carrying out at Princeton research on the effect of vibrations for the Civil Aeronautics Authority. I believe that for comparative evaluation of vibration effects, the M.I T. apparatus as designed, is entirely adequate. I would like to wish good luck to Mr. Lowe in the continuation of his work and will be interested to hear further about it.

THE THEORY OF STRESSES AND DISPLACEMENTS IN LAYERED SYSTEMS AND APPLICATIONS TO THE DESIGN OF AIRPORT RUNWAYS

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SYNOPSIS

In foundation and particularly in airport design and construction, the engineer is dealing basically with layered soil deposits. The theory of stresses and displacements in a two-layer system was developed in accordance with the methods of the mathematical theory of elasticity and is presented in order to reveal some of the fundamental relations existing between the physical factors, which control the load-settlement relations, and in order to provide a practical method of analysis for the design of airport runways. The theory reveals the controlling influence of two important ratios on the load-settlement characteristics of the "two-layer system," namely: (1) the ratio r/h_1 of the radius bearing area to the thickness of the reinforcing or pavement layer; and (2) the ratio E_2/E_1 of the modulus of the subgrade to that of the pavement. For practical design purposes, the theoretical results have been evaluated numerically and expressed in Basic Influence Curves, giving values of the settlement coefficient F_w in terms of these basic ratios The settlement coefficient is applied as a simple multiplying or correction factor to the familiar Boussinesq Equation for surface settlement at the center of a circular flexible bearing area. The practical design problem for airport runways involves the selection of suitable and economical types of pavement construction and the determination by means of the influence curves for the "two-layer system" of the thickness required to give adequate support to airplane wheel loads and reasonable length of service.

THE TWO-LAYER SYSTEM THEORY—ASSUMPTIONS AND CONDITIONS

The "two-layer system" theory is presented first of all in order to provide a basis for a better understanding of the nature of the real phenomena, and to reveal some of the fundamental relations existing between the physical factors which control the load-settlement relations Second, it is intended to provide a practical method of analysis for the design of airport runways

Boussinesq solved the problem of stresses and displacements in a uniform deposit for concentrated load applied at the surface. The scientific approach in the present problem involved the rigorous development of a theory of stresses and displacements in the more general case of a "two-layer system" by the methods of the mathematical theory of elasticity, which is believed to be correct. The general solution of the "two-layer" problem required that the necessary assumptions of the theory of elasticity be made, and that certain essential boundary and continuity conditions be satisfied, but did not require any radical simplifying assumptions beforehand as to the nature of the distribution of stresses on the subgrade or of their relation to displacements.

It must be realized that all theories deal with ideal materials and ideal conditions, which are only imperfectly satisfied in natural soil deposits. Judgment as to realm of

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validity and of application of the theory should be made primarily on the basis of actual performance and experience, that is, the extent to which the theory is adequate to explain the real phenomena and to which it is in reasonable agreement with it.

The "two-layer system," illustrated in Figure 1, consists of a surface or pavement layer 1 of a certain thickness h_1, which rests continuously upon and reinforces a weaker subgrade layer 2. A surface load is applied, uniformly distributed over a flexible bearing area of radius r. The application of the theory of elasticity to the solution of the problem required the following assumptions and conditions:

(1) The necessary assumptions of the theory of elasticity were made that the soils of each of the two layers are homogeneous, isotropic, elastic materials, for which Hooke's law is valid. While these assumptions are

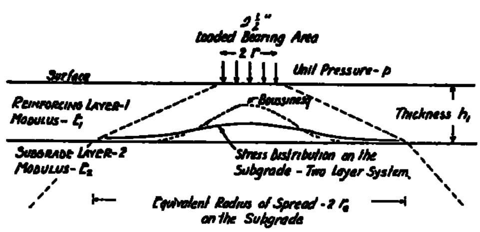

Figure 1. Two-Layer System

only imperfectly satisfied in natural soil deposits, the evaluation of full-scale load tests should yield average strength properties of the soils, which are fairly representative within the range of permissible settlements.

(2) The surface reinforcing layer 1 is assumed to be weightless and to be infinite in extent in the horizontal direction, but of finite thickness h_1. The subgrade layer 2 is assumed to be infinite in extent both horizontally and vertically downward.

(3) The solution of the problem must satisfy certain necessary boundary conditions, namely, that the surface of layer 1 must be free of normal and shearing stresses outside the limits of the loaded area, and that at infinite depth the stresses and displacements in the subgrade layer 2 must be equal to zero

(4) Most important of all, the solution for the "two-layer" problem must satisfy certain essential continuity conditions of stress and displacement across the interface between layer 1 and layer 2 It is assumed that the two layers are continuously in contact and act together as an elastic medium of composite nature, as shown in Figure 2. Furthermore, it is assumed that the subgrade provides initially a continuous uniform support for the pavement layer, which is really the primary condition to be achieved in good construction practice. Continuity requires that the normal and shearing stresses and the vertical and horizontal displacements must be equal in the two layers at the interface. Only in the horizontal radial stress σ_r will there be a discontinuity across the interface. This follows from the fact that, since the horizontal displacements u_1 and u_2 must be equal,

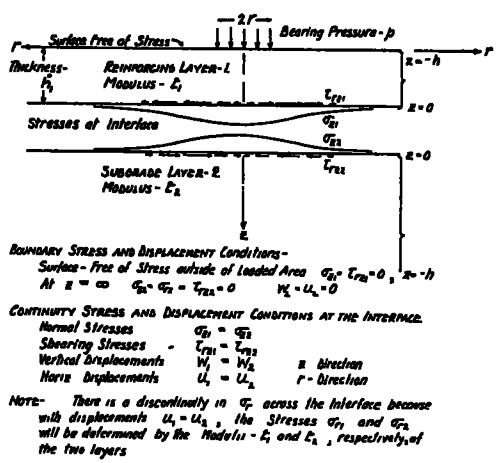

Figure 2. Boundary and Continuity Conditions of Stress and Displacement for a Two-Layer System.

the radial stress σ_{r1} and σ_{r2} either side of the interface will be different and must be determined by the moduli E_1 and E_2, respectively.

For all types of flexible pavement construction, the continuity conditions will probably be reasonably satisfied and the layered system should act substantially in accordance with the theory in the vicinity of the applied loading. From the very nature of soil deposits, conditions must be expected to vary considerably over a large area. The primary problem is not so much to determine average conditions, as it is to make reasonably certain that possibly the most unfavorable conditions are known over a given area that may give rise to "soft spots" in runways, taxiways, and aprons

The theory is not intended to apply to corner or edge loadings on concrete pavements. However, if the subgrade is continuously in contact with the concrete pavement and provides reasonably uniform support, which is the primary aim of good construction practice, and if the load is applied near the center of a fairly large slab, the concrete pavement and subgrade should act substantially in accordance with the theory.

(5) In order to obtain a practical solution of the problem and to reduce the complications, it was necessary to assume that Poisson's ratio was either 1/2 or 0 in both layers. The value of 1/2 was used, because it was considered to be somewhat more representative of the actual conditions. Furthermore, very little is known about this property for soils or what values should be used.

THE THEORY OF THE TWO-LAYER SYSTEM

In developing the theory of the "two-layer system," the stress and displacement equations of elasticity for the three-dimensional problem were employed, which were derived by Love to satisfy the equations of equilibrium and compatibility of the theory of elasticity (A. E. H. Love, "Treatise on the Mathematical Theory of Elasticity," 1923, p. 274) (S. Timoshenko, "Theory of Elasticity," 1934, p. 309, Eq. 172–4). The approach used in the solution of the "two-layer" problem followed somewhat along the lines suggested in the Theory of Elasticity, pp. 44–47, by Timeshenko.

The Mathematical Theory of Elasticity. A. E. H. Love and S Timoshenko.

The equations of elasticity for the three-dimensional problem of axial symmetry

(a) Equations of Equilibrium.

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0$$

(b) Equation of Compatibility.

$$\nabla^4 = 0$$

$$\nabla^2 = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial_r} + \frac{\partial^2}{\partial z^2}\right]$$

Equations of Elasticity:

(c) Stress

$$\sigma_z = \frac{\partial}{\partial z}\left[(2-\mu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right]$$

$$\sigma_r = \frac{\partial}{\partial z}\left[\mu\nabla^2\phi - \frac{\partial^2\phi}{\partial r^2}\right]$$

$$\sigma_\theta = \frac{\partial}{\partial z}\left[\mu\nabla^2\phi - \frac{1}{r}\frac{\partial\phi}{\partial r}\right]$$

$$\tau_{rz} = \frac{\partial}{\partial r}\left[(1-\mu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right]$$

(d) Displacement.

$$W = \frac{1+\mu}{E}\left[(1-2\mu)\nabla^2\phi + \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}\right]$$

$$u = -\frac{1+\mu}{E}\left[\frac{\partial^2\phi}{\partial r^2}\right]$$

Strain components $\frac{\partial w}{\partial z}$ and $\frac{\partial w}{\partial r}$ made compatible.

Two-Layer System Theory.

The stress function ϕ of equation (e) was found to satisfy the compatibility equation (b) and involved Bessel Functions in order to satisfy the differential operation ∇^2. The following stress and displacement equations were then obtained, assuming Poisson's Ratio equal to 1/2.

Stress Function

$$\phi = J_0(mr)[Ae^{mz} - Be^{-mz} + Cze^{mz} - Dze^{-mz}] \quad \text{(e)}$$

Equations for Surface Layer 1.

$$\sigma_{z1} = -m J_0(mr)[A_1 m^2 e^{+mz} + B_1 m^2 e^{-mz} + C_1 m^2 z\, e^{+mz} + D_1 m^2 z\, e^{-mz}]$$

$$\sigma_{r1} = [m J_0(mr) - \frac{J_1(mr)}{r}]\,[A_1 m^2 e^{mz} + B_1 m^2 e^{-mz} + C_1 m e^{mz} + C_1 m^2 z e^{mz} - D_1 m e^{-mz} + D_1 m^2 z e^{-mz}] + m J_0(mr)[C_1 m e^{mz} + D_1 m e^{-mz}]$$

$$\tau_{rz} = m J_1(mr)[A_1 m^2 e^{mz} - B_1 m^2 e^{-mz} + C_1 m e^{mz} + C_1 m^2 z e^{mz} + D_1 m e^{-mz} - D_1 m^2 z e^{-mz}] \quad \text{(f)}$$

$$W_1 = \frac{3}{2E_1} m J_0(mr)[A_1 me^{mz} - B_1 me^{-mz} + C_1 mze^{mz} - D_1 mze^{-mz}]$$

$$u_1 = \frac{3}{2E_1} m J_1(mr)[A_1 me^{mz} + B_1 me^{-mz} + C_1 e^{mz} + C_1 mze^{mz} - D_1 e^{-mz} + D_1 mze^{-mz}]$$

Equations for Subgrade Layer 2. (g)

A similar set of equations was obtained with coefficients A_2, B_2, C_2, and D_2 and a Modulus E_2.

Two-Layer System. Equations of Stress and Displacement.

The constants in equations (f) and (g) were evaluated to satisfy the boundary and continuity conditions for the "two-layer system" for a surface loading with a distribution $\sigma_z = -mJ_0(mr)$. These equations all involve a coefficient of the Strength Properties $N = \left[\frac{E_1 - E_2}{E_1 + E_2}\right]$

At the Interface. Layer 1. $z = 0$.

The following checks were made on these equations as to their correctness of form

(1) If E_1 and E_2 are equal, that is, a homogeneous deposit throughout, the strength coefficient N equals zero, and equations (h) to (m) reduce to the familiar forms from which the Boussinesq equation may be readily derived

(2) If E_2 becomes infinite, that is a rough rock surface at the base of Layer 1, the strength coefficient N equals minus one (−1), and equation (h) for the normal stress at the rock surface becomes identical with that given first by Biot (M. A. Biot, "Effect of Certain Discontinuities on Pressure Distribution in a Loaded Soil" Physics, Dec 1935, p. 367) and later more completely by Pickett (G. Pickett, "Stress Distribution in a Loaded Soil with Some Rigid Boundaries", *Proceedings*, Highway Research Board, vol. 18, Pt II, p. 35, 1938). Also σ_z equals σ_r at the rock surface and $W = 0$.

Two-Layer System. Evaluation of Displacement Equation.

The numerical evaluation was completed only for the settlement equation (m) at the surface of Layer 1, because it was considered to have the greatest immediate and practical importance in airport problems. By a Bessel function expansion of the load function, expressed in terms of the parameter "m", (simi-

$$\sigma_{z1} = -m J_0(mr)[1 - N]\left[\frac{(1 + mh)e^{mh} - N(1 - mh)e^{-mh}}{e^{2mh} - 2N(1 + 2m^2h^2) + N^2e^{-2mh}}\right] \tag{h}$$

$$\sigma_{r1} = -m J_0(mr)\left[\frac{(1 - mh)e^{mh} - N(1 + 3mh)e^{mh} - N(1 - 3mh)e^{-mh} + N^2(1 + mh)e^{-mh}}{e^{2mh} - 2N(1 + 2m^2h^2) + N^2e^{-2mh}}\right] - \frac{J_1(mr)}{r}[1 + N]\left[\frac{mh\, e^{mh} - Nmh\, e^{-mh}}{e^{2mh} - 2N(1 + 2m^2h^2) + N^2e^{-2mh}}\right] \tag{i}$$

$$W_1 = -\frac{3}{2E_1} J_0(mr)[1 + N]\left[\frac{(1 + mh)e^{mh} - N(1 - mh)e^{-mh}}{e^{2mh} - 2N(1 + 2m^2h^2) + N^2e^{-2mh}}\right] \tag{j}$$

At the Surface of Layer 1. $z = -h$

$$\sigma_{z1} = -m J_0(mr)[1] \tag{k}$$

$$\sigma_{r1} = -m J_0(mr)\left[\frac{e^{2mh} - 2N(1 - 2m^2h^2) + N^2e^{-2mh}}{e^{2mh} - 2N(1 + 2m^2h^2) + N^2e^{-2mh}}\right] + \frac{J_1(mr)}{r}\left[\frac{2N\, m^2h^2}{e^{2mh} - 2N(1 + 2m^2h^2) + N^2e^{-2mh}}\right] \tag{l}$$

$$W_1 = -\frac{3}{2E_1} J_0(mr)\left[\frac{e^{2mh} + 4Nmh - N^2e^{-2mh}}{e^{2mh} - 2N(1 + 2m^2h^2) + N^2e^{-2mh}}\right] \tag{m}$$

lar to the Fourier expansion) the settlement equation at the surface of Layer 1 was obtained for an arbitrary loading as a superposition of loadings having a distribution $\sigma_z = -pmJ_0(mr)$, which was equivalent to a concentrated load P at the surface of Layer 1.

$$W_1 = \frac{1.5P}{2\pi E_1}\int_0^\infty \cdot \left[\frac{e^{2mh} + 4Nmh - N^2e^{-2mh}}{e^{2mh} - 2N(1 + 2m^2h^2) + N^2e^{-2mh}}\right] J_0(mr)\, dm \qquad \text{(n)}$$

This equation was successfully evaluated by replacing the integrand with a series of a finite number of terms having the well known integral forms of equations (o) A sufficient number of terms of the infinite series obtained by dividing the numerator of equation (n) by the denominator were used in each case to insure the desired degree of accuracy in the evaluation of the equation. The coefficients C_1 to C_{10} etc. in the equivalent finite series of equation (p) were evaluated by suitable methods (similar to relaxation methods), so that the value of the finite series was within less than one per cent (1%) of the value of the integrand in that range of values of $\alpha = mh$ for which the integrand contributed at least 99 per cent of the total value of the integral in equation (n). These coefficients are functions of the strength ratio E_2/E_1. They were evaluated for a practical range of 12 values of this ratio, varying from 1/2 to 1/10,000 (the latter representing concrete).

Integral Forms:

$$\int_0^\infty e^{-a\alpha}J_0\left(\frac{r}{h}\alpha\right) d\alpha = \left[a^2 + \left(\frac{r}{h}\right)^2\right]^{-\frac{1}{2}}$$

$$\int_0^\infty \alpha^n e^{-a\alpha}J_0\left(\frac{r}{h}\alpha\right) d\alpha = (-1)^n \frac{d^n}{da^n}\left[a^2 + \left(\frac{r}{h}\right)^2\right]^{-\frac{1}{2}} \qquad \text{(o)}$$

Equivalent Settlement Equation:

$$W_1 = \frac{1.5P}{2\pi E_2}\int_0^\infty \left[\frac{1-N}{1+N} + C_1(1 + 2\alpha + 2\alpha^2)e^{-2\alpha} + C_2(1 + 4\alpha + 8\alpha^2 + 8\alpha^3 \cdots)e^{-4\alpha} + C_3(1 + 6\alpha + 18\alpha^2 + 32\alpha^3 \cdots)e^{-6\alpha} + C_4(1 + 8\alpha + 32\alpha^2 \cdots)e^{-8\alpha} + \cdots + C_{10}(1 + 24\alpha \cdots)e^{-24\alpha}\right] \cdot J_0\left[\frac{r}{h}\alpha\right]\frac{d\alpha}{h} \qquad \text{(p)}$$

Two-Layer System. Basic Settlement Equation.

Equation (p) was then integrated term by term by means of the integral forms of equations (o), leaving for the present the coefficients C_1 to C_{10} etc. as undetermined coefficients The surface settlement equation for a concentrated load P, applied at the surface of a two-layer system then becomes:

$$W_1 = \frac{1.5P}{2\pi E_2}\Bigg[\left[\frac{1-N}{1+N}\cdot\frac{1}{r}\right] + C_1\left[\frac{1}{2h[1 + (r/2h)^2]^{\frac{1}{2}}} + \frac{1}{2h[1 + (r/2h)^2]^{3/2}} - \frac{1}{2^2h[1 + (r/2h)^2]^{3/2}} + \frac{3}{2^2h[1 + (r/2h)^2]^{5/2}}\right] + C_2\left[\frac{1}{4h[1 + (r/4h)^2]^{1/2}} + \frac{1}{4h[1 + (r/4h)^2]^{3/2}} - \frac{2}{4^2h[1 + (r/4h)^2]^{3/2}} + \frac{6}{4^2h[1 + (r/4h)^2]^{5/2}} - \frac{18}{4^3h[1 + (r/4h)^2]^{5/2}} + \frac{30}{4^3h[1 + (r/4h)^2]^{7/2}}\right] + \cdots + C_{10}\left[\frac{1}{24h[1 + (r/24h)^2]^{1/2}} + \frac{1}{24h[1 + (r/24h)^2]^{3/2}}\right]\Bigg] \qquad \text{(q)}$$

The Basic equation of settlement at the center of a circular bearing area for a two-layer System was then obtained by integrating equation (q) over a circular area for a uniformly distributed load of

$$P = 2\pi \int_0^r p \cdot r\, dr.$$

Basic Settlement Equation:

$$w_c = \frac{1.5pr}{E_2}\left[\left[\frac{1-N}{1+N}\right] + C_1\left[\frac{0.5r/h}{[1+(r/2h)^2]^{1/2}} + \frac{0.25r/h}{[1+(r/2h)^2]^{3/2}}\right] + C_2\left[\frac{0.25r/h}{[1+(r/4h)^2]^{1/2}} + \frac{0.125r/h}{[1+(r/4h)^2]^{3/2}} + \frac{0.0937r/h}{[1+(r/4h)^2]^{5/2}}\right] + C_3\left[\frac{0.167r/h}{[1+(r/6h)^2]^{1/2}} + \cdots\right] + \cdots + C_9\left[\frac{0.050r/h}{[1+(r/20h)^2]^{1/2}}\right] + C_{10}\left[\frac{0.042r/h}{[1+(r/24h)^2]^{1/2}}\right]\right] \qquad (r)$$

$$w_c = \frac{1.5pr}{E_2}\cdot F_w\left[\frac{r}{h_1}, \frac{E_2}{E_1}\right] = \frac{1.5pr}{E_2} F_w \qquad (s)$$

It is important to note that the basic settlement equation (s) reduces to the extremely simple form of the Boussinesq equation with a multiplying or correction coefficient F_w, which is a function $F_w = F_w\,[r/h_1, E_2/E_1]$

Since equation (r) is far too cumbersome and unusable for ordinary computations, it was evaluated numerically by substituting the values of the coefficients C_1 to C_{10} etc. for the 12 values of the basic ratio E_2/E_1 and for a practical range of values of the basic ratio r/h_1 from 0 to 20, using prepared tabulations of the functions of r/h The overall accuracy of the numerical evaluation is believed to have been held well within 2 per cent error, by carrying a sufficient number of terms in equation (p) and a sufficient number of significant figures in all computations

LOAD-SETTLEMENT RELATIONS FOR THE TWO-LAYER SYSTEM

The theory of the "two-layer system" reveals the controlling influence of two important ratios on the load-settlement characteristics of the system, namely: (1) the ratio r/h_1 of the radius of bearing area to the thickness of the reinforcing or pavement layer 1; and (2) the ratio E_2/E_1 of the modulus of the subgrade to that of the reinforcing layer 1. For practical design purposes the results of the theory of the "two-layer system" are expressed in the basic influence curves of Figures 3 and 4, which give values of the settlement coefficient F_w for a practical range of values of these basic ratios. In accordance with equation (2), the settlement coefficient F_w is applied as a simple multiplying or correction factor to the familiar Boussinesq equation for a homogeneous deposit, giving the surface settlement at the center of a bearing area of radius r, loaded with a uniformly distributed load p.

Boussinesq Settlement Equation. Poisson's ratio μ equals 1/2.

Flexible bearing area (a)

$$W = \frac{2(1-\mu^2)}{E}\, pr = 1.5\,\frac{pr}{E}$$

Rigid bearing area (b) (1)

$$W = \frac{\pi(1-\mu^2)}{2E}\, pr = 1.18\,\frac{pr}{E}$$

Two-Layer System Settlement Equation. Poisson's ratio μ equals 1/2.

Flexible bearing area (a)

$$W = 1.5\,\frac{pr}{E_2}\, F_w$$

Rigid bearing area (b) (2)
(assumed coefficient 1.18)

$$W = 1.18\,\frac{pr}{E_2}\, F_w$$

It must be emphasized that the dimensions of the modulus E used in the theory of the "two-layer system" are stress/strain (lb per sq. in. divided by inches per inch), which are in accordance with the usual concepts and notations of the theory of elasticity.

From equation (1) or (2), the equivalent strain for a bearing area on the surface of the

ground is equal to the settlement divided by a characteristic length of the bearing area:

$$\text{Strain} = \frac{p}{E} = \frac{w}{1.5r} \quad \text{or} \quad \frac{w}{1.5rF_w}$$
$$\text{Modulus } E = \frac{p}{w/(1.5r)} \quad \text{or} \quad \frac{p}{w/(1.5rF_w)} \tag{3}$$

This in effect states that with respect to the stress-settlement relations for a surface loading, the equivalent height of soil column is 1 5 times or (1 5 F_w) the radius of the bearing area, as compared with 4 times the radius of the specimen in the unconfined compression test. It is to be noted that the modulus thus defined is a stable quantity and is independent of the size of bearing area and of the Two-Layer Settlement coefficient F_w.

One test of the correctness of the form of the settlement equation (2) for the "two-layer system" is that it must reduce to the Boussinesq equation (1) at the limits. For r/h_1 approaching zero, that is, with h_1 very large, the deposit becomes a homogeneous one all of layer 1, and the settlement coefficient F_w reduces numerically to the value E_2/E_1, as shown by the values of the intercepts on the vertical axis for r/h_1 equal to zero for the influence curves of Figure 3. The settlement equation (2) then reduces to

$$W = \frac{1.5pr}{E_2} \times \frac{E_2}{E_1} = \frac{1.5pr}{E_1}$$

At the other limit with r/h_1 approaching infinity, and either with r very large or with h_1 very small, the deposit reduces to a homogeneous one all of the subgrade layer 2, no pavement. The settlement coefficient F_w becomes equal to unity as shown in Fig. 3 and settlement equation (2) reduces to equation (1). For all other values of the basic ratios, the values of the settlement coefficient are defined by the basic influence curves.

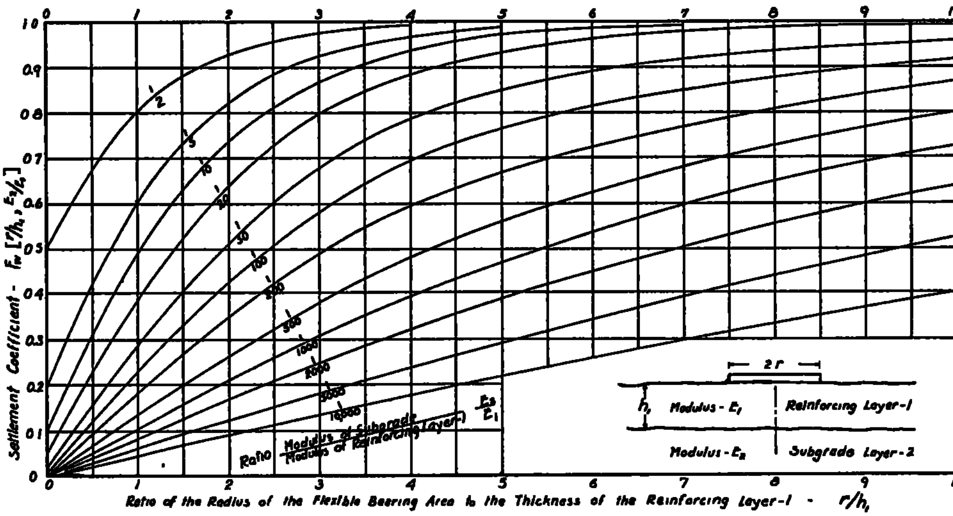

Figure 3. Load-Settlement Characteristics of a Two-Layer Soil System. Influence Curves and Basic Relations for the Settlement Coefficient

$$F_w[r/h_1, E_2/E_1]$$

Settlement at the Center of a Flexible Bearing Area.

$$W_c = 1.5 \frac{P \cdot r}{E_2} F_w[r/h_1, E_2/E_1].$$

LOAD-SETTLEMENT CHARACTERISTICS OF THE TWO-LAYER SYSTEM

In the design of airport runways the problem of primary importance is to determine the effectiveness of a pavement layer in reinforcing and restraining a given subgrade. Basi-

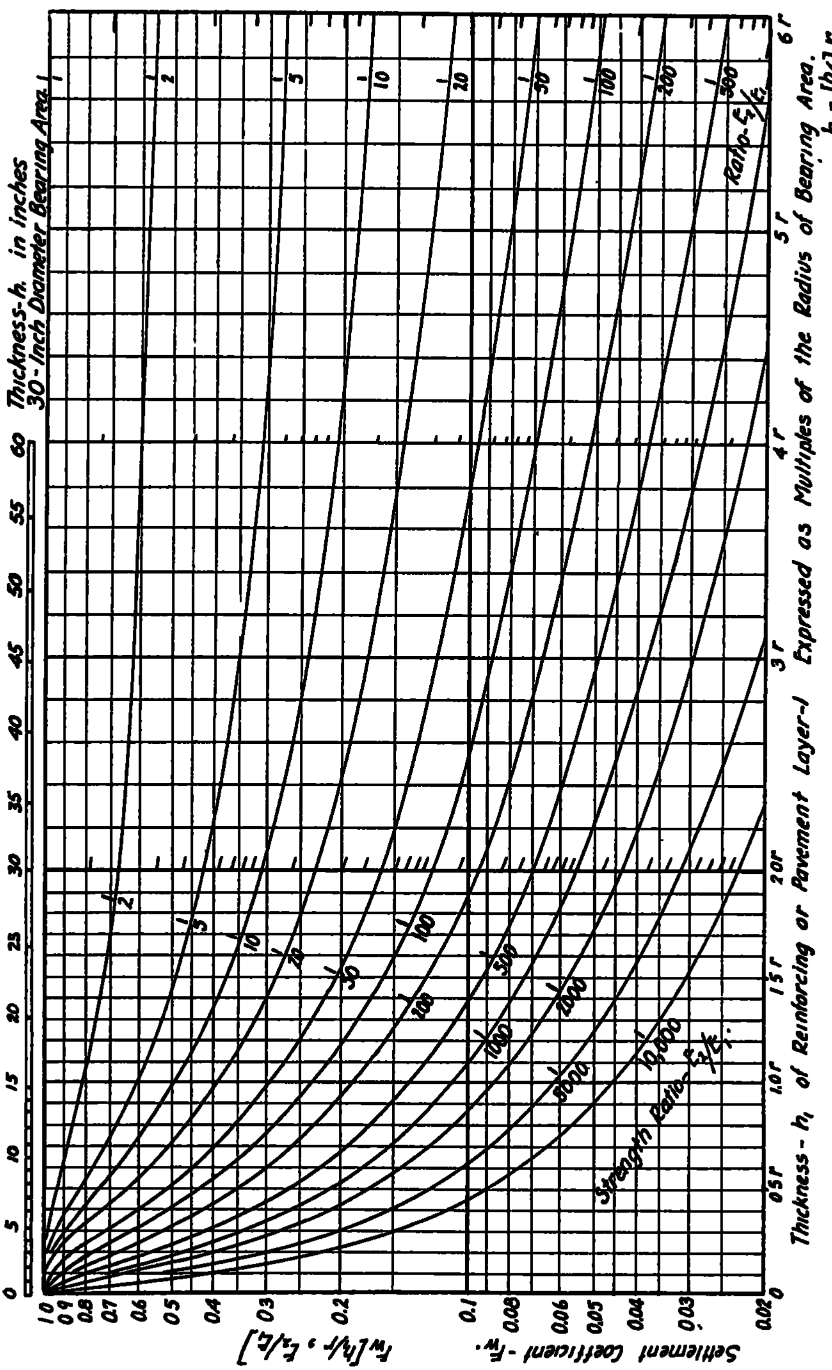

Figure 4. Influence Curves of the Settlement Coefficient — F_w for the Two-Layer System. Basic Load-Settlement Relation

$$W_c = 1.5 \frac{P \cdot r}{E_2} \cdot F_w . \qquad F_w = \frac{W E_2}{1.5 P r} .$$

cally the criterion for design must be the limitation of settlement for various types of pavements to values sufficiently small as to insure against objectionable consolidation of the base coarse layer and deformations in the subgrade, which would result in eventual failure of the runway surface under the action of repeated dynamic wheel loadings. Design thus requires the selection of suitable pavement materials and the determination of the thickness of pavement required to meet this criterion and to provide adequate support to wheel loads on a given subgrade

First of all it is important to note that any horizontal line on Figure 4 for a constant value of F_w defines equivalent systems for equal settlement, the Boussinesq factor in the reinforcing and restraining a given subgrade is profoundly affected by the values of the basic ratios r/h_1 and E_2/E_1, as illustrated in Figures 5 and 6. If the thickness of layer 1 is doubled (or if the strength is doubled) for a constant value of the factor ($1.5\ p\ r/E_2$), the second system is not, in general, twice as effective as to supporting capacity (pressure and settlement). Also with increasing radius of bearing area, the effectiveness of layer 1 decreases considerably, and the strength properties of the weaker sub-grade control more and more the load-settlement relations. For bearing areas larger than about 48 in. in diameter, Figure 7 shows that the influence of size becomes less important, as experience has shown. But it is also evident that the influence of the size of the bearing area is a function of the strength ratio E_2/E_1 as well.

TABLE 1
EQUIVALENT SYSTEMS
EQUIVALENT SYSTEMS FOR EQUAL SETTLEMENT AND SUPPORTING CAPACITY

Given: Diameter of Bearing Area 18 in Contact Pressure 50 psi.
Settlement limited to 0 2 in.
Required value of the settlement coefficient

$$F_w = \frac{w\, E_2}{1\ 5\ p\, r} \qquad \text{(Eq. 2a)}$$

Subgrade Modulus	E_2	F_w
(a)	2000 psi	0 593
(b)	500 psi	0 148

Required Values of the Thickness h_1 of the Reinforcing Layer 1 for Equivalent Systems Designated by the Strength Ratio E_2/E_1

E_2/E_1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{50}$	$\frac{1}{100}$	$\frac{1}{200}$	$\frac{1}{500}$	$\frac{1}{1000}$	$\frac{1}{2000}$	$\frac{1}{5000}$
(a) h_1 E_2-2000	32 Compacted Subgrades	9	7 Gravel	5 2	3 7 Crushed Stone Bases Flexible Bituminous Pavements	2 8	2 0		Approximate Type of Construction		
(b) h_1 E_2-500	Inches		90 Compacted Subgrades	35	20 Gravel	15	12.8 Crushed Stone Bases Flexible Bituminous Pavements	8 5	6 5	5 2	4 0 Concr. Pavemt.

settlement equation ($1\ 5\ p\ r/E_2$) remaining constant. An important aspect of the design problem of airport runways, from the practical and economic standpoint, is the selection of a suitable type of pavement construction from a number of possible equivalent systems of equal supporting value and equal settlement A study is made in Table 1 of equivalent types of construction for comparative purposes, assuming in one case a relatively good subgrade and in the other a relatively weak subgrade It is to be noted that the strength properties of the subgrade, have an enormous influence on the load-settlement characteristics of the "two-layer system"

Second, it is important to note that the effectiveness of the pavement layer 1 in

Third, the importance of the pressure reducing and spreading effect of a given reinforcing layer 1 on the pressure transmitted to the subgrade is shown clearly in Figure 8 by computing an equivalent radius of bearing area r_e, as if the same total load P were applied directly to the subgrade, such that the center settlement is the same as in the "two-layer system," but with the intensity of the uniformly distributed pressure accordingly greatly reduced

$$\text{Equivalent Radius } w = \frac{1.5Pr}{r^2E_2}\cdot F_w = \frac{1\ 5Pr_e}{r_e^2 E_2} \qquad (4)$$

$$r_e = r/F_w$$

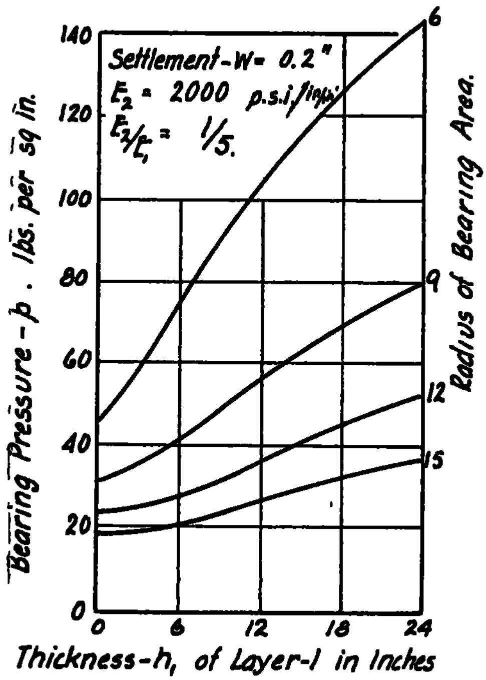

Figure 5. Influence of Size of Bearing Area and Thickness of Layer 1 on the Intensity of Bearing Pressure.

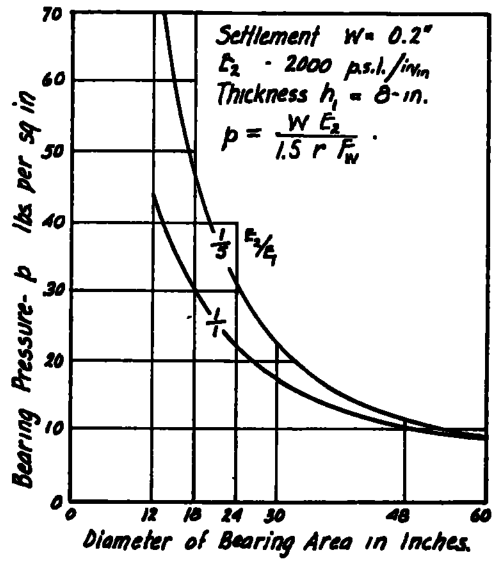

Figure 7. Influence of Size of Bearing Area on the Intensity of Bearing Pressure for Constant Settlement.

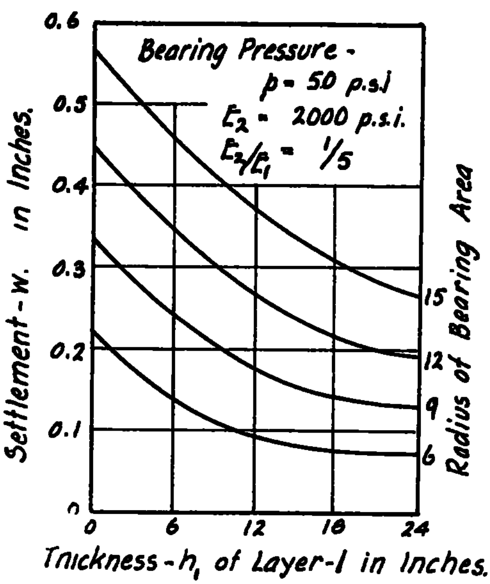

Figure 6. Influence of Size of Bearing Area and Thickness of Layer 1 on the Settlement

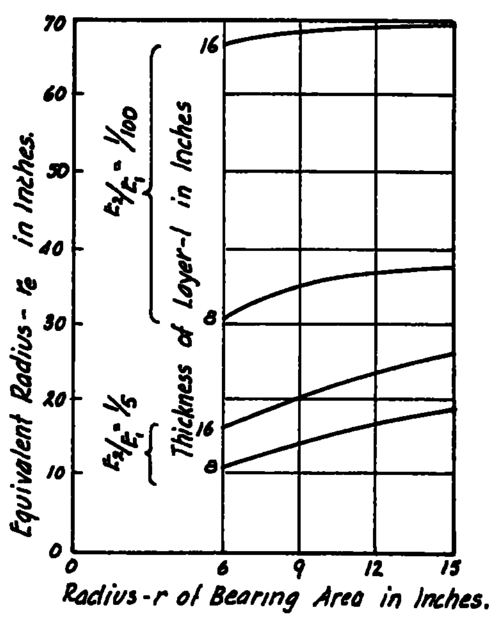

Figure 8. Equivalent Radius of Bearing Area on the Subgrade $r_e = r/F_w$

The curves of Figure 8 indicate that the pressure spreading effect is very much greater than heretofore supposed and markedly increases as the ratio E_2/E_1 varies from 1/2 to 1/10,000. This indicates that the strength properties of the subgrade have a somewhat smaller influence on the load-settlement phenomena of the system as the strength of the reinforcing layer increases, particularly for concrete pavements, which have relatively high flexural strengths.

LOAD TEST ANALYSES

It is of first importance to determine what the strength properties are of the various types of natural and stabilized subgrade, base courses, and pavements, which are representative of actual conditions of compaction and saturation. Table 1 of "equivalent systems" shows that there may be very important differences on the strength properties. It is especially important to determine the influence of the restraints exerted by various types of pavements upon the strength properties of the subgrade and base course. This requires evaluating as accurately as possible the results of carefully conducted series of loading tests on different sizes of bearing area, or on different thicknesses of pavement layer. The success with which load tests can be evaluated depends first of all upon how closely the theory used approximates actual conditions, and second, upon the consistency of the load test data in the different test locations.

A few examples of load test data from a number of different sources are analyzed, not only to illustrate the practical application of the "two-layer system" method of analysis, but also to show that the theory is adequate to explain the real phenomena and that it is in good agreement with it. The critical physical factors to be determined for design purposes are the proper values of the subgrade modulus E_2 and of the reinforcing or pavements modulus E_1, which will be representative of the actual values under the given conditions of compaction and saturation.

Example 1. Improvement of Subgrade by Compaction.

Experience has shown that a large percentage of pavement failures are due to poor subgrade conditions and excessive subgrade deformations, primarily as a result of saturation of the subgrade. The improvement of subgrade support by good compaction at optimum moisture content is the most essential and generally the most economical measure to be taken.

Subgrade: Load test on the natural subgrade on a 30-in diameter rigid bearing plate.

Adopted maximum value of settlement, w = 0 2 in.

Corresponding bearing pressure, p = 48 psi.

Subgrade Modulus: (Eq. 1b)

$$E_2 = \frac{1.18 \times 48 \times 15}{0\ 2}$$

$$E_2 = \frac{4270 \text{ lb. per sq. in.}}{\text{in. per in.}}$$

Two-Layer System- Load test on top of an 8-in layer of compacted subgrade, compacted at optimum moisture content.

Settlement, w = 0 2 inches. Pressure, p = 64 lb. per sq. in

Solve Eq 2b for the Settlement Coefficient, F_w

$$F_w = \frac{0.2 \times 4270}{1.18 \times 64 \times 15} = 0.755$$

Thickness of layer 1, $h_1 = (h/r)r = (8/15)r =$ 0 533 r

Interpolating in the Influence Curves of Fig. 4 for the values of F_w = 0 755 and h_1 = 0 533 r, the value of E_2/E_1 equals 1/7 The modulus of the compacted layer 1, E_1 = 4270 × 7 =

$$E_1 = \frac{29{,}900 \text{ lb per sq. in}}{\text{in. per in.}}$$

This represents a very good increase in strength of the subgrade soil by compaction, and is an indication of a fairly well-graded subgrade soil

Example 2 Reinforcing Effect of a Compacted 6-in. Crushed Stone Base Course.

Subgrade. Load test on the natural subgrade on 30-in rigid plate

Settlement, w = 0 2 in Bearing pressure, p = 58 lb. per sq in.

Subgrade Modulus, $E_2 = \dfrac{1.18 \times 58 \times 15}{0\ 2} =$ 5130 lb. per sq. in.

Two-Layer System-Load test on top of 6-in crushed stone base course.

Settlement, w = 0.2 in., Pressure, p = 98 lb. per sq. in.

$$F_w = \frac{wE_2}{1.18pr} = \frac{0.2 \times 5130}{1\ 18 \times 98 \times 15} = 0.593$$

$$h_1 = (6/15)r = 0\ 4r$$

Interpolating in the influence curves of Figure 4 $E_2/E_1 = 1/60$
Base Course Modulus, $E_1 = 5130 \times 60 =$ 307,800 lb. per sq. in.

in per in

This represents a rather an exceptional quality of base course construction.

Example 3. Flexible Bituminous Pavement and Base Course Construction

The data were taken from "Foundations of Flexible Pavements" by O. J. Porter, *Proceedings*, Highway Research Board, Vol 22, 1942, Figure 24D, page 120, giving the results of loading tests on a test section of runway of a western airport. Values were interpolated from the curves of deflections at the top of the pavement for a static wheel (tire) loading of 10,000 lb.

Subgrade: Plastic clay. Moisture content—30 per cent. C.B R.—3 to 5 per cent

Load Test: Truck tire loading on top of pavement. Average radius of contact—r about 7.3 in. Wheel load—10,000 lb. Contact pressure—p 60 psi.

(1)				
Combined thickness, in.	10	20	30	40
$(h/r)r$ h_1	1 37r	2 74r	4 12r	5.49r
Settlement, in w	0.19	0 105	0 075	0.058
$E_w = \frac{wE_2}{1\ 5pr}$.	$\frac{E_2}{3450}$	$\frac{E_2}{6250}$	$\frac{E_2}{8750}$	$\frac{E_2}{11,300}$

(2) Trial Values of E_2 and F_w

Values of E_2	Corresponding Values of F_w			
800	0 232	0.128	0 092	0 071
700	0.203	0 112	0 080	0 062
600	0.174	0 096	0 069	0 053
500	0.145	0 080	0 057	0 044

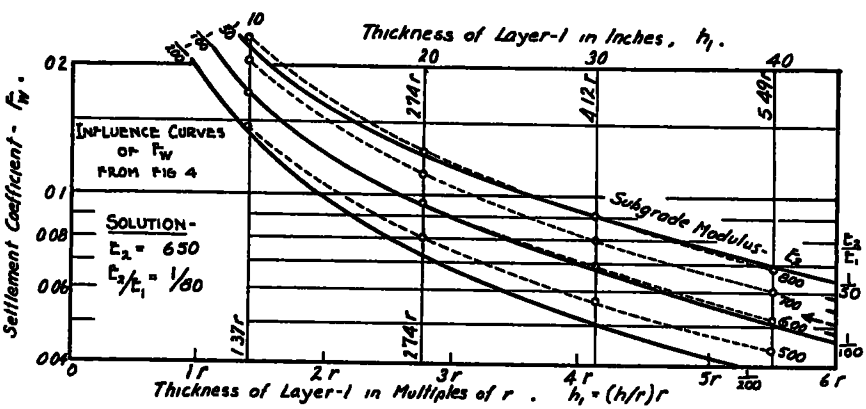

Figure 9. Example 3. Determination of E_2 and E_2/E_1 for the Two-Layer System

The evaluation of the strength properties of the subgrade and combined pavement and base course is made by simple graphical trial methods, similar to a solution of simultaneous equations The solution requires data either on three or more sizes of bearing areas, or on different thicknesses of pavement and base course construction. The data must be reasonably consistent and comparable as to strength properties of the respective layers in the several locations tested.

For flexible pavements there is generally not sufficient difference in the strength properties of the surface wearing course and the base course to permit evaluating them separately. Therefore in this case it is permissible to use the total combined thickness in the analysis of loading tests.

Pavement: 3 in of asphaltic concrete.

Base Course: Variable thicknesses of compacted crushed gravel.

A sheet of tracing paper is laid over the influence curves of Figure 4 and the values of F_w corresponding to the trial values of E_2 are plotted, as shown in Figure 9, and smooth curves are drawn through the plotted points. The consistency of the data is immediately apparent, and in this case is exceptionally good. The values of E_2 and E_2/E_1 can then be readily interpolated between the plotted curves, which most consistently parallel the influence curves of Figure 4. The combined accuracy in the determination of E_2 and E_2/E_1, as expressed in the relation F_w/E_2 in the settlement equation is more important than extreme accuracy in the determination of either separately. This example illustrates the remarkable closeness with which the theory of the "two-layer system" approximates the real phenomena. Otherwise it would not be possible to superpose actual test data on the theoretical influence curves and obtain any

such agreement. Subgrade Modulus, $E_2 = \frac{650 \text{ lb per sq. in.}}{\text{in per in.}}$. Modulus of combined Asphalt Pavement and Crushed Gravel Base Course,

$$E_1 = 650 \times 80 = \frac{52{,}000 \text{ lb. per sq in.}}{\text{in. per in.}}$$

Example 4. Flexible Bituminous Pavement

The data were taken from "Load Tests on Flexible Surfaces" by W. S Housel, *Proceedings*, Highway Research Board, Vol 21, 1941, Fig. 1, page 122, giving load test results Values were interpolated from the load-deflection curves as follows:

Pavement: 3 in. of dense-graded bituminous wearing course.
Base Course: 6 in. of compacted well-graded gravel.
Subgrade: Plastic clay fill.

subgrade controls more and more the load-settlement characteristics of the system

(1) Trial Values of E_2 and F_w for the Graphical Solution.

h_1	2 25r	1.61r	1 31r
$F_w = \frac{wE_2}{1\ 18pr}$	$\frac{E_2}{2480}$	$\frac{E_2}{1980}$	$\frac{E_2}{1710}$
Values of E_2	Corresponding Values of F_w		
500	0 202	0 252	0 292
600	0 242	0 305	0 351
700	0 282	0 354	0 409
800	0 322	0 404	0 467

The graphical solution in Figure 10b yields the following values: Subgrade Modulus, $E_2 = \frac{650 \text{ lb. per sq. in}}{\text{in per in}}$. Modulus of Combined Bituminous Pavement and Base Course,

$$E_1 = 650 \times 12 = \frac{7{,}800 \text{ lb. per sq. in}}{\text{in. per in.}}$$

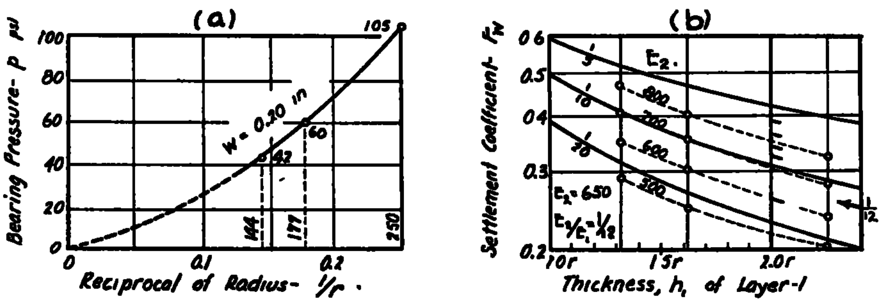

Figure 10. Example 4. (a) Two-Layer Curve p vs $1/r$. (b) Determination of E_2 and E_2/E_1, Two-Layer System

Load Tests: Top of pavement on rigid bearing plates
Combined thickness of pavement construction: 9 in.

Radius, in	r	4 0	5 6	6 9
Reciprocal	$1/r$	0 25	0 177	0.144
Thickness, in	$(h/r)r$	2 25r	1 61r	1 31r
Settlement, in	w	0 20	0 20	0 20
Pressure, psi	p	105 0	60 0	42 0

In order to reveal the nature of the load-settlement phenomena for the "two-layer system," values of the pressure p for the selected value of settlement $w = 0.20$ in are plotted against the reciprocal of the radius, $1/r$ in Figure 10a The curvature observed is characteristic of the "two-layer system," where a relatively weak subgrade is reinforced with a pavement layer of considerable strength As the bearing area increases in size the weaker

Example 5 Penetration Asphalt Pavement on Compacted Run-of-Bank Gravel Base Course.

The data in Examples 5 and 6 were obtained through the courtesy of Madigan-Hyland Co , Ltd , Engineers and Architects, Hinsdale, N. H , and is used with their kind permission. The data are very complete in detail and very consistent The load tests were made on a runway test section under the direction of Dr. Ek Koo Tan, in charge of the testing laboratories

In order to anticipate the most serious conditions of the subgrade, the weakest section was chosen for the test section, and the drainage ditches were flooded to a constant depth of 1 ft -4 in below the base of the pavement for one week prior to making the load tests, in order to permit capillary saturation of the subgrade

A. *Subgrade* Load test on top of natural clay. 30-in rigid plate Dark gray silty

Clay, some medium to fine Sand, trace of fine Gravel Medium Plasticity. LL.—42.4, PL.—25.2, PI —17.2.

Unconfined compressive strength-50 lb. per sq. in
Settlement, in., $w = 0.20$
Corresponding pressure, lb per sq. in $p = 4.3$
Unconfined Subgrade Modulus, $E_3 = \frac{1.18 \times 4.3 \times 15}{0.20} =$

$$E_3 = \frac{380 \text{ lb. per sq. in}}{\text{in. per in.}}$$

B. *Base-Course.* Compacted Run-of-Bank Gravel. Two-Layer System. Gray coarse to fine gravel, some coarse to fine sand, trace silt. Maximum density, 130 lb. per cu. ft. average. Optimum moisture, 11.0 per cent. 95 to 100 per cent compaction in place in the base course.

(1) Load Test Data.

Section No.		6	1	5
Thickness of base course, in		12	24	38
$h_1 = (h/r)r$		0.8r	1.6r	2.53r
Settlement, in	w	0.2	0.2	0.2
Pressure, psi	p	16.5	28.4	38.4
Subgrade moisture content, %				
Before flooding		27.0	27.0	
After flooding		27.6	28.0	

(2) Trial Values of E_3 and F_w for the Graphical Solution.

Section No	6	1	5
h_1	0.8r	1.6r	2.53r
$F_w = \frac{wE_3}{1.18pr}$	$\frac{E_3}{1460}$	$\frac{E_3}{2500}$	$\frac{E_3}{3400}$
Values of E_3	**Corresponding Values of F_w**		
500	0.34	0.20	0.147
600	0.44	0.24	0.176
700	0.48	0.28	0.205
800	0.55	0.32	0.235

The graphical solution, similar to that of Example 3, Figure 9 yields the following values: $E_3 = 650$ and $E_3/E_2 = 1/25$
Effective Subgrade Modulus, $E_3 = 650$ lbs. per sq in /in. per in.
Run-of-Bank Base Course Modulus, $E_2 =$

$$650 \times 25 = \frac{16{,}200 \text{ lb per sq. in.}}{\text{in. per in}}$$

C *Penetration Asphalt Pavement.* Three-Layer System. Load test at third point of 30-ft. asphalt pavement 1.5 in. wearing course, 4 in. of penetration asphalt macadam, 1 1/4 to 3-in. stone, compacted and filled with choke stone; 4 in of water-bound macadam Total thickness, $h_1 = 9.5$ in Age, 12 days.

Total thickness of pavement, h_1	h_1	9.5 in.	0.63r
Thickness of Run-of-Bank Base Course	h_2	38 in	2.53r
Settlement, in.	w	0.2	
Pressure, psi.	p	52.5	

Three-Layer System Analysis. For a "three-layer system" an approximate analysis is made in the following manner. The settlement equation now contains an additional multiplying coefficient f_w for the "three-layer system," similar to F_w for the "two-layer system."
"Three-Layer System" settlement equation,

$$w = 1.18\,pr\left[\frac{F_w f_w}{E_3}\right] \tag{5}$$

The "Three-Layer" Coefficient,

$$f_w = \frac{w}{1.18\,pr}\left[\frac{E_3}{F_w}\right] \tag{6}$$

The solution requires three steps, as follows:

(a) At the lower limit with the pavement thickness $h_1 = 0$ (no pavement) the "two-layer" load-settlement equation is:

$$w = 1.18\,pr\left[\frac{F_w}{E_3}\right] \tag{1b}$$

From part *B* above for the load tests on top of the base course:

$E_3 = 650$, $E_3/E_2 = 1/25$, and $E_2 = 16{,}200$

The Two-Layer coefficient from the influence curves of Figure 4:

$$F_w = F[2.53r, 1/25] = 0.180$$

(b) At the other limit with the pavement thickness $h_1 = \infty$ a homogeneous deposit all of the pavement layer, the value of the coefficient f_w for $h_1 = \infty$ is unknown and is to be found. For a finite thickness of $h_1 = 9.5$ in. of the pavement the value of the "three-layer" coefficient f_w is obtained from Eq 6.

$$fw = \frac{0.2}{1.18 \times 52.5 \times 15}\left[\frac{650}{0.180}\right] = 0.78$$

This means that the pavement layer reduces the settlement to 78 per cent of that for the base course alone

(c) In the approximate analysis it is assumed that the "three-layer" coefficient f_w follows an influence curve similar to those of F_w for the "two-layer system" in Fig. 4, as h_1 increases from zero to infinity. Actually these curves may be somewhat flatter. But the approximation appears to give values of about the right order of magnitude.

For $h_1 = 0$ "two-layer system," $E_3/F_w = 650/0.18$

For $h_1 = \infty$ "one-layer system," all pavement, $\left[\frac{E_3}{F_w f_w}\right]$ must become equal to E_1 of the pavement.

The Argument A_3 obtained from the influence curves of Figure 4 instead of E_2/E_1 is defined as follows:

"Three-Layer" Argument,

$$\frac{\left[\frac{E_3}{F_w}\right]}{\left[\frac{E_2}{F_w f_w}\right]} = \frac{\left[\frac{E_3}{F_w}\right]}{\left[\frac{E_1}{1}\right]} = A_3 \qquad (7)$$

For the finite thickness of pavement, entering the curves of Figure 4 with the values $h_1 = 0.63r$ and $f_w = 0.78$, the point falls on the influence curve designated $A_3 = 1/5$

Substituting in equation 7, $E_1 = 5 \times \frac{650}{0.18} = 18{,}000$

The Penetration Macadam Modulus, $E_1 = \frac{18{,}000 \text{ lb. per sq in.}}{\text{in. per in.}}$

This is only slightly greater than that of the base course modulus $E_2 = 16{,}200$ obtained in Part B above. Therefore in most cases one is justified in using the combined thickness of asphalt pavement and base course in computing thicknesses and supporting capacity.

The importance of the *Restraints* offered by the base course layer upon the effective strength properties of the clay subgrade can now be judged by comparing the value obtained in Parts A and B. Load tests on the unconfined subgrade gave an unconfined subgrade modulus of 380 compared to an effective subgrade modulus of 650 from load tests on the base course. It is evident that the base course layer offers a very considerable restraint upon the subsurface lateral deformations in the clay subgrade because of the shearing restraint at the interface, and upon the tendency for upheaval of the subgrade outside the limits of the bearing plate because of the load spreading and reinforcing effect of the base course layer, as indicated in Figure 8 Hence the effective strength properties are accordingly considerably increased.

Example 6. Concrete Pavement. "Three-Layer System."

The data for the subgrade and base course is the same as for Example 5, with the thicknesses as given.

Concrete pavement: Class: E-3 concrete. No reinforcing steel.

Age at the time of load test: 42 days.

Modulus of Elasticity of the concrete, $E_1 = 3{,}000{,}000$

In the analysis of concrete pavements, it is not permissible to use the combined thickness of concrete pavement and base course, because of the great difference in the strength properties. Therefore it is necessary to consider the pavement construction as a "Three-Layer System."

(1) Test Data. Center load test on top of concrete pavement 30-in. diameter rigid bearing plate.

Section No.		6	1	5
		15-ft slab	30-ft.	20-ft
Thickness of concrete, in	h_1	8 0 53r	8 0 53r	7 0 46r
Thickness of base course, in	h_2	12 0 8r	24 1 6r	38 2 53r
Settlement, in	w	0 05	0 05	0 05
Pressure, psi	p	42 7	63 0	70 5

(2) Two-layer Coefficient F_w, as if concrete was laid directly on the subgrade, the base course thickness, $h_2 = 0$ as one limit.

$E_1 = 3{,}000{,}000$

E_3 the subgrade modulus to be found by the trial method.

Trial values of E_3	Corresponding values of $F_{w3} = F[h/r, E_3/E_1]$		
Section No	6	1	5
1000	0 142	0 142	0 160
1500	0 162	0 162	0.185
2000	0 178	0 178	0 200
2500	0.190	0.190	0 215

(3) "Three-Layer" Coefficient, Eq. 6. Example 5. Trial Values.

Section No.	6	1	5
$f_w = \frac{w}{1.18\,p\,r}\left[\frac{E_3}{F_{w3}}\right]$	$\frac{E_3}{15{,}000 F_w}$	$\frac{E_3}{22{,}300 F_w}$	$\frac{E_3}{24{,}900 F_w}$
Trial Values of E_3	Corresponding Values of f_w		
1000	0 466	0 316	0.251
1500	0 613	0 415	0 325
2000	0 744	0 504	0 402
2500	0 870	0.583	0 466

The graphical solution is shown in Figure 11, which yields the value of $E_3 = 1500$ and the three-layer" argument, $A_3 = 1/7$. If the law governing the "three-layer system" was very

different from that for the "two-layer System," and could not be approximated quite closely by the influence curves of Figure 4, then the trial values of f_w could not be superposed and exhibit the same general form and agreement as indicated in Figure 11.

It is evident that the rigid concrete pavement offers much greater restraints upon all materials underlying the pavement, because of the greater load spreading and reinforcing effect and the relatively high flexural strength. These restraint effects are disclosed only by

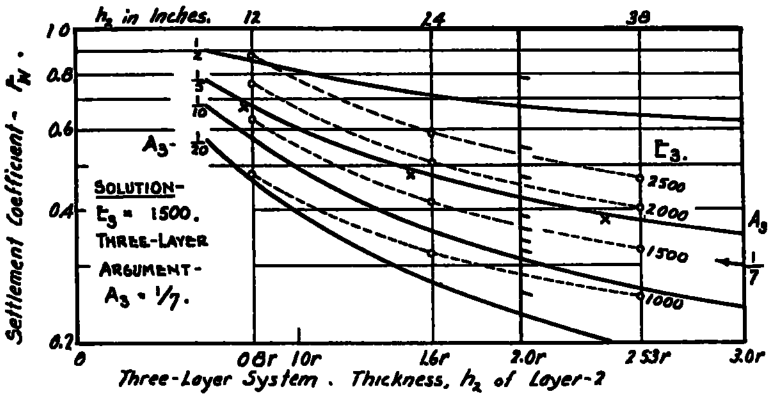

Figure 11. Example 6. Determination of E_3 and the Argument, A_3 for the Three-Layer System

(4) Determination of the Base Course Modulus by trial to approximately satisfy the "three-layer" argument A_3

$$E_3 = 1500 \qquad A_3 = \frac{\left[\frac{E_3}{F_{w3}}\right]}{\left[\frac{E_2}{F_{w2}}\right]} = \frac{1}{7}$$

Sec 5	$\frac{1500}{0.185} \times 7 = 56{,}600$ $\frac{E_2}{F[0.46,\ E_2/E_1]} = \frac{24{,}000}{0.43} = 55{,}800$
Sec 1 Sec. 6	$\frac{1500}{0.163} \times 7 = 64{,}800$ $\frac{E_2}{F[0.53,\ E_2/E_1]} = \frac{24{,}000}{0.38} = 63{,}200$

Since these values check sufficiently close, the "three-layer" analysis yields the following effective strength properties:

Concrete Modulus, E_1 = 3,000,000
Effective Base Course Modulus, E_2 = 24,000
Effective Subgrade Modulus, E_3 = 1500

The importance of the *restraints* offered by a rigid concrete pavement upon the effective strength properties of the base course and subgrade is apparent by comparing these values with those obtained from Example 5 for the same sections.

Clay Subgrade, unconfined, E_3 = 380
Base Course, Two-Layer. Effective, E_3 = 650, E_2 = 16,200
Concrete Pavement, Three-Layer. Effective, E_3 = 1500, E_2 = 24,000

analyzing and comparing load tests made at different stages of construction, because the strength properties of many soils are a function of the stress and restraint conditions imposed, as in the tri-axial compression test.

PRACTICAL DESIGN CONSIDERATIONS

These few examples not only serve to illustrate the possibilities and the practical applications of the method of analysis for layered systems, but also show that the theory of the "two-layer system" is in reasonably good agreement with the real phenomena.

The practical problem of design of airport runways involves the selection of suitable and economical types of pavement construction, and the determination of the thickness required to provide adequate support on a given subgrade. The airplane wheel loads and tire sizes are known, and therefore the design values of contact pressure and average radius of contact are known.

It is now proposed to widen the scope and application of the design diagrams now used by the army engineers for determining the combined thickness of flexible bituminous pavements, which were developed empirically from experience and test data (California bearing ratio experience, etc.). This practical approach can now be placed on a broader theoretical basis in accordance with the basic

concepts of the theory of the "two-layer system."

A tentative form of design diagram is suggested in Figure 12, in which are incorporated certain basic criteria for design as practical modifications of the present diagrams used in determining the required combined thickness limitation should be, especially with regard to repeated dynamic loading effects in order to insure satisfactory service and length of life. This should be given careful study because the settlement limitation has a most important influence on the thickness requirements, as shown in Figure 6.

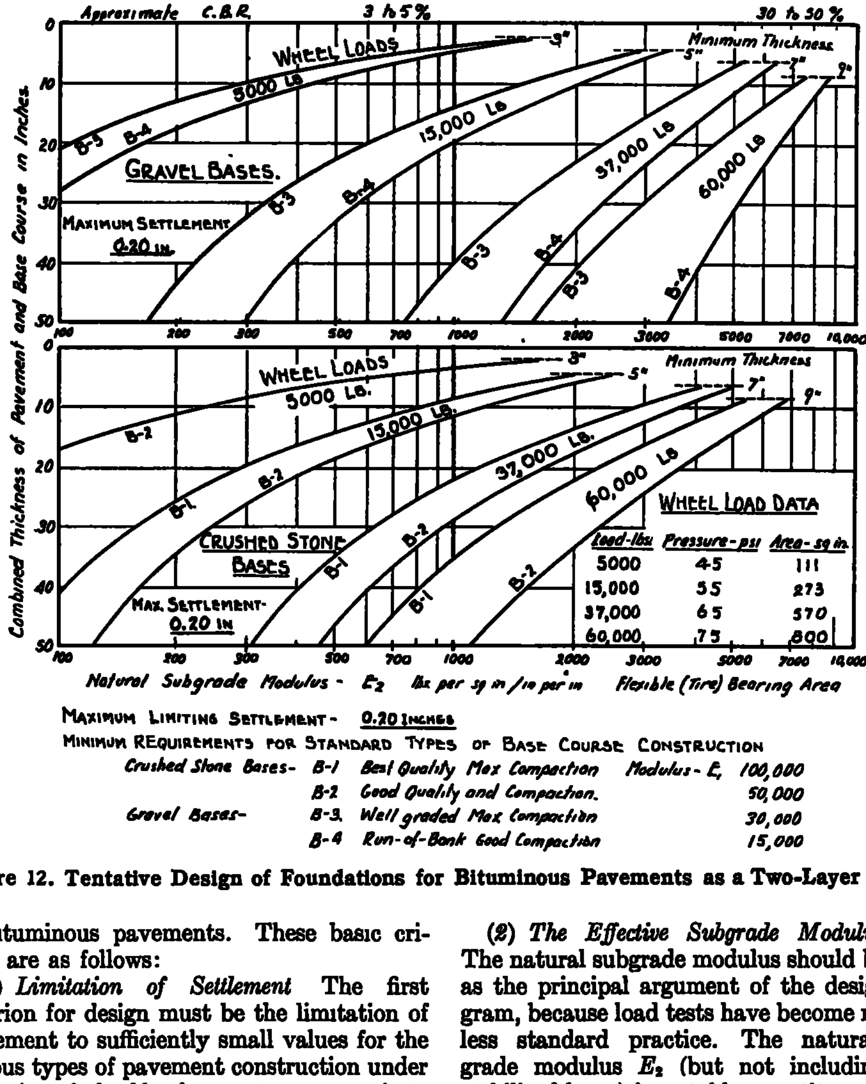

Figure 12. Tentative Design of Foundations for Bituminous Pavements as a Two-Layer System

of bituminous pavements. These basic criteria are as follows:

(*1*) *Limitation of Settlement* The first criterion for design must be the limitation of settlement to sufficiently small values for the various types of pavement construction under the action of wheel loads so as to insure against (a) objectionable consolidation in the base course itself, and (b) excessive deformations in the subgrade which would result in the eventual failure of the runway pavement. Experience will indicate what the settlement

(*2*) *The Effective Subgrade Modulus, E_2.* The natural subgrade modulus should be used as the principal argument of the design diagram, because load tests have become more or less standard practice. The natural subgrade modulus E_2 (but not including any stabilized layer) is a stable quantity, and permits the evaluation of the load-settlement relations for layered systems. On the other hand the coefficient of subgrade reaction k, now commonly used, is limited as to its practical usefulness, because it is not a stable quan-

tity, but is a function of the size of the bearing area and of the layered system coefficient F_w, which therefore remains unknown However, it is possible, if the necessary factors are known, to convert quite readily, the k-values by means of Equation 2 to the more basic modulus E_2, which is in accord with theoretical concepts.

Coefficient of Subgrade Reaction—

$$k = \frac{p}{w} = \frac{E_2}{1.5r} \quad \text{or} \quad \frac{E_2}{1.5rF_w} \tag{8}$$

Subgrade modulus

$$E_2 = (1.5rk) \quad \text{or} \quad (1.5rkF_w)$$

The coefficient will be 1.5 or 1 18, depending or whether a flexible or rigid bearing plate was used in the k-value determination, respectively.

(3) *Standards of Base Course and Pavement Construction* It is obvious from the few examples analyzed and from Table 1 of tentative equivalent types of construction, that the quality and strength properties of different types of base course construction are by no means equal and can vary enormously. The problem of first importance is to determine the range of the effective strength properties of the various types of stabilized subgrade, base course, and pavement constructions that can be consistently achieved by good

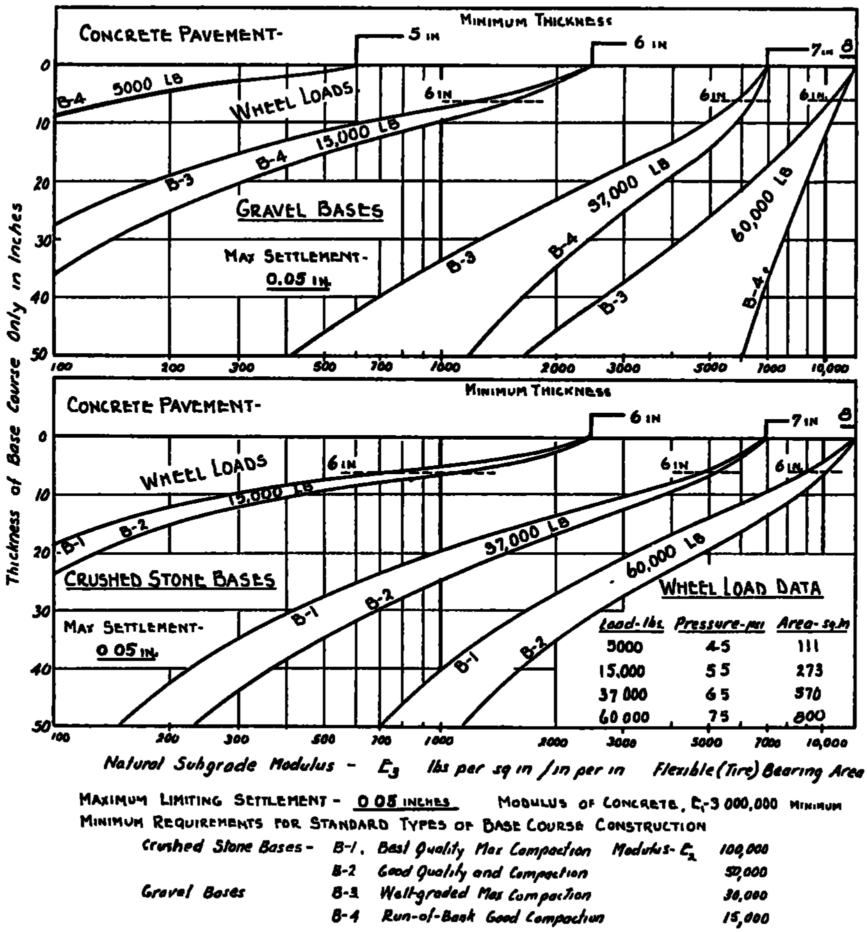

Figure 13. Tentative Design of Foundations for Concrete Pavements as a Three-Layer System

construction practice. It should be possible to set up for design and construction purposes minimum standards of good construction practice on the basis of load test data and performance, and to specify certain minimum requirements for their strength properties. These minimum standards may be designated, for example, as indicated tentatively in Figure 12. Thus the type of base course construction can be selected with due regard for economic and practical considerations, and the thickness required for a given subgrade condition can be determined accordingly. The tentative design curves of Figure 12 show that there are very important differences in the thickness requirements for the assumed minimum standards of base course construction. For the light wheel loads of training planes, for example, stabilized subgrades with suitable wearing courses may be entirely satisfactory. But for the heaviest wheel loads now anticipated the very best types of base course construction possible today are required for economic reasons and in order to provide adequate support and to insure satisfactory service and length of life for bituminous pavements.

A similar tentative design diagram is also constructed for concrete pavements in Figure 13, employing the approximate methods of the "three-layer system." It is not permissible in this case to determine a combined thickness. However, because of the practical limitations of the thickness of concrete pavements in order to reduce temperature warping stresses, certain minimum and maximum thicknesses of concrete pavements may be specified for the different wheel loads. The thickness of base course required to give adequate support for a given subgrade condition can then be determined.

With these design criteria specified for a given runway project load tests can be made on a runway test section for preliminary design purposes Field load test checks of quality and strength can then be made as a matter of routine during construction (1) on the natural subgrade to determine tentatively the thickness of base course and pavement required, (2) on top of the base course to check the quality and strength at this stage of construction, and finally and most important (3) on top of the finished pavement to check the effective supporting capacity of the whole pavement construction. The permissible maximum settlement at each stage in the construction will be different, and can be readily determined by the layered system method of analysis and tabulated for the different standard types of pavement construction for field load test checks.

The importance and the constancy of the restraint effects exerted by different types of pavements can be learned by comparison of the effective strength properties at the different stages in the construction with those evaluated for the finished pavement by the "three-layer" method of analysis. Design, based on the effective strength properties, if the restraint effects are found to be fairly constant and stable, would be the more economical. If the restraint effects are neglected, and the design is based on the unconfined subgrade modulus, it should be on the safe side, but may be over conservative. Experience and performance should indicate what is the best practice to follow in design and construction.

Thus the design of airport runways can be placed on a broader theoretical basis in accordance with the basic concepts of the theory of the Two-Layer System.

Acknowledgements. The writer wishes to express his thanks to Dr. R. D. Mindlin and Dr. M. G. Salvadori, of the Department of Civil Engineering, Columbia University, New York City, for helpful suggestions on certain theoretical aspects of the paper.

DISCUSSION ON THE THEORY OF STRESS AND DISPLACEMENTS IN LAYERED SYSTEMS AND APPLICATIONS TO THE DESIGN OF AIRPORT RUNWAYS

MR. L A. PALMER, *Bureau of Yards amd Docks, U. S. Navy Department:* Consideration of the practical application of Professor Burmister's contribution suggests a number of questions. First of all the mathematical development is confined to a two-layered system, pavement and subgrade. Actually, the pavement itself is not single-layered and

for airports it may be comprised of at least 3 distinct layers, surface course, base course and compacted sub-base, all having different moduli and load distributing characteristics. A homogeneous subgrade would then constitute a fourth layer. The subgrade itself usually lacks homogeneity. For example, the top two feet of subgrade may consist of silty sand underlaid by clay Thus actually the problem involves at least four layers and more often than not five or more layers must be reckoned with.

Recognizing this hopelessly complicated condition, Mr. E. S. Barber (1)[1] and this writer chose to apply the formulas for stresses and deflections derivable from the point load expressions of Boussinesq. M G. Spangler and H. O. Ustrud (2), later followed the same procedure. Various other contributors, W. S. Housel, B. E. Gray, A. T. Goldbeck and others have introduced simplifying assumptions and have considered an "angle of spread" from wheel load through pavement to subgrade.

A second consideration from a practical standpoint is the possible order of magnitude for the E of pavement components. Burmister indicates values as high as 350,000 lb. per sq. in Such a value could be characteristic of soil cement and possibly also of dense asphaltic concrete, but it would not seem that beds of granular material could have more than a tenth of this value.

A third point to consider is the application of the theory to concrete pavements. This involves the assumption that the stress trajectories in concrete pavements and subgrade are continuous This does not seem reasonable and is not necessary in the well known Westergaard (3) analysis. For a single concentrated load, the assumed equality between pressure of pavement on subgrade and the product, deflection multiplied by k, modulus of subgrade reaction as Westergaard has used it, should be approximately correct. It is in the case of a distributed load on a rigid slab that this assumed proportionality is most questionable for in this case the deflection at any given point is affected by loads at distances away from the point as well as the load that is immediately over it. The Westergaard analysis has been put to severe trial and has not been found wanting. One hesitates to replace it with a new theory unless it is shown that the assumptions in the new theory are axiomatic and that the computed stresses in the concrete slab check closely those obtained by the Westergaard analysis

Fourth, J. H. Griffith (4) in 1929 and O. K. Froelich (5) in 1932 proposed the use of a "concentration factor" to account for the discrepancies between observed pressures and those computed from the ordinary theory of Boussinesq. From their theory it is possible to compute vertical pressure at all points under the load and not for points only on the axis of the loading. On Page 327, Table 1, of Vol. 20 *Proceedings*, the Highway Research Board, the pressures observed by Spangler and those computed by Griffith's theory, concentration factor taken as equal to 5, are in fairly good agreement This is not merely coincidental It should also be noted that the pressures computed by the ordinary Boussinesq theory ($n = 3$) are actually less, not greater than the observed values. The abnormal spreading of the load through the granular material as would be indicated by the $\frac{E_2}{E_1}$ ratio was not in evidence. With reference to the Ames experiments, Spangler proposed the formula,

$$p_z = p_o e^{-k^2 r^2}$$

p_z = vertical pressure in subgrade at any point

p_o = vertical pressure points on axis of loading

r = radial distance of point from axis.

k = a constant.

Also

$$p_o = \frac{CP}{t}$$

C is a function of P, t and E of the subgrade.

P = total wheel load

t = pavement thickness

Such formulas not only fit the facts but have also the advantage of simplicity.

In any theory in this connection, an essential boundary condition is that the total load on any horizontal plane below the load, P, must equal P. To satisfy this condition it is necessary to find an expression for the vertical pressure applicable at all points under the load. No solution is complete without this.

[1] Figures in parentheses refer to list of references at end of Mr. Palmer's discussion.

Considering the great degree of variability in the subgrade over a runway area of 200,000 sq yd. or more one hesitates to tie too closely to any classical theory. This writer believes that theory in the design of flexible pavements can at the best serve only to indicate trends and to possibly reduce the number of loading tests. Therefore, any theory proposed and used must be the simplest sort, involving a minimum of tests and computations

In a separate discussion, Mr. E. S Barber has shown agreement between computed values according to Professor Burmister's theory and that presented by Mr. Barber and this writer (1). This agreement is neither coincidental nor significant What we need more to know is whether or not the computed values can be checked experimentally

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(2) M. G. Spangler and H. O Ustrud, "Wheel Load Distribution Through Flexible Type Pavements," *Proceedings*, Highway Research Board, Vol. 20, 1940 (see also, Vol. 21, 1941 and Vol. 22, 1942).
(3) H. M. Westergaard, "Stresses in Concrete Runways of Airports," *Proceedings*, Highway Research Board, Vol. 19, 1939.
(4) J. H. Griffith, "Pressures Under Substructures," *Engineering and Contracting*, March 1929.
(5) O. K. Froelich, "Drukverdeeling in Bouwground," *De Ingénieur*, April 15, 1932.

Mr. E. S Barber, *Public Roads Administration:* This interesting paper derives the displacements in a two-layer system from the theory of elasticity for use in the design of pavements by a method similar to that proposed by L. A Palmer and the writer in the 1940 Proceedings of the Board, Vol. 20, p. 279. In this latter method the pavement was assumed to be incompressible and to have no flexural rigidity. In subsequent discussion (p 330 of the 1940 Proceedings) the writer proposed to consider the rigidity of the pavement by substituting for it a thickness of subgrade of equal stiffness which is $(C_p/C_s)^{\frac{1}{3}}$ times the pavement thickness. This device was proposed on the basis of the stiffness factor for slabs and was checked against the elastic displacement due to a point load on a two-layer system presented by K. Marguerre in Ingenieur-Archiv, vol. 4, p. 332, 1933. The resulting formula for pavement thickness, t, was

$$t = \frac{a}{(C_p/C_s)^{1/3}}\left(\left(\frac{1{\cdot}5pa}{C_sS_s}\right)^2 - 1\right)^{\frac{1}{2}}$$

where

p = pressure on circular loaded area of radius, a

C_p and C_s = modulus of elasticity of pavement and subgrade, respectively

S_s = displacement at top of subgrade

Solved for S_s, this becomes

$$S_s = \frac{1.5pa}{C_s}\left(1 + \frac{(C_p/C_s)^{2/3}}{(a/t)^2}\right)^{-\frac{1}{2}} = \frac{1.5pa}{C_s}F_s$$

TABLE 1
COMPARISON OF DISPLACEMENT FACTORS COMPUTED FROM TWO THEORIES

$C_p/C_s = E_1/E_2$	$a/t = r/h_1$	F_s	F_p	$F_s + F_p$	F_w
10,000	10	0 42	0 00	0 42	0 40
10,000	5	0 23	0 00	0 23	0 22
10,000	1	0 05	0 00	0 05	0 05
100	10	0 91	0 00	0 91	0.91
100	5	0.73	0 00	0 74	0 76
100	1	0.21	0 01	0 22	0 23
2	1	0 62	0.19	0.81	0.80

The displacement within the pavement which was originally neglected may be taken as

$$S_p = \frac{1.5pa}{C_s}F_p \quad \text{where} \quad F_p = \frac{1 - F_s}{C_p/C_s}$$

The factor for the total displacement is then $F_s + F_p$ and is compared in Table 1 with its equivalent, F_w, from Figure 3 of the present paper. It is evident that the two methods are practically the same numerically

Dr A Casagrande, *Graduate School of Engineering, Harvard University:* Professor Burmister has presented a valuable contribution by making numerical solutions of an important problem readily available to the engineer. All too often mathematicians leave their solutions in a form which still requires cumbersome and time-consuming computations for each case of application. For example, Marguerre's general solutions of the

same problem,[1] which the writer has used in analyses of foundations, have been evaluated numerically only for a frictionless interface, and even these are not sufficiently numerous.

Since in foundation engineering, when dealing with compressible fine-grained soils, the assumptions of Hooke's law and of perfect continuity at the interface are usually tolerably well fulfilled, detailed numerical solutions of the theory of elasticity for two-layer systems are of important practical value

When dealing with pavements and coarse-grained base courses on soft subgrades, the following complications should be considered.

(1) Because for coarse-grained materials Hooke's law is clearly not applicable, the actual distribution of normal stresses will show considerably greater stress concentrations on the subgrade than computed by the theory of elasticity.

(2) Assumption of perfect continuity at the interface may be very much on the unsafe side. The greater the difference in resistance to deformation between the two layers, the greater are the shear stresses theoretically transmitted at the interface (e g concrete pavements directly on clay). But if these computed shear stresses approach the shear strength of the softer material, then the maximum normal stress on the soft material will become much greater than the computed value Numerical solutions for the maximum shear stress at the interface for perfect continuity, and of the maximum normal stresses on the interface, both for perfect continuity at the interface as well as for a frictionless boundary, are needed to establish the extreme limits between which the actual conditions will lie.

(3) Most flexible pavements have not enough tensile strength to resist the considerable tensile stresses which develop, according to the theory of elasticity, for relatively large ratios of E_1/E_2 and r/h. Even if no tension cracks develop, the computed results will be on the unsafe side because the "modulus of elasticity" for tension is much smaller than for compression, which is another way of saying that the material for practical purposes can not take tension.

The cumulative effect of the foregoing is certainly serious enough to deserve careful investigation before translating load tests into recommendations for design in the manner suggested by the author. However, even if the writer cannot share the author's enthusiasm for the application of his solution to the design of pavements, the writer does believe that this approach deserves the attention of all engineers who are conducting research on the design of pavements, and strongly recommends that additional numerical solutions should be prepared as a basis for further studies which are necessary for a safe application of this method.

Irrespective of the theoretical method of evaluation of load tests, there remains the important question as to what extent individual static load tests reflect the results of thousands of dynamic load repetitions under actual traffic Experience and large-scale traffic tests have already indicated that various types of soils react differently, and that the results of static load tests by no means bear a simple relation to pavement behavior. There appears to be hardly any other way to find the answer to such questions than by means of large-scale traffic tests.

Mr. T. A. Middlebrooks, *U. S. Engineer Department, War Department:* As our knowledge is increased concerning the stress-strain characteristics of soils and pressure distribution in soil masses, it can be expected that a truly scientific method for the design of flexible pavements will be devised. Methods for designing pavements which are now in use or proposed can be divided into four different types, as:

a. Empirical methods based on experience and use of soil classification only. Methods of this type are used by most highway departments.

b. Empirical methods based on experience and the use of a physical test for obtaining a relative evaluation of the soils' supporting capacity The California Bearing Ratio and the North Dakota Cone Test methods are of this type.

c. Empirical methods based on experience and the use of the true stress-strain characteristics of the soils involved, as determined by the triaxial or other tests. A

[1] K. Marguerre, "Spannungsverteilung und Wellenausbreitung in der kontinuierlich gestuetzten Platte," Ingenieur-Archiv, Vol IV, 1933.

method of this general type is being used by the Kansas Highway Department.

d. Scientific methods based on the actual stress conditions imposed and the true stress-strain characteristics of the soils and pavement material involved. This is the ideal method of designing pavements, but there is no method of this type which has been perfected sufficiently for general use.

Professor Burmister's application of the theory of elasticity to the design of airport pavements more nearly falls under method *d.* It is not a purely scientific method, however, since the allowable or critical deflection must be obtained empirically. There is no question that truly scientific methods are the ideal approach to the problem. However, our knowledge of the pressure distribution in soils and the true stress-strain characteristics of soils is not sufficient at this time to warrant the adoption of such methods. This does not detract from Professor Burmister's paper, since its presentation at this time gives direction to what appears sometimes to be a general confusion of ideas.

The results obtained from the method in its present form indicate that amazingly thin bases can be used when the base is composed of a high type granular material. This is not consistent with the writer's experience. Although the data available at the present time are not all conclusive, it is the writer's opinion that the thickness of base will not vary greatly with the type of granular material as long as this material has adequate strength to prevent failure within itself This conclusion on variation of thickness will hold true, I believe, as long as the material does not have appreciable flexural strength. When the base material has definite flexural strength, of course it can be expected that the thickness can be decreased to some extent, depending upon the type of material and construction. In this connection, it should be recognized, however, that low flexural strength may actually be detrimental if the material is brittle, resulting in a low critical deflection.

One of the largest variables in the use of a scientific method of this type is the allowable deflection. It is noted that the author has used 0.2-in. allowable deflection in all his examples. This deflection is too high for a large number of repetitions. Experience to date indicates that the critical deflection will vary from approximately 0.05-in to 0 15-in. depending upon the type of subgrade, the type of base material, wheel load, and probably other factors It would probably be more accurate in dealing with the asphalt surface to use the curvature of the surface as a criterion rather than deflection However, it must be recognized that use of either deflection or curvature is only a convenient means of approximating the effect of the stresses induced on the materials involved. Actually, what we want to know is whether or not the stresses induced exceed the strength of the materials. As long as the strength of the materials involved is not exceeded, the amount of deflection is of little consequence in this case where we are dealing with shear failures.

In view of the difficulty, if not impossibility, of arriving at the correct critical deflection to be used for the varying conditions to be encountered, it would appear highly desirable to approach the problem from the standpoint of shearing stress versus shearing strength rather than deflection versus modulus of deformation. Considerable work has been done by Biot, Hall, and others on the distribution of shearing stresses, since Carother's original paper However, I do not believe that the distribution of shearing stresses has been used to any great extent in pavement design. Additional work along these lines would be highly beneficial to the profession.