximum unit shearing stress.

Cohesive materials are considered to be in the plastic state and to deform according to the laws of plastic flow. Stress is independent of deformation beyond the yield point and the material in the zone of the shear failure surface will attain a uniform, maximum, shearing stress before final failure by slipping takes place.

The usual conventional mathematical notation and arrangement are used in the

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derivation with the X axis horizontal and parallel to the ground surface, and the Y axis vertical. The following list shows some of the more important symbols:

S = Unit shearing stress.
S_M = Maximum unit shearing stress.
N = Normal stress.
h = Vertical height of embankment.
h_{cr} = Critical height of embankment at which failure is impending.
l = Length of the curved failure surface.
W = Weight of the entire failure section.
w = Unit weight.
θ = Deflection from the vertical. (In general.)
α = Slope of failure surface at given point.
$x' = \dfrac{dx}{dy}$

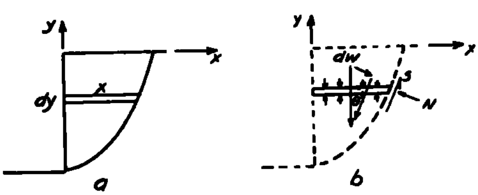

Figure 1

THE DIFFERENTIAL EQUATION OF THE FAILURE SURFACE

The first step in a logical mathematical analysis of embankment stability is the determination of the most probable zone, or surface, of failure. In order to obtain a solution in general terms a vertical embankment of cohesive material is first conceived to exist with sliding failure impending on a perfectly general surface of maximum unit shearing stress, extending from some point on the upper ground surface to the toe of the vertical face. See Figure 1-a.

From the figure of unit thickness outlined by this general curve, a horizontal section of x length and dy height, Figure 1-b, is selected for further consideration. By replacing the surrounding portions of the embankment by their reactions a free body $(w)\ x\ dy$ was obtained, supported in space by forces identical to those existing in the embankment. Since by definition normal forces are not pertinent here the stability of the section can be expressed:

$$dW \cos\theta = S\, dl. \quad \ldots\ldots(1)$$

Or algebraically:

$$(w)\ x\ dy \frac{1}{\sqrt{1+(x')^2}} = S\sqrt{1+(x')^2}\, dy \ldots.(2)$$

Note: It should be pointed out that the forces in the section are not concurrent, and although the summation of the horizontal and vertical forces equals zero, a residual unbalanced couple is set up This is the origin of the overturning couple referred to later in the solution.

Integrating (2) over the entire figure:

$$\int_0^y \frac{x\,dy}{\sqrt{1+(x')^2}} = S_w \int_0^y \sqrt{1+(x')}\, dy$$

Or:

$$S = \frac{{}_w\int_0^y \frac{x\,dy}{\sqrt{1+(x')^2}}}{\int_0^y \sqrt{1+(x')^2}\,dy} \quad . \qquad (3)$$

Differentiating this expression for maximum shear S.

$$O = \frac{{}_w\int_0^y \sqrt{1+(x')^2}\, dy \left[\frac{x}{\sqrt{1+(x')^2}}\right] - {}_w\int_0^y \frac{x\,dy}{\sqrt{1+(x')^2}}\left[\sqrt{1+(x')^2}\right]}{V^2}$$

Or:

$$\frac{{}_w\int_0^y \frac{xdy}{\sqrt{1+(x')^2}}}{\int_0^y \sqrt{1+(x')^2}\,dy} = \frac{xw}{1+(x')^2} \quad (= S_{Max.})$$

$$S_{Max.} = \frac{xw}{1+(x')^2} \qquad (4)$$

From the solution of this differential equation proceeds the actual equation of the surface of maximum shear failure.

$$ax = \frac{a^2}{4}(y + b)^2 + 1 \ldots \quad . \quad (5)$$

By proper application of the boundary conditions the surface of sliding shear failure is shown to be that portion of a second degree parabola between the axis of the curve and the latus rectum. These two lines coincide with the upper ground surface and the vertical face of the embankment respectively. See Figure 2 below.

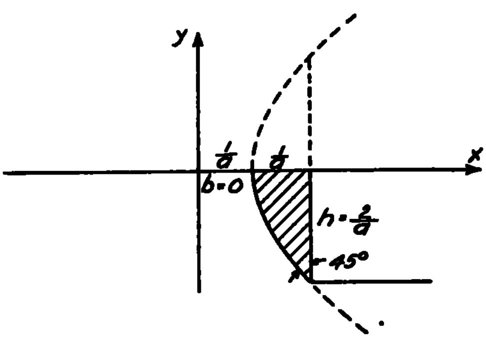

Figure 2

THE VALUE OF CRITICAL HEIGHT

The most practical aspect of the embankment analysis from the practical viewpoint is the value of critical height, or the prediction of maximum embankment height consistent with stability.

A slightly approximate solution may be made considering resultant shear as the vector resultant of a string polygon made up of uniform maximum shearing vectors along the failure surface.

From Figure 3:

$$W \cos\theta = S\, h/\cos\theta$$

$$S = W \cos^2\theta/h$$

$$h_{cr} = \frac{3.76S}{w} \ldots \quad \ldots \quad (6)$$

This value is nearly identical with that obtained by Fellenius in the "Swedish Circle" method:

$$h_{cr} = \frac{3.86S}{w} \quad \ldots\ldots \quad (7)$$

However a rigid derivation of resultant normal and tangential forces by the exact methods of integral calculus reveals these forces to be non-concurrent with the gravitational force, and indicates the presence of a large overturning couple. The origin of this couple was indicated earlier during the derivation of the differential equation for the sliding surface. It is necessary, to maintain equilibrium conditions, to provide an equal, opposite, reactive couple in the supporting medium

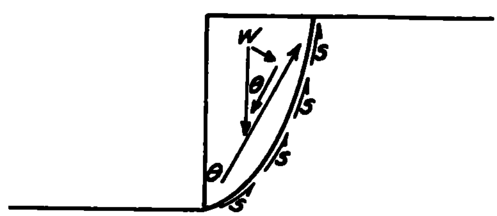

Figure 3

of the surrounding mass. This involves tensile stresses along the upper region of the failure surface of an amount equal to approximately one half the magnitude of maximum unit shear.

The validity of formulas (6) and (7) is thus dependent on the ability of cohesive materials, which are not ordinarily considered capable of entertaining tension, to carry large tensile stresses.

The importance of this tendency of the material to separate, or pull apart, near the upper ground surface cannot be overlooked. The downward propagation of a tension crack tends to destroy shearing resistance, increasing the shearing intensity on the remaining unseparated surface, thus causing sudden failure in supposedly stable embankments. Further analysis shows the magnitude of this overturning couple to decrease as the tension crack deepens (Fig. 4). In the extreme case the value of critical height is reduced slightly more than one half when the zone of separation penetrates to the depth of 0.517 h, where actual non-tension equilibrium is achieved. Here

$$h_{cr} = \frac{1.76S}{w} \quad\ldots\ldots\ldots\ldots(8)$$

This effect is strikingly illustrated in the photographs of gelatin models, Figures 5–8 inclusive.

An exact solution for critical height is obtainable by solving directly for $S_{\mathrm{Max.}}$ from the basic differential equation:

$$S_{\mathbf{Max.}} = \frac{xw}{1 + (x')^2} \quad\ldots\ldots\ldots(4)$$

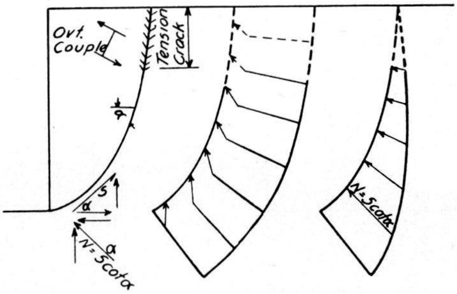

Figure 4

Where: x, y, equal ordinates of any point on the Max. Shear Curve, say h, $h.$, $1/a = 2/h$, $b = 0$, $x' = a/2\ (y + b)$. Then:

$$S_{\mathrm{Max.}} = \frac{h(w)}{2}$$

And:

$$h_{cr} = \frac{2S}{w} \quad\ldots\ldots\ldots\ldots(9)$$

This is a familiar, conservative relation much used in practical soil mechanics.

EXPERIMENTAL MODEL STUDIES

In connection with the foregoing mathematical demonstration model studies were made to provide physical corroboration and to emphasize the significance of the mathematical findings. Verification was obtained by means of gelatin models, of the effect of tension on the sliding surface, and of the outline of the failure surface itself.

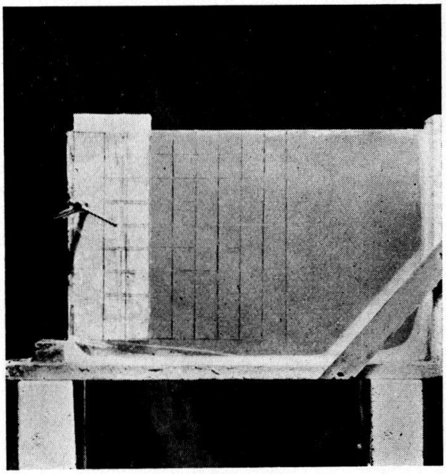

Figure 5

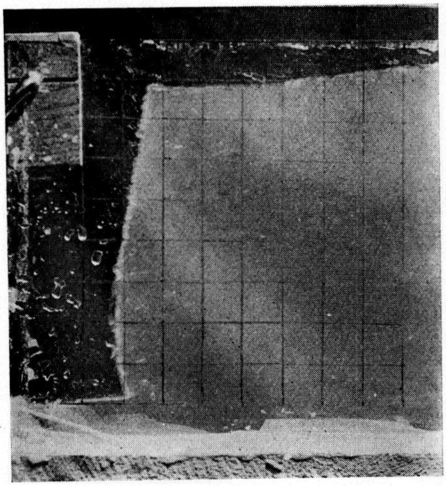

Figure 6

Figures 5–8 show the interesting behavior of these models as a portrayal of the failure of a vertical embankment.

Figure 5 illustrates the preparation of the model by molding gelatin between parallel glass plates lubricated with vaseline. Figure 6 illustrates the model after the wooden supporting block has been removed from the vertical face of the gelatin.

Figure 7 was photographed after some period of time during which the gelatin was allowed to soften under a slowly rising temperature. This figure shows the formation of several tension cracks in the top surface, the first and largest being almost exactly a distance from the face equal to one half the height of the gelatin mass. Figure 8 represents an advanced stage of the failure and shows that after initial separation due to tensile stress on the upper part of the critical surface the unstable portion of the mass slides down the surface. It is this later stage of failure which is more or less in agreement with the conventional conception of failure under rotation. It is significant, however, that when tension is developed the shearing resistance may be destroyed and the critical height reduced, leading to sudden failure.

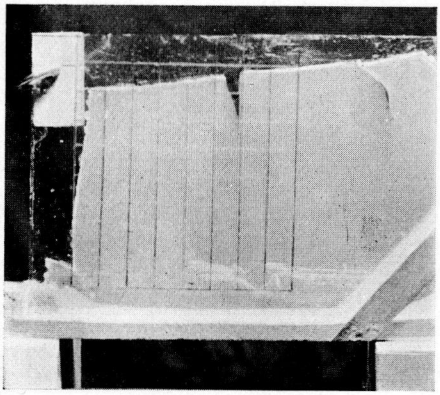

Figure 7

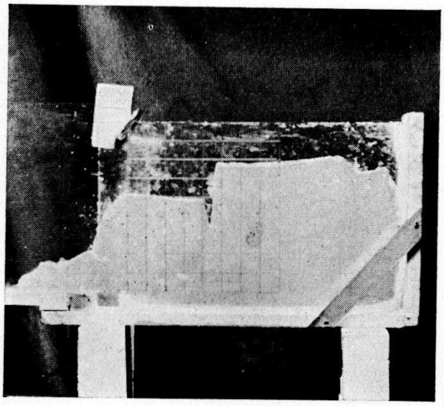

Figure 8

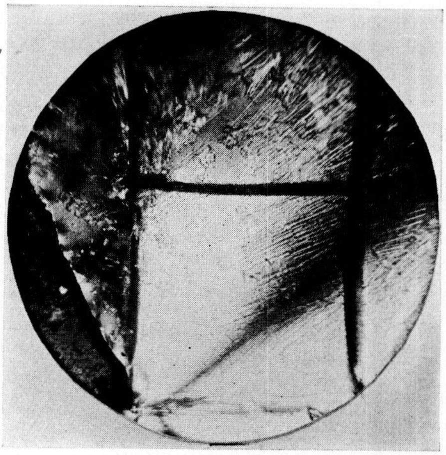

Figure 9

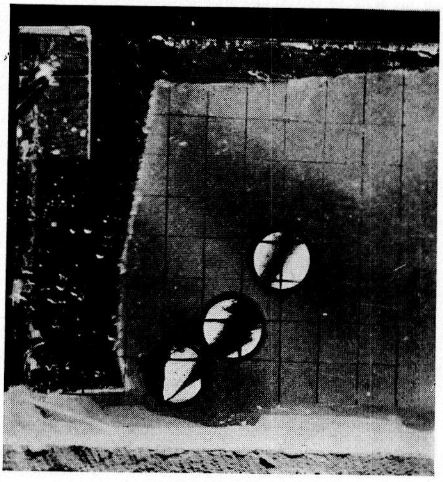

Figure 10

These photographs of progressive failure of the gelatin model were also supplemented by photo-elastic studies. Photographs were taken to locate the fringe of constant maximum shearing stress, although no quantitative results were attempted due to practical difficulties in calibrating the medium employed. Figure 9 is a photograph of the isochromatic or fringe in a field at the very bottom of the vertical face Similar photographs were made at several points along the theoretical surface of maximum shear and these have been superimposed upon the photograph of the full model in Figure 10.

These experiments were of a qualitative nature but never the less clearly illustrate the action of tensile forces and other characteristics of embankment failure, as demonstrated in the mathematical investigation.

CONCLUSIONS

In a brief word of recapitulation attention is invited to the similarity between many of the findings of this analysis of cohesive embankments at the point of incipient failure, observed natural phenomena, and accepted axioms in other fields of engineering mechanics. The second degree parabola failure curve together with the 0.5 ratio between height and breadth bears a marked resemblance to those generally observed in natural embankments. In the 45 deg. slope of the slip surface at the toe appears the conventional slope of the maximum shear plane in a unit cube. The same slope occurs also in all second degree parabolas at the intersection of the latus rectum.

The conception of initial failure due to tension is of definite importance in safe design practice and in the study of the mechanics of embankment failure. Current design methods which presume to mobilize shear along the entire sliding surface are subject to occasional discrepancies in practice. Many cohesive soils, when moisture content and other conditions are favorable, display a certain quasi-tension or "suction," and may for a time develop shearing resistance over the whole surface. Under other circumstances, however, such as the bank of a recently drained reservoir, a tension crack may precipitate sudden shear failure. Embankments based on critical heights near 4 S/w are best suited for temporary work and require an element of judgment in their use Conservative design values in the range of 2 S/w are recommended for permanence and safety.

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