A MATHEMATICAL ANALYSIS OF SOME PHASES OF THE FLEXIBLE SURFACE DESIGN PROBLEM

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SYNOPSIS

A brief review points out the lack of applicable mathematical relationships between the analysis of the variables and the design of flexible surfaces or other structures involving soil or aggregate masses This is followed by the development of a possible fundamental equation through the application of the Law of Similitude to the problem in its simplest form Since the well known Housel relationship between soil support and perimeter area for a given deflection seems to be a special form of this general equation, the prospects that it may be substantiated seem excellent. If this equation is proved correct it will have some interesting corollaries, such as the possibility that laboratory tests can be extrapolated to field use. The definite need for flexible test plates instead of the rigid so far used is another possible consequence from the implications of the formula

The derivation of the relationship between the increase in supporting strength and layer thickness for foundation materials over a subgrade, where the load tests indicate Housel's relationship is being maintained, throws further light on the design situation, as it affords an indication of which type of design formula may be required to most nearly represent the conditions. It indicates that the supporting power of the foundation is directly proportional to the foundation material strength, from which it would follow that methods which specify set foundation thicknesses regardless of the foundation material quality, are incorrect

In conclusion warning is given that the formulae must be fully confirmed by experiment, but it is felt that the relationships derived may be quite helpful in indicating the proper trend of future investigation and methods to use in analyzing the results.

Aggregate structures for the spreading of stresses or the carrying of loads were early used by man. Despite the apparent simplicity of such structures, their analysis for design purposes is among the most complex problems of mechanics. The growing importance of highways and airports, and the economic benefits which will result from the development of a rational method of design of flexible surfaces (including in this classification all which are composed of discrete particles of soil or aggregate functioning as a mass through frictional interlocking or the cementing effect of fluid binding materials), have given this subject outstanding prominence today.

Even in simplified form, the problem is not readily susceptible to mathematical analysis of the stress strain relationships. The classic solution for the distribution of stresses resulting from a uniform load applied over a circular area to a completely elastic, homogeneous, and isotropic medium has been of some help in the study of cohesive soils; but if anything, it has been misleading in its attempted applications to actual structures with properties differing greatly from these assumed characteristics. No solution to a similar attack, for an elastic layer of different structural characteristics imposed upon the base medium—unless the Westergaard analysis for a rigid slab can be so classified—is known to the writer, but a very general analysis of this type would undoubtedly be far more helpful. Perhaps the determination of the trend of the variables involved, through a systematic experimental study by the use of materials exhibiting these ideal properties, will be a compromise solution.

As a result of this general mathematical situation, the usual approach to this problem has been either through field tests or the application of empirical formulae to experience data, with no solution from either attack generally accepted as satisfactory today. However it does appear possible to throw some light on certain aspects of the situation through a fundamentally rational mathematical attack, and this paper is concerned with such analysis.

FUNDAMENTAL LOAD BEARING RELATIONSHIP

In the actual applications, with which we are primarily interested in practice, the load is applied to the structure through pneumatic tires, resulting in a more or less uniform intensity of loading Test procedures to evaluate the load bearing characteristics of soils and aggregates have in general resorted to rigid application surfaces, such as steel plungers or plates, due to their immediate availability and the apparent difficulties involved in using an application surface which gives uniform pressure distribution—that is, a so-called "flexible bearing plate" A number of engineers, including the writer, have questioned the correctness of applying the results from such a loading technique to field design, feeling that a uniform correlation might not exist between the two load application methods, and certainly the assumption of a direct relationship is quite questionable. Since the ideal way to determine the load bearing capacity of a material when used in a structure is through the application of actual loads, the load bearing test attack seems the most rational and is consequently receiving the maximum of attention at the present time In the course of so doing, this matter of rigid versus flexible bearing plates is being given attention, although it is not yet certain that the mechanical difficulties of working out a suitable plate have yet been solved in such fashion as to permit the obtaining of all the data needed. Certainly any analysis which will assist on this point, and incidentally throw some basic light on probable relationships between the variables involved in the carrying of loads by such structures, should be helpful

The simplest load bearing situation is the application of a uniform loading over a known circular area, resulting in some measurable deflection of the soil or aggregate surface. If the deflection at the center of the plate is used as a criterion of this situation, we then have three variables. In general, the aggregate structure resists by its ability to withstand direct compression and frictional movement or shear. If we assume that the soil or aggregate mass can therefore be completely characterized from a structural standpoint by coefficients which express these two properties, we have two more variables or a total of five. Through the application of Rayleigh's "Law of Similitude," it appears that if the dimensions of the soil constants can be determined—that is, if the constants can be properly chosen—we might throw some light on the inter-relationships of these various quantities in this case.

Since the soil acts more or less as a confined mass, it seems reasonable to characterize the compression resistance on the basis of resistance per unit of volume or pounds per cubic foot. Note that this differs from the usual factor, the modulus of elasticity in compression expressed in pounds per square foot. the use of the latter would also require the bringing in of Poisson's ratio. The shearing tendency, which occurs on planes, seems reasonably to be expressible in resistance per unit of area or pounds per square foot. These are accordingly assumed.

Another important assumption is that the problem is statistically determinate in the range of pressures considered; or in other words, that static equilibrium results from the application of any force in this range.

The resulting application of dimensional equations to the problem is given in Section 2 of the Appendix, following Nomenclature in Section 1. This derivation indicates that a possible solution of the problem—that is, a relationship between the variables—can take the form of

$$p = \frac{ak_sD}{r} + bk_cD \qquad \text{Equation 1}$$

As pointed out in the Appendix, this appears to be the familiar perimeter area relationship used by Housel but in a more general form.

This equation indicates that the pressure will be directly proportional to the deflection for any plate, yet we know from observation that this is rarely the case This however may be easily explained A basic assumption in the derivation was that the soil had constant characteristics Quite evidently if consolidation has not taken place, all the structural soil properties will change as the squeezing together of the particles in the soil or aggregate mass occurs However once this consolidation has been completed (as a result of compaction and/or use) so that none additional occurs from the application of loads within the design range, it can be expected that the deflection will be directly proportional to the pressure. Furthermore field or laboratory proof that this occurs will serve as some evidence of the correctness of the equation. Evidently, to take care of this situation, the equation should be expressed in the differential form.

$$\frac{dp}{dD} = \frac{ak_s}{r} + bk_c \qquad \text{Equation 2}$$

This is the most general form of the relationship, holding for any variation of k_s and k_c with D, if the other assumptions made remain valid However in view of the objections by some to the differential form, as well as the wide acquaintanceship with the equation used by Housel, Equation (1) instead of Equation (2) will be referred to hereafter, with understanding of its limitations, since this does not affect the validity of these further conclusions

Certain corollaries immediately follow from any demonstrated correctness of Equation (1). If the relationship is proven for any range of the variables—or more particularly, the dimensionless groups—it immediately follows that it will probably hold through their entire variation. This follows because the experimental verification for any set condition not merely is applicable to those values but for any others which result in the same dimensionless product value If the relationship is proven for a number of quite different soils (as may be hoped for from results to date) over a considerable range of plate radii, it is obvious that the probable range of the dimensionless quantity so substantiated will be quite large. In other words, if the Housel relationship proves to stand up with flexible plates (below the failure point of the soil), it should be possible to extrapolate it to a zero value of the perimeter area ratio Also all other lines on the perimeter area ratio graph for analogous structures over the same subgrade will then pass through this same zero value; because obviously, for the plate of infinite radius, the only supporting factor will be the subbase compressive resistance, which must in the end carry (and hence determine) the load support value regardless of any added layers to improve the load distribution From the standpoint of the application of laboratory or field bearing tests with small loads to the design for the large units which may be applied to such surfaces, this is an extremely important conclusion.

There is another deduction to be made from this relationship. If the assumptions specified are correct and Equation 1 is confirmed by experiment, the application of a uniform pressure makes possible the need for only two dimensionless groups, from which the relationship follows If the load however is not uniformly applied over the surface, as occurs with a rigid plate, another dimensionless quantity characterizing the pressure distribution must be brought in We then have three dimensionless groups, the relationship of which must be determined by experiment. It is quite evident that, if the pressure distribution coefficient for a rigid plate holds constant over a range, a straight line may result, but this is only true while these circumstances maintain and this straight line relationship may change at some point to some other one Even in the range where the straight line relationship holds, the correlation with the straight line using a flexible plate is still not known, though it can possibly be ultimately established by experiment At the present time it is obviously unwise to attempt to apply the results of bearing tests obtained from rigid plates where the load conditions involve flexible load application equipment. The obviously erroneous results which would have been obtained had field bearing tests with rigid plates been extrapolated to the design conditions in certain instances are thus explained.

If the additive properties of the various resisting effects (shearing, etc) maintain and some experimental information on the geometry of the pressure distribution can be obtained, this attack through dimensional analysis has possibilities for a better understanding of the soil reaction under rigid plates. Even the deformation curve under flexible plate loadings may throw some light on this subject. Since the characterization of subgrade reaction (or the suitability of the various assumptions made concerning it), seems to be the remaining point to be settled in connection with the rational design of rigid slabs, this whole matter may prove to be of interest to the designers of such structures as well as of flexible surfaces.

In general, it would seem that the foregoing general relationship, if found applicable as Housel's work leads us to hope, will be of great benefit in permitting a more rational analysis of the problem. Some correlation can be perhaps established between the constants in the Housel formula and the fundamental properties of the soil. The relationship between the changes resulting from compaction and those in these fundamental soil constants may also be of use in a study of this practical matter of consolidation, its prediction and control.

EFFECT OF FOUNDATION THICKNESS

Another point in the present discussion of flexible surfaces is in connection with the rate at which the load is distributed or spread by layers of supporting aggregate, such as a crushed stone foundation. For example, the Gray formula, which was an early attempt to give a more rational design attack, assumes a constant 45 deg angle for this slope The Klinger formula likewise specifies a constant angle but allows for its variation with the material. Some insight may be thrown on this situation by some elementary mathematical analysis, as shown by paragraph three of the Appendix

In considering the application of Equation (3) to the conditions possibly existing, there appear to be three cases requiring analysis, remembering always the primary assumption that the Housel formula is found applicable to all layers.

The first case will occur where an additional layer of the subgrade material is added Obviously this will not change the soil constants nor will the area subjected to shear or other action be altered. Consequently $\frac{dm}{dt}$ should equal zero, and there will be no increase in permissible pressure This conclusion would have to result, because the basic assumptions behind Equation 1 call for an infinite depth and this addition of a soil layer of the same material will consequently not alter the situation in any respect.

The second case will occur where a layer of a different but granular material—or perhaps more exactly, material offering resistance only through compression and shearing effects—has been added. Obviously the assumptions behind Equation 1 as derived are no longer present and therefore it does not directly apply. It may come in only through the implication that the combined structure acts exactly as though it were composed of one material, showing the resistance required by Equation (1). The addition of such a layer under these circumstances could increase the area of the planes in the foundation layer subjected to shear in direct proportion to the thickness of this layer If this happens, the resultant shearing effect, or the effective shearing constant in the substitute Equation 1 found applicable, would change directly with the thickness, even though the shearing properties of the foundation material remained unchanged. In this case then $\frac{dm}{dt}$ would be equal to a constant

Obviously in this second case we have the equivalent of a resistance by shearing, with this shearing effect in proportion to the foundation layer thickness This is exactly in line with the thickness formulae proposed by Housel and others Conversely the finding that $\frac{dm}{dt}$ is equal to a constant implies that the straight line formula will clearly represent the situation, although it is an implication only that the added layer functions through compression and shearing effects, with actually the latter the only one increasing the load support value

Where this second case exists, as indicated by the finding of a constant value of $\frac{dm}{dt}$, the angle of distribution—the tangent of which is equal to $\frac{dr}{dt}$ in Equation (3) (Appendix)—obviously increases as the bearing value decreases This is a very interesting conclusion. Its immediate corollary is that the weaker the subgrade the more effective a foundation layer of given thickness in increasing its strength. This is of great practical interest when the rather large thickness called for by recent empirical formulae for flexible surfaces where the subgrade is quite weak, is considered.

The third case occurs when effects other than compression and shear resistance are obtained through the foundation layer This primarily implies the additional presence of cohesion, which through horizontal shear gives bending resistance up to the limit of the cohesive strength of the foundation material. Obviously the assumptions made in deriving Equation (1) will be greatly exceeded in this situation; if the relationship characteristics of Equation (1) still hold, such are simply experimental findings though not necessarily unreasonable. Where these cohesive effects are appreciable, the increase in shearing areas will still be proportional to the thickness, but the bending resistance will increase at a faster rate, consequently $\frac{dm}{dt}$ will not be a constant as in Equation (2) but presumably will increase with the thickness The actual amount of the change in thickness will have to be experimentally determined but Equation (3) indicates that this increase will have to be roughly proportional to the pressure increase for the Gray or Klinger type formulae to hold even approximately. Since the increase in thickness of the foundation layer where such effects exist will involve not merely a greater load bearing capacity by the foundation layer but in addition a more rapid spread of the effective radius of load distribution to the subgrade, the relationships may obviously be quite complex

It will be noted, that in the second case the strengthening effect of the foundation layer is proportional to its shearing strength, and in the last to its shearing strength plus or multiplied by an additional factor due to cohesion—most likely the combination of the two It is obvious that the load distributing effect of each and every layer in the foundation is directly dependent upon the strength characteristics of the material used It then follows that design methods which in specifying the foundation thickness do not allow for the strength of the foundation material, or which only call for high bearing strength in the top layers and imply that the characteristics of the lower layers are less or not important, are incorrect in principle if Equation (3) is valid for the conditions.

If the straight line relationship for different flexible bearing plates maintains, even with foundation layers over subgrades, as the findings to date indicate may be the case, though far from conclusively, it is evident that Equation (3) offers a means to analyze the situation and determine which of the proposed structural formulae most nearly represents the actual facts.

CONCLUSIONS

In conclusion some warning must be given The attack suggested has been primarily stochastic. The assumptions made, while reasonable, are not obviously demanded or even apparently more or less necessary Consequently the equations derived cannot be taken as literal physical laws until substantiated by evidence. They do however seem quite in line with the trend of the data so far made available If verified they should serve as an excellent tool for better understanding the phenomenon of load bearing by aggreagate masses, planning the experimental work necessary, and analyzing the relationships indicated by it.

APPENDIX

1. *Nomenclature*

- W . Total load on plate, lb.
- r . Radius of plate, ft
- p Resulting uniform pressure applied to surface of structure, lb. per sq ft
- D . Deflection resulting (at any specified point, say center of plate), ft.
- k_c : Soil compression coefficient, lb per cu ft
- k_s : Soil shearing coefficient, lb. per sq. ft.
- m · Housel perimeter shear constant, lb. per ft.
- n : Housel pressure constant, lb per sq ft.
- t : Thickness of added structure layer, ft.
- a, b: Constants, dimensionless

2. *Development of a possible structural equation through dimensional analysis*

For the case of a uniform pressure applied to the surface of a homogenous isotropic soil (or aggregate mass) of unlimited area and depth through a circular (flexible) plate, where the soil can be characterized completely (from the structural standpoint) by two constants representing its resistance to direct compression and shear respectively, five variables are involved. These variables can be defined in a dimensional system involving the funadmental units of mass, length, and time. The table of quantities following lists these variables and their dimensions

Since there are five arguments (variables and/or dimensional constants) and three dimensional units, from the TT theorem two dimensionless products which together involve all these arguments can be anticipated The relationship between these dimensionless groups must then be determined from experimental data or experience.

TABLE OF QUANTITIES FOR DIMENSIONAL ANALYSIS

Description	Symbol	Dimensions
Plate radius	r	(L)
Plate pressure (uniform)	p	$(ML^{-1}T^{-2})$
Plate deflection	D	(L)
Soil pressure coefficient	k_c	$(ML^{-2}T^{-2})$
Soil shearing	k_s	$(ML^{-1}T^{-2})$

Dimensionless quantities possible are $\left(\frac{k_sD}{pr}\right)$ and $\left(\frac{k_cD}{p}\right)$. These must be related in some manner which can only be conclusively established by experiment

If the shearing characteristic alone furnished the resistance, the first dimensionless group could be set equal to a constant and the resulting pressure from the deflection in question correspondingly determined: similarly if the compression characteristic alone acted. The simplest and most reasonable relationship to be anticipated is that these two effects would be additive according to some constant weighting. This is the same as saying that the most likely relationship would be:

$$a\frac{(k_sD)}{(pr)} + b\frac{(k_cD)}{(p)} = 1$$

or

$$p = \frac{ak_sD}{r} + bk_cD \qquad \text{Equation 1}$$

This is obviously a more general form of Housel's[1] perimeter area relationship, with $m = \frac{ak_sD}{2}$ and $n = bk_cD$. It has the advantage that the variables involved are clearly portrayed.

[1] Housel, W S.—*Proceedings*, The Association of Asphalt Paving Technologists, January, 1942, Vol. 13, p. 84.

3. *Change of structural variables with change in thickness of structure*

The basic equation is assumed to be (for a constant deflection)

$$p = \frac{m}{r} + n = \frac{W}{\pi r^2}$$

or

$$\frac{W}{\pi} = mr + nr^2 = r(m + nr)$$

Differentiating with respect to the structure thickness, we have

$$0 = (m + nr)\frac{dr}{dt} + r\left(\frac{dm}{dt} + r\frac{dn}{dt} + n\frac{dr}{dt}\right)$$

or

$$0 = \frac{W}{\pi r^2}\frac{dr}{dt} + \frac{dm}{dt} + r\frac{dn}{dt} + n\frac{dr}{dt}$$

Consideration of the support given to infinitely large plates indicates that n will remain constant regardless of the change in t, if we ignore the increment in deflection contributed by the direct compression of the layer of thickness dt, hence $\frac{dn}{dt} = 0$ and

$$(p + n)\frac{dr}{dt} + \frac{dm}{dt} = 0$$

or

$$\frac{dr}{dt} = \frac{-\frac{dm}{dt}}{p + n} \qquad \text{Equation 3}$$