# APPLICATIONS OF THE MOHR CIRCLE AND STRESS TRIANGLE DIAGRAMS TO TEST DATA TAKEN WITH THE HVEEM STABILOMETER 

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The paper is in two parts. In Part 1 a brief historical outline of the Mohr circle diagram is given, together with a summary of its principal applications. In the development of the mathematical theory, which is restricted to the most important case of combined stress in a plane, such as occurs in the triaxial compression test, expressions giving values of the normal and tangential components of stress acting on a general plane in terms of two principal forces acting on mutually perpendicular planes are graphically derived from Mohr's circle of stress diagram, and by analytical processes. Theories of failure, including Mohr's theory of rupture, are discussed briefly, and Mohr's circle of rupture diagram is applied to the analysis of three typical sets of data taken with Hveem's stabilometer in the triaxial compression test for stability. A method of representation of Hveem's stability formula and the calculation of Hveem stability by means of the Mohr diagram is included.

In Part 2, an independent method of graphical stress analysis, applicable primarily to materials whose properties conform with Coulomb's equation is presented. This method, which is called the "Stress Triangle Diagram," is equivalent in results to the Mohr Circle diagram and other methods in common use, but possesses greater simplicity of form and permits greater speed in construction and interpretation than the Mohr diagram. Its correctness is established by comparison of expressions for the normal and tangential components of stress on the plane of failure derived from the geometry of the diagram with equivalent expressions resulting from analytical operations. It is shown by contrast with the Mohr diagram that this method possesses the additional advantage of permitting greater facility in the derivation of several useful relations between the stresses involved.

## Part 1

## THE MOHR CIRCLE DIAGRAM

Briefly stated, the Mohr circle diagram is one of a number of graphical solutions applicable to a variety of problems in statics involving stresses in equilibrium, either compressive or tensile, in either two or three dimensions, and of second moments, such as moments of inertia. It may also be applied in a slightly modified form to the analysis of strains. In general it may be used, with certain modifications, to find the values of any two or three variables, whose values with respect to a set of rectilinear co-ordinate axes are known, under a transformation by rotation of the axes. A special case, however, in which the highway research engineer is concerned is that occurring in the triaxial compression test, viz., to find the normal and tangential components of stress acting on a
general plane through a given point of a body from measured values of two principal stresses acting on mutually perpendicular planes passing through the point.

The development of this excellent graphical solution, substantially in its present form, is due to Otto C. Mohr (1) ${ }^{1}$, an architectural engineer and professor at the Dresden Politechnikum Institute, about 1868. He also developed other graphical and semi-graphical methods, some of which were later rediscovered independently by others (2), such as the Area-Moment construction (3) for determining the form of a flexed beam, the Mohr correction diagram (4) used to correct an assumed displacement of the joints of a structure in solving truss problems, the Mohr-Land
${ }^{1}$ Italicized numbers in parentheses refer to a list of explanatory footnotes and references at the end of the paper.
construction ( $(\mathbf{5})$ for moments of inertia, etc., which should not be confused with the conventional Mohr stress circle diagram. Closely tied in with the Mohr circle construction, however, is the Mohr theory of rupture which will be discussed briefly in a later section.

From the available literature it would appear that Mohr's graphical methods were not used extensively by others for some time after their discovery. Mohr's original article on the subject was published in Zivilingenieur, p. 113, (1882). An account of the Mohr circle construction was given by Levy (6) and it was discussed and extended by Canevazzi in Italy ( $\uparrow$ ) and by Culman and Ritter in Switzerland ( $($ ). In England, J. J. Guest discussed the Mohr circle in connection with other graphical methods and Mohr's theory of rupture in an article published in 1900 (9). Theories of rupture and, incidentally, Mohr constructions were treated by W. A. Scobels and by von Mises (10) in 1913; also by Westergaard (11) in 1924, and Nadai (12), in 1931. The Mohr circle of stress and circle of rupture constructions have also been incorporated, along with other graphical methods, in a number of college texts on statics.

During the past decade there appears to have been a revival in the study of the Mohr diagram and in its practical applications in various fields of industry. For instance, it has been used in the calculation of stresses and strains in airplane construction, as indicated in articles by J. A. Wise (18), 1940, Niles and Newell (14), 1938, H. W. Sibert (15), 1939, and in highway research by several investigators. It has been applied to problems in soils research by Terzaghi (16), Casagrande (1i), Palmer and Barber (1s), Hogentogler and Barber (19), J. D. Watson (20), and others; and to problems in connection with stability of bituminous mixtures by such investigators as J. Ph. Pfeiffer (21) and V. A. Endersby (22).

Theory of the Mohr Circle Diagran-Theoretical treatments of the Mohr diagram are numerous, but variations in nomenclature and method of mathematical derivations render it difficult for the average reader to get a clear picture of the subject.

The three dimensional case of the Mohr stress circle diagram has received detailed treatment by Westergaard (11), Nadai (23),
and Timoshenko (24); several other writers have treated this case in some detail. The analytical derivation for the general case of plane stress involving initial values of both normal and tangential components of stress ${ }^{2}$ on two mutually perpendicular planes (not principal planes) is given in several college texts on statics. This problem will be stated and the derivation of the fundamental relations shown under Derivations, 1. The highway research engineer is concerned primarily with the special case of plane stress occurring in the triaxial compression test, in which two mutually perpendicular stresses, known as principal stresses, are given, with the normal and tangential stress components on an oblique plane through their intersection required. Derivation of the equations and applications of the Mohr diagram herein will be restricted to this case. It can be shown by deduction from the general case of three dimensional stress (23) that, since the intermediate principal stress equals the minor principal stress in the triaxial compression test, the plane diagram will correctly represent all combinations of stress. Two forms of the expressions for the normal and tangential stress components will be developed by analytical processes, and one of these forms will be derived by geometrical means from the Mohr diagram, for purposes of comparison. The values of the angle of shear $\alpha$ for which the normal and tangential components are at maximum and minimum values will be deduced both from the equations and from the diagram, and the results compared.

The analytical derivation of expressions for the normal and tangential components of stress acting on a general plane through a given point of a body in terms of two mutually perpendicular principal compressive stresses acting on principal planes passing through the point are obtained as follows:

Figure 1 represents an elementary prism bounded by the principal planes $A C F D$ and $F C B E$, an oblique plane $A B E D$, and two parallel vertical planes $D E F$ and $A B C$, in a body under combined stress from two unit compressive principal stresses $\sigma_{1}$ and $\sigma_{4}$. It is seen from the assumed conditions of static

[^0]equilibrium that the total normal and tangential components of reactive stress on the oblique plane $A B E D$ are given by the equations
\[

$$
\begin{align*}
\sigma d s d z= & \sigma_{1} d x d z \cos \alpha+\sigma_{3} d y d z \sin \alpha  \tag{1}\\
\tau d s d z= & \sigma_{1} d x d z \sin \alpha- \\
& \sigma_{3} d y d z \cos \alpha, \text { respectively.... } \tag{2}
\end{align*}
$$
\]



Figure 1
Dividing both sides of equations (1) and (2) by the area $d s d z$ and substituting $\cos \alpha$ for $\frac{d x}{d s}$ and $\sin \alpha$ for $\frac{d y}{d s}$, they reduce to

$$
\begin{equation*}
\sigma=\sigma_{1} \cos ^{2} \alpha+\sigma_{3} \sin ^{2} \alpha \tag{3}
\end{equation*}
$$

and $\tau=\sigma_{1} \cos \alpha \sin \alpha-\sigma_{3} \sin \alpha \cos \alpha$

$$
\begin{equation*}
=\left(\sigma_{1}-\sigma_{3}\right) \sin \alpha \cos \alpha . \tag{4}
\end{equation*}
$$

Obviously, the required stresses $\sigma$ and $\tau$ could be obtained by substitution of given values of $\sigma_{1}, \sigma_{3}$, and $\alpha$ in equations (3) and (4) and subsequent reduction, but the same results may be obtained more expeditiously in most cases by the simple graphical solution known as the Mohr stress circle diagram.

From a point () in a straight line ON (Fig. 2) the principal stresses $\sigma_{1}$ and $\sigma_{3}$ (assumed to equal 960 psi and 200 psi , respectively, for the purpose of illustrating the procedure) are laid off to any convenient scale (in a positive
direction for compressive stréss, or in the opposite direction for tensile stress). A circle with center at distance $\frac{\sigma_{1}+\sigma_{3}}{2}(=580 \mathrm{psi})$ from $O$ is then drawn through the terminals of $\sigma_{1}$ and $\sigma_{3}$. The co-ordinates of the intersection of this circle with a straight line through the terminal of $\sigma_{3}$ with inclination $\alpha$ to $O N$, are then the required values of the normal component $\sigma$ and the tangential component $\tau$ of stress on this plane measured to the same scale as were $\sigma_{1}$ and $\sigma_{3}$ (i.e., 330 psi and 285 psi for the particular value of $\alpha$ selected). In Figure 2 it is apparent that circle $A P B$ is the locus of points whose coordinates are values of the normal and tangential components of stress on every plane perpendicular to the plane of the paper passing through a given point of the body. So, for any arbitrarily selected value of $\alpha$ (for example, for $\alpha_{p^{\prime}}=60$ deg. selected merely for illustration), the values of the normal and tangential components of stress in the chosen units are determined from the stress circle. to be $\sigma_{p^{\prime}}=390 \mathrm{psi}$ and $\tau_{r^{\prime}}=329 \mathrm{psi}$ on this particular plane. If $\sigma_{1}=\sigma_{\mathfrak{3}}$, the stress circle cannot be constructed, but the physical interpretation is obvious: the specimen must be a perfect fluid, with zero shear stress on every plane, i.e., the stress is hydrostatic.
On substitution of the trigonometrical identities $\sin \alpha \cos \alpha=\frac{\sin 2 \alpha}{2}, \cos ^{2} \alpha=$ $\frac{1+\cos 2 \alpha}{2}$, and $\sin ^{2} \alpha=\frac{1-\cos 2 \alpha}{2}$, equations (3) and (4) reduce to

$$
\begin{equation*}
\sigma=\frac{\left(\sigma_{1}+\sigma_{3}\right)}{2}+\frac{\left(\sigma_{1}-\sigma_{3}\right)}{2} \cos 2 \alpha \tag{5}
\end{equation*}
$$

and

$$
\begin{equation*}
\tau=\frac{\left(\sigma_{1}-\sigma_{3}\right)}{2} \sin 2 \alpha \ldots \tag{6}
\end{equation*}
$$

forms which are better suited for comparison with expressions obtained from the diagram.

As may be seen by inspection of the right triangle $C P O^{\prime}$ in Figure 2,

$$
\begin{aligned}
& \sigma=O O^{\prime}-C O^{\prime}=\frac{\left(\sigma_{1}+\sigma_{3}\right)}{2}+ \\
& \quad \frac{\left(\sigma_{1}-\sigma_{3}\right)}{2} \cos 2 \alpha, \text { and } \\
& \tau=O^{\prime} P \sin 2 \alpha=\frac{\left(\sigma_{1}-\sigma_{3}\right)}{2} \sin 2 \alpha,
\end{aligned}
$$

expressions which are identical with equations (5) and (6).

It may also be noted that, by substitution of $960 \mathrm{psi}, 200 \mathrm{psi}$, and 60 deg for $\sigma_{1}, \sigma_{3}$, and $\alpha$, respectively, in equations (5) and (6) one obtains

$$
\begin{array}{r}
\sigma=\frac{1}{3}(960+200)+\frac{1}{2}(960-200) \cos 120 \\
=390 \mathrm{psi}, \text { and }
\end{array}
$$

$\tau=\frac{1}{2}(960-200) \sin 120=329$ psi, which
results in close agreement with those read directly from the diagram. (See $P^{\prime}$, Fig. 2)
tion of the resulting equation is substituted back in the original equations (5) and (6) to find their maximum and minimum values, one has, from equation (5)

$$
\begin{aligned}
\frac{d \sigma}{d \alpha} & =\frac{d}{d \alpha}\left[\frac{\left(\sigma_{1}+\sigma_{3}\right)}{2}+\frac{\left(\sigma_{1}-\sigma_{3}\right)}{2} \cos 2 \alpha\right] \\
& =-\left(\sigma_{1}-\sigma_{8}\right) \sin 2 \alpha=0
\end{aligned}
$$

and from (6)

$$
\begin{aligned}
\frac{d \tau}{d \alpha} & =\frac{d}{d \alpha}\left[\frac{\left(\sigma_{1}-\sigma_{3}\right)}{2} \sin 2 \alpha\right] \\
& =\left(\sigma_{1}-\sigma_{3}\right) \cos 2 \alpha=0
\end{aligned}
$$



Figure 2. Mohr Circle Diagram

Now, the values of $\alpha$ for which $\sigma$ and $\tau$ are maxima and minima may be found either from equations (5) and (6) using calculus, or by inspection of the diagram. Inspection of the graph shows that $\tau=0$ when $\alpha=0$ or 90 deg . and is a maximum of $\frac{\sigma_{1}-\sigma_{3}}{2}$ in absolute value at $2 \alpha=90 \mathrm{deg}$ or $270 \mathrm{deg}(\alpha=$ 45 or 135 deg ); and that $\sigma$ is a minimum $\left(=\sigma_{3}\right)$ at $\alpha=90$ deg and a maximum $\left(=\sigma_{1}\right)$ at $\alpha=0$.

Employing the standard method in which the first derivatives of $\sigma$ and $\tau$ from equations (5) and (6) with respect to $\alpha$ are set equal to zero, and the value of $\alpha$ obtained from solu-
from which
or

$$
\alpha=\frac{1}{2} \sin ^{-1} 0=0 \text { or } 90 \mathrm{deg}
$$

$$
\alpha=\frac{1}{2} \cos ^{-1} 0=45 \text { or } 135 \mathrm{deg}
$$

Substitution of the first of these values of $\alpha$ back into equation (5) gives

$$
\begin{gathered}
\sigma=\frac{\left(\sigma_{1}+\sigma_{3}\right)}{2}+\frac{\left(\sigma_{1}-\sigma_{3}\right)}{2} \\
=\sigma_{1} \text { (maximum value) and } \\
\sigma=\frac{\left(\sigma_{1}+\sigma_{3}\right)}{2}-\frac{\left(\sigma_{1}-\sigma_{3}\right)}{2}=\sigma_{3} \text { (minimum value) }
\end{gathered}
$$

Also substitution of the values of $\alpha$ obtained from the differentiation in equation (6) gives

$$
\tau=\frac{\left(\sigma_{1}-\sigma_{3}\right)}{2} \text { (maximum value) }
$$

and

$$
\tau=-\frac{\left(\sigma_{1}-\sigma_{3}\right)}{2}(\text { minimum value })
$$

Equations (5) and (6) are merely the parametric forms of the equation of a circle called Mohr's stress circle (Fig. 2). If a series of
and tangential components of compressive stress acting on the planes referred to. Lay off $\sigma_{v}$ and $\sigma_{x}$ (assuming $\sigma_{y}>\sigma_{x}$ ) to scale to the right of $O$ on $\overline{O N}$. At the terminal point $C$ of $\sigma_{x}$ erect $\overline{C P}$ perpendicular to $\overline{O N}$ in a positive direction, equal in magnitude to $\tau_{x y}$; and, at terminal $D$ of $\sigma_{y}$ erect $\overline{D P}$ perpendicular to $\overline{O N}$ equal to $\tau_{y x}$ in magnitude but in the negative direction. Since the tacitly


Figure 3. Mohr Circle Diagram
such circles are plotted from the several simultaneous values of principal stresses taken in a triaxial compression test on a material which conforms with Coulomb's relation, for example, their common envelope is the locus of all points whose co-ordinates are the normal and tangential stress components on a plane having a constant state of elasticity and making a given angle with the major principal plane.

Solution of the inverse problem of finding the principal planes and stresses, having given the normal and tangential stress components on two mutually perpendicular arbitrary planes passing through a given point, is easily effected by means of the Mohr diagram as follows:

Let $\sigma_{x}, \tau_{x y}$ and $\sigma_{y}, \tau_{y x}$ (Fig. 3) be the normal
assumed condition of equilibrium requires that $\tau_{x y}=-\tau_{y x}$, a circle passing through the points $P$ and $P^{\prime \prime}$ with center at $\frac{\sigma_{y}+\sigma_{x}}{2}$ may be constructed. This stress circle so constructed is the locus of points whose coordinates are the normal and tangential components of stress on planes making angles $\alpha$ and $90+\alpha$ with the given planes (where $\alpha$ is half the angle described by the radius vector $\overline{O^{\prime} P}$ in Fig. 3); and the normal and tangential stress components on a plane, making an angle $\overline{P^{\prime} P O^{\prime}}(=\alpha)$ with the given plane, are represented graphically by $\overline{O C^{\prime}}$ and $\overline{C^{\prime} P^{\prime}}$, respectively, (see Derivations, 1 , for proof). The intersections of the stress circle with $\overline{O N}$ determine the maximum and
minimum values of normal stress corresponding to $\tau_{x y}=-\tau_{y x}=0$. But, by definition, these maximum and minimum stresses are principal stresses previously denoted by $\sigma_{1}$ and $\sigma_{3}$, and the planes are the principal planes. Now if the point $P$ (Fig. 2 and 3) is located on a plane of failure, then the tangent to the circle at this point will intersect $\overline{O N}$ at the angle $\phi$, called the angle of internal friction, and the relation between the general stresses $\sigma_{y}, \sigma_{x}$, and $\tau_{x y}$, and the parametric constants $K$ and $\phi$ may be expressed by the equation

$$
\begin{align*}
& \sqrt{\left(\frac{\sigma_{y}-\sigma_{x}}{2}\right)^{2}+\tau_{x y}^{2}}- \\
& \quad \frac{\sigma_{y}+\sigma_{x}}{2} \sin \phi-K \cos \phi=0 \ldots \tag{7}
\end{align*}
$$

(See Derivations, 3, for proof.)
Theories of Failure-Before proceeding to the application of the Mohr Circle diagram in the analysis of experimental data from the triaxial compression test, it will be necessary to review briefly the various theories of failure in a material under stress and, especially, Mohr's theory of rupture. When a body is subjected to external stress, it undergoes some change in form or volume, (i.e., strain). For low values of stress the corresponding strain may be temporary; but when the stress reaches a certain higher value, depending on the nature of the material and the conditions of the experiment, including the particular combination of stresses acting, the strain may become permanent. When this point is reached the body is said to have failed. This point of failure, or rupture, or yield point, is rather indefinite in many instances, both as to definition and experimental determination, and its selection is subject to some degree of arbitrariness. In a sense failure is a progressive phenomenon, especially in plastic and semi-plastic materials (as asphaltic concrete) since there is always some relative displacement of the ultimate particles of the substance in any deformation, however small, thus indicating that the internal forces have been overcome to some extent, and since recovery in such materials is never absolutely complete. Strictly speaking, however, the term failure usually implies a permanent break or rupture such as is exhibited by
brittle materials. In view of the foregoing remarks, it is clear that in applying the Mohr diagram construction to the analysis of data in any specific case it is essential to define failure, plane of failure or rupture, and other terms used in order that the results may be properly interpreted in the light of these definitions.
From experience it appears that failure or rupture, in isotropic substances at least, nearly always takes place on a fairly definite set of planes characteristic of the material and of the physical conditions under which it is tested.

Various theories have been advanced as to the limiting conditions of stress existing in the material at the failure point. Three of these earlier theories, which have been proven incorrect or incomplete, are the maximum stress theory (26), the maximum strain theory (27), and the maximum shear theory (2S). As indicated by the titles these theories assume that failure is determined by the maximum values of the normal stress, of strain, and of shear stress, respectively. In Navier's theory which is merely an extension of the maximum shear (Coulomb's) theory, it is assumed that the limiting shear stress is affected by the normal component of stress on the plane of rupture and is proportional to it. Two other cases with limited applications are: (1) the theory of constant energy of distortion (29) (applicable to ductile metals) which asserts that the difference between the major and minor principal stresses is a constant, i.e., that the maximum shear stress is constant at the yielding point for a given material; and (2) Brandtzaeg's theory (30) (applicable to concrete) in which failure is assumed to occur along planes running in various directions rather than along a definite direction.
Mohr's theory, which is an extension of the maximum shear theory, makes the hypothesis that the limiting shear stress depends on the normal stress acting on the plane of failure and also that failure is determined by the maximum difference in magnitude between the principal stresses. Specifically, he assumes that the line of rupture is independent of the intermediate principal stress, an assumption partially invalidated by experiments of v. Kármán (31) and R. Böker (32), and that the same line of rupture will result from
the maximum differences between various combinations of principal critical stresses. He also assumes that the angle between the line of rupture and the axis of shear stress in a Mohr diagram is equal to the angle between the planes of rupture in the material at failure. Graphically and mathematically this means that the series of Mohr stress circles constructed from various paired values of major and minor principal stresses taken at the point

Mohr circle of rupture. The fact that we are dealing with the tangent to the circle rather than the secant makes it possible to express the relations between stresses and angles in more simple and practical forms.

For the purpose of simplifying the solutions as they are usually given, an expression for the resultant $R$ of the normal and tangential components of stress on the plane of failure will be derived at this point.


Figure 4. Mohr Circle Diagram
of incipient failure (called Mohr's circles of rupture) will have a common envelope which is the locus of points whose co-ordinates are the values of the normal and tangential components of stress on the plane of failure. Thus Mohr's circle of rupture is merely a special case of his circle of stress, viz., that circle of stress constructed from limiting or critical values of the principal stresses measured at the instant of incipient failure, or point of plastic yield, of the substance under stress.

Mohr's Circle of Rupture Diagram-Some Important Mathematical Relations-With slight modifications and extensions the mathematical theory of the Mohr circle of stress may be used to represent the relations in the

From the plane geometry theorem, which states that a tangent from a point to a circle is the mean proportional between the whole secant drawn through the same point and its external segment, it follows (Fig. 4, Circle (2)) that

$$
\begin{align*}
\overline{M P} & =R=\sqrt{\overline{M B} \cdot \overline{M A}} \\
& =\sqrt{\left(\sigma_{1}+\overline{M O}\right)\left(\sigma_{3}+\overline{M O}\right)} \\
& =\sqrt{\left(\sigma_{1}+K \operatorname{ctn} \phi\right)\left(\sigma_{3}+K \operatorname{ctn} \phi\right)} \tag{8}
\end{align*}
$$

since $\overrightarrow{M O}$, (the intrinsic pressure) is

$$
\begin{aligned}
\overline{M O} & =\frac{K}{\tan \phi} \\
& =K \operatorname{ctn} \phi,(\text { from right triangle } \overline{M O H}) . .(9
\end{aligned}
$$

When $K=0$, this resultant stress reduces to

$$
\begin{equation*}
R=\sqrt{\sigma_{1} \sigma_{3}} \tag{10}
\end{equation*}
$$

If one now starts with the identity (Fig. 4) $\overline{O O^{\prime}}=\overline{M O^{\prime}}-\overline{M O}$ and substitutes the value just obtained for $\overline{M P}$, and expresses $\overline{O O^{\prime}}$, $\overline{M O^{\prime}}$, and $\overline{M O}$ in terms of the principal stresses, cohesion, and angle of internal friction, the result is two equations;

$$
\frac{\sigma_{1}+\sigma_{3}}{2}=\frac{\sigma_{1}-\sigma_{3}}{2 \sin \phi}-K \operatorname{ctn} \phi \ldots .(11)
$$

and

$$
\frac{\sigma_{1}+\sigma_{3}}{2}=\frac{\sqrt{\left(\sigma_{1}+K \operatorname{ctn} \phi\right)\left(\sigma_{3}+K \operatorname{ctn} \phi\right)}}{\cos \phi}-
$$

$$
\begin{equation*}
K \operatorname{ctn} \phi \tag{12}
\end{equation*}
$$

which express existing relations between the factors of cohesion, angle of internal friction, and the major and minor principal stresses.

When $K=0$, they reduce to values for the $\sin \phi$, and $\cos \phi$ in terms of the principal stresses alone, thus;

$$
\begin{equation*}
\sin \phi=\frac{\sigma_{1}-\sigma_{3}}{\sigma_{1}+\sigma_{3}} \ldots \ldots \tag{13}
\end{equation*}
$$

and

$$
\begin{equation*}
\cos \phi=\frac{2 \sqrt{\sigma_{1} \sigma_{3}}}{\sigma_{1}+\sigma_{3}} \ldots \cdots \tag{14}
\end{equation*}
$$

Another useful relation, $\sigma_{1}=\sigma_{3} \tan ^{2} \alpha$, the derivation of which has been attributed to Terzaghi (33), may be obtained from the above equations by making use of Mohr's assumed relation between the angle of internal friction and the angle of shear, $\phi=2 \alpha-90$ (Derivations, 2).

A third form of the expressions for $\sigma$ and $\tau$ in terms of the principal stresses alone (not derived or referred to in the literature on the subject, so far as can be ascertained) which is easily derived from the previously developen relations (8), (9), (10), (13), and (14), and which is simple in form and susceptible of easy computation, may be obtained as follows:

From right triangle $O P C$ in Figure 2, it is seen, from the defining equation for the sine and cosine functions, that

$$
\sigma=\overline{O P} \cos \phi \text { and } \tau=\overline{O P} \sin \phi
$$

Substituting

$$
\begin{aligned}
\overline{O P}=R=\sqrt{\sigma_{1} \sigma_{3}} & (\text { From }(10)) \\
\sin \phi=\frac{\sigma_{1}-\sigma_{3}}{\sigma_{1}+\sigma_{3}} & (\text { From }(13)) \\
\cos \phi=\frac{2 \sqrt{\sigma_{1} \sigma_{3}}}{\sigma_{1}+\sigma_{3}} & (\text { From }(14))
\end{aligned}
$$

in the above equation gives

$$
\begin{equation*}
\sigma=\frac{2 \sigma_{1} \sigma_{3}}{\sigma_{1}+\sigma_{3}} \tag{15}
\end{equation*}
$$

and

$$
\begin{equation*}
\tau=\sqrt{\sigma_{1} \sigma_{3}} \frac{\sigma_{1}-\sigma_{3}}{\sigma_{1}+\sigma_{3}} \cdots \tag{16}
\end{equation*}
$$

As is apparent both from the equations and from the diagrams, the Mohr circles of rupture may be constructed from various given pairs of quantities other than the principal stresses. Besides the inverse problem, which has already been solved, the slope of the common tangent, or the angle of internal friction, and either of the intercepts, furnishes the means for construction of the circles of rupture whose diameters give the compressive and tensile strengths (for which one of the principal stresses is zero). This same data plus one of the principal stresses makes it possible to construct the corresponding Mohr circle of rupture, and so find the other principal stress, the secondary stresses, the other intercept, etc. Also, if it can be safely assumed, from supplementary information, that no part of the shear stress on the plane of failure is independent of the normal stress on this plane (e.g., in case of dry sand), then one Mohr circle of rupture plotted from a single pair of critical principal stresses is sufficient for the complete graphical solution, the line of failure in such a case being the tangent to the circle through the origin of the diagram (Fig. 2).
Application to Triaxial Compression Test Data-In practical application of the Mohr diagram method to the analysis of any given type of data certain precautions should be observed and certain limitations noted. Mohr's theory of rupture, on which his graphical representation is predicated, is applicable primarily to cases of plastic failure in isotropic materials whose properties con-
form to the Coulomb formula. It does not hold exactly for brittle substances such as marble or concrete (34); but it has been found from numerous experiments to be fairly consistent with tests on some isotropic brittle materials and ductile metals, and on such semiplastic materials as soils and asphaltic concrete for the combined stresses existing in the triaxial compression test under ordinary conditions. It should be noted that neither Mohr's theory nor any other of the several theories of rupture outlined in the preceding section takes account of the effect of time rate (3亏) or duration of loading, or of possible changes in the crystalline structure of the material.
It may be seen from the derivations of the fundamental equations of plastic equilibrium (see equations (15) and (16) and Derivations) by means of Mohr's diagram, that the validity of Coulomb's relation ( $\tau=\sigma \tan \phi+K$ where $\phi$ and $K$ are constants of the material) is tacitly assumed. For materials having approximately constant values of cohesion and angle of internal friction, such as dry sand or asphaltic concrete with flint aggregate, this assumption appears to be justified, for the Mohr envelope of the circles of rupture is found to closely approach a straight line having the above equation. However, the noticeable departure from the straight line common tangent in many instances indicates the necessity for extreme care in interpreting the results of this graphical method. The deviation of the Mohr envelope from the straight line relation may be interpreted as the effect of a change in cohesion, due to compaction of the specimen, or as a consequence of changes in structural properties of the material due to internal rearrangement of aggregate particles in the mix, etc., which is a necessary accompaniment of the changing consolidation. In effect these alterations constitute a change in the nature of the substance and in its stability during the course of the test. In cases of this type, the magnitude of the deviation of the envelope from the theoretical linear form, and its rate of change, or of such other mathematically related quantities as shear resistance, angle of shear, etc., may have significance as a possible means of evaluating the results of the triaxial test. Such a method of attack is being tentatively investigated by this laboratory for the purpose of obtaining
better correlation between test data and practical performance of bituminous mixtures. It is quite probable that Mohr's theory, as well as the maximum energy theory and Brandtzaeg's theory, will have to be revised and extended so as to include all the observed secondary effects. In fact, two or three other more comprehensive theories have already been formulated which explain many of the observed discrepancies between the present theories and experimental results for certain types of soils (36). Coulomb's equation must certainly be revised in view of the experiments by Krey-Tiedemann and Hvorslev on cohesive soils, described in Reference 36 in which it is shown that cohesion is a function of preconsolidation pressure and of void ratio. From the form of the Mohr envelope obtained from triaxial test data on most asphaltic concrete specimens, it is also obvious that cohesion, and in some cases angles of shear and of internal friction, are not constants characteristic of the material, but are functions of preconsolidation pressure, and of applied test pressures, void ratio, etc., which may vary in a rather complicated fashion throughout a single test.

Validity of the Mohr Diagram Method as Applied to Hveem Stability Test Data-Because of the lateral pressure always present in the triaxial test actual deformation in this direction, to any great extent, is prevented; but it can be shown that the displacement reading in the stability test with Hveem's stabilometer is an indication of, and indeed a function of a small more or less permanent lateral deformation. Displacements taken on actual bituminous specimens before and after a run gave values which differed by an average of 0.56 turns of the displacement pump ( 0.112 cu in .). This means that the volume of the space containing the liquid within the chamber about the sides of the specimen has been reduced by 0.112 cu in . due in part to a permanent lateral deformation produced in the specimen during the test, thus proving that a yield point has been exceeded. That this condition of yield or incipient failure exists at all pressures used in the test, and to approximately the same degree, is indicated by the form of the curves in Figure 5 (upper branch) showing lateral pressure plotted against vertical pressure for a
similar specimen. The lateral pressure, which is obviously due to, and a function of, lateral deformation is seen, from the direction of curvature of the graph, to increase faster than vertical pressure at all values of pressure, thus proving the existence of a state of failure throughout the entire test. Moreover, the fact that the rate of variation in curvature is practically constant shows that the extent or degree of failure is approximately constant throughout the test.
the deviation between the upper and lower branches is also proof of a residual increase in diameter of the specimen, the horizontal distance between the branches being a measure of this retained deformation.

It appears, therefore, that the use of the Mohr circle method for analysis of data taken in the Hveem stability test is justified if the results are properly interpreted (37).

Applications of the Mohr diagram to three typical sets of test data are exhibited $\mathrm{a}^{\mathrm{n}}$


Figure 5. Pressure Hysteresis Curve-One Cycle

As additional proof that the yield point is exceeded during the triaxial test, two representative sample specimens of asphaltic concrete were tested maintaining a constant lateral pressure of 60 psi , and subjecting the specimens to a series of vertical loads ranging from 16 psi to 2300 psi in increments of 40 to 80 psi , and computing the vertical and horizontal volume deformations from readings on Ames dials. Load deformation curves plotted from the resulting data on log-log paper indicated yield points at approximately $150-200 \mathrm{psi}$ vertical load.

The position and form of the lower branch of the pressure hysteresis curve shown in Figure 5 merely serves to confirm the conclusions already deduced. The direction of

Figures 6, 7, and 8. In Figure 6 the specimen had a low stability ( $16 \%$ ); in Figure 7 it was high ( $61 \%$ ); and in Figure 8 an intermediate value of 39 percent was selected. Two circles were constructed in each graph in accordance with the previously outlined procedure for construction of the Mohr stress circle: the first, No. (1), having a vertical principal stress of 400 psi and the second, No. (2), a lateral principal stress of 200 psi . The envelope is assumed to be a straight line, the common tangent $\overline{H D}$. As is obvious from inspection of the diagrams, $\overline{D C}$ in Circle No. (1) of each diagram represents the value of shear resistance or shear stress in the plane of yield at a major principal stress of 400 psi . The drawings reveal that it is made up of two
parts, one of which ( $\overline{D Q}$ in Fig. 7) is proportional to the normal stress $\sigma$ acting on the plane of failure; the other component $\overline{Q C}$ or $\overline{O K}$, being a composite resistance (caused by interlocking of mineral particles, viscosity, adhesion, surface tension, true cohesion, etc., ) which is independent of the normal stress. The diameters of the smaller circles, No. (3)
tangent to the line of rupture to the left and right of this point, respectively, may also be included in the diagram when used for soil analysis. In this connection attention is called to a somewhat different treatment of Mohr diagram analysis (38) in which the analytical geometry terminology of poles and polars is employed.


Figure 6. Mohr Circle Diagram
and (4), drawn through the origin and tangent to the line of yield $\overline{H D}$ on either side of the origin are the values of compressive and tensile strength, respectively, since they are the values of axial stresses required to produce failure at a lateral pressure of zero. The intercept of the line of yield on the horizontal axis is the intrinsic pressure.

Two other special circles of rupture, which may be called Rankine circles of active and passive earth pressure, passing through the terminal of the static earth pressure and

Representation and Measurement of Hveem Stability by Means of Mohr Diagram-Hveem's stability formula (39), which was apparently used to plot the flow curves in the contour chart (Fig. 9), used in calculating relative stability from data taken with the Hveem stabilometer, is

$$
S=\frac{22.2}{\frac{R D}{400-R}+0.222}
$$

in which $S=$ relative stability (\%)
$R=$ pressure gauge reading (horizontal stress) at an applied load of 400 psi
$D=$ turns displacement ( 1 turn $=$ $0.1 \mathrm{in} .=0.2 \mathrm{cu} . \mathrm{in}$.)
By deduction it is apparent that $D$ is a function of lateral deformation of the specimen during the test; and application of dimension formulae shows that the constant
plication of both numerator and denominator of the right hand member by the factor $\frac{400-R}{0.222}$ and division of both members of the equation by 100 , into the form

$$
S=\frac{400-R}{400-R\left(1-\frac{D}{0.222}\right)}
$$

in which $S$ is the relative fractional stability.


Figure 7. Mohr Circle Diagram
0.222 has the dimensions of a displacement, which, from its size and position in the equation, appears to be the displacement corresponding to some highly rigid condition (hypothetical) of the specimen selected as a basis for the calculation of percentage. More specifically, it is the value which will give a stability of 100 percent when $R=0$ in H veem's stability formula.

This formula may be transformed by multi-

This form of the expression permits greater facility in calculation, especially when used in conjunction with Table 1 in which the function, $1-\frac{D}{0.222}$ is given for values of displacement $D$, ranging from 0.60 -turn to 2.59-turns displacement. Incidentally, this form also lends itself to representation in the Mohr diagram. It will be noted that the


Figure 8. Mohr Circle Diagram


Figure 9. Relative Stability
numerator in the formula is represented by the diameter of the Mohr circle of rupture passing through a vertical principal stress $\sigma_{1}$ of 400 psi and the corresponding lateral principal stress $R$, ( $\sigma_{3}$ ) (see Circle No. (1) of Fig. 6, 7, and 8 ); and that the denominator may be represented by the diameter of another circle (No. (5) in Fig. 6, 7, and 8) passing through the same vertical principal stress of 400 psi and a lateral principal stress given by the term $R\left(1-\frac{D}{0.222}\right)$, which is readily computed by the use of Table 1. The ratio of the diameter of Circle No. (1) to that of
gram with its base line along the axis of abscissae and its center at $O^{\prime}$ (center of the circle whose diameter is $400-R$ ), and the angular position on the protractor of the point whose rectangular co-ordinates are the radii $O^{\prime} F$ and $O^{\prime \prime} B^{\prime \prime}$ with respect to origin $O^{\prime}$ is noted. Since the protractor scale has been calibrated to read the ratio of these co-ordinates corrected for specimen height, the corrected value of the required stability in percent is thus obtained directly from the angular scale. If the point located on the rectangular system by the radii of the circles falls off the protractor, an arm pivoted at the

## TABLE 1

GIVING $\left(1-\frac{D}{0.222}\right)^{2}$ AS A FUNCTION OF $D^{b}$

| D | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.6 | 1.70 | 1.75 | 1.79 | 1.84 | 1.88 | 1.93 | 1.97 | 2.02 | 2.06 | 2.11 |
| 0.7 | 2.15 | 2.20 | 2.24 | 2.29 | 2.33 | 2.38 | 2.42 | 2.47 | 2.51 | 2.56 |
| 0.8 | 2.60 | 2.65 | 2.69 | 2.74 | 2.78 | 2.83 | 2.87 | 2.92 | 2.96 | 3.01 |
| 0.9 | 3.05 | 3.10 | 3.14 | 3.19 | 3.23 | 3.28 | 3.32 | 3.37 | 3.41 | 3.46 |
| 1.0 | 3.50 | 3.55 | 3.59 | 3.64 | 3.68 | 3.73 | 3.77 | 3.82 | 3.86 | 3.91 |
| 1.1 | 3.95 | 4.00 | 4.04 | 4.09 | 4.14 | 4.18 | 4.23 | 4.27 | 4.32 | 4.36 |
| 1.2 | 4.41 | 4.45 | 4.50 | 4.54 | 4.59 | 4.63 | 4.68 | 4.72 | 4.77 | 4.81 |
| 1.8 | 4.86 | 4.90 | 4.95 | 4.99 | 5.04 | 5.08 | 5.13 | 5.17 | 5.22 | 5.26 |
| 1.4 | 5.31 | 5.35 | 5.40 | 5.44 | 8.49 | 5.53 | 5.58 | 5.62 | 5.67 | 5.71 |
| 1.5 | 5.76 | 5.80 | 5.85 | 5.89 | 5.94 | 5.98 | 6.03 | 6.07 | 6.11 | 6.16 |
| 1.6 | 6.21 | 6.25 | 6.30 | 6.34 | 6.39 | 6.43 | 6.48 | 6.52 | 6.57 | 6.61 |
| 1.7 | 6.60 | 6.70 | 6.75 | 6.79 | 6.84 | 6.88 | 6.93 | 6.97 | 7.02 | 7.06 |
| 1.8 | 7.11 | 7.15 | 7.20 | 7.24 | 7.29 | 7.33 | 7.38 | 7.42 | 7.47 | 7.51 |
| 1.9 | 7.56 | 7.60 | 7.65 | 7.69 | 7.74 | 7.78 | 7.83 | 7.87 | 7.92 | 7.96 |
| 2.0 | 8.01 | 8.05 | 8.10 | 8.14 | 8.19 | 8.23 | 8.28 | 8.32 | 8.37 | 8.41 |
| 2.1 | 8.46 | 8.50 | 8.55 | 8.59 | 8.64 | 8.68 | 8.73 | 8.77 | 8.82 | 8.86 |
| 2.2 | 8.91 | 8.95 | 9.00 | 9.05 | 9.09 | 9.14 | 9.18 | 9.23 | 9.27 | 9.32 |
| 2.3 | 9.36 | 9.41 | 9.45 | 9.50 | 9.54 | 8.69 | 9.63 | 9.68 | ${ }^{9.72}$ | 9.75 |
| 2.4 | 9.81 | 9.86 | 9.90 | 9.95 | 9.99 | 10.04 | 10.08 | 10.13 | 10.17 | 10.22 |
| 2.5 | 10.26 | 10.31 | 10.35 | 10.40 | 10.44 | 10.49 | 10.53 | 10.58 | 10.62 | 10.67 |

[^1]Circle No. (5), then, is the fractional stability which, when multiplied by 100 , is the relative percent stability sought.

- The computation of this ratio may be done by any one of three methods. In the first, a statistician's ruler (logarithmic scale) may be applied to a logarithmic scale printed on the margin of graph paper (Fig. 8) in the manner of the slide rule process of division. In another method which is more rapid and convenient, the lower cycle on the fixed scale of a slide rule was calibrated and marked so as to read values of stability corrected for specimen height directly, thus obviating the necessity of this extra step in the usual process of calculation from charts. By a third method, a special protractor calibrated to read the corrected stability is applied to the dia-
center of the protractor may be used to locate its angular position. The necessary and sufficient condition for use of this method is that the two rectangular scales be the same.


## Derivations-

1. Stress Components for the General Case of Combined Stress.

Expressions for the normal and tangential components of stress on an arbitrary plane will first be derived directly by the analytical method, and then shown to be equivalent to expressions derived by the geometrical method from a Mohr diagram. Referring to Figure 10, it will be seen that the assumed conditions for equilibrium when both tangential and normal stress act on planes $A C F D$ and $C B E F$ yield the following summations for the total
normal and tangential components of stresses on plane $A B E D$ :
$\sigma d s d z=\sigma_{1} \cos \alpha d x d z+\sigma_{3} \sin \alpha d y d z+$
$\tau_{y x} \cos \alpha d y d z+\tau_{x y} \sin \alpha d x d z \ldots \ldots$ (1a) and

$$
\tau d s d z=\sigma_{1} \sin \alpha d x d z-\sigma_{3} \cos \alpha d y d z-
$$

$$
\begin{equation*}
\tau_{x y} \cos \alpha d x d z+\tau_{y x} \sin \alpha d y d z \tag{2a}
\end{equation*}
$$

$\qquad$


Figure 10
Dividing each equation by the area $d s d z$, and collecting terms (after setting $\left|\tau_{x y}\right|=\left|\tau_{u x}\right|$ ), the unit values of normal and tangential stress components are obtained, viz.,
$\sigma=\sigma_{1} \cos ^{2} \alpha+\sigma_{3} \sin ^{2} \alpha+2 \tau_{x y} \sin \alpha \cos \alpha$,
and
$\tau=\sigma_{1} \sin \alpha \cos \alpha-\sigma_{3} \sin \alpha \cos \alpha+$

$$
\tau_{x y}\left(\sin ^{2} \alpha-\cos ^{2} \alpha\right.
$$

But since $\sin \alpha \cos \alpha=\frac{\sin 2 \alpha}{2}$ and $\sin ^{2} \alpha-$ $\cos ^{2} \alpha=-\cos 2 \alpha$ and also $\sin ^{2} \alpha=\frac{1-\cos 2 \alpha}{2}$ and $\cos ^{2} \alpha=\frac{1+\cos 2 \alpha}{2}$ these expressions may be written (using general subscripts $x$ and $y$ ))

$$
\sigma=\frac{\sigma_{y}+\sigma_{x}}{2}+\frac{\sigma_{y}-\sigma_{x}}{2} \cos 2 \alpha+\tau_{x y} \sin 2 \alpha
$$

and

$$
\tau=\frac{\sigma_{y}-\sigma_{x}}{2} \sin 2 \alpha-\tau_{x y} \cos 2 \alpha
$$

It will now be shown from the geometry of Figure 3, that $\sigma$ and $\tau$ have the above values.

In the right triangle $O^{\prime} P^{\prime} C^{\prime}$ it is seen from the definition of the sine and cosine functions of an angle, that

$$
\begin{aligned}
& \tau=\overline{C^{\prime} P^{\prime}}=\overline{O^{\prime} P^{\prime}} \sin C^{\prime} O^{\prime} P^{\prime} \\
&=\sqrt{\left(\frac{\sigma_{y}-\sigma_{x}}{2}\right)^{2}+\tau_{x y}^{2}} \\
& \sin \left[2 \alpha-\sin ^{-1} \frac{\tau_{x y}}{\left.\sqrt{\left(\frac{\sigma_{y}-\sigma_{x}}{2}\right)^{2}+\tau_{x y}^{2}}\right]}\right.
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma= & \overline{O C^{\prime}}=\overline{O O^{\prime}}+\overline{O^{\prime} C^{\prime}} \\
= & \frac{\sigma_{y}+\sigma_{x}}{2}+\sqrt{\left(\frac{\sigma_{y}-\sigma_{x}}{2}\right)^{2}+\tau_{x y}^{2}} \\
& \cos \left[2 \alpha-\cos ^{-1} \frac{C O^{\prime}}{\sqrt{\left(\frac{\sigma_{y}-\sigma_{x}}{2}\right)^{2}+\tau_{x y}^{2}}}\right]
\end{aligned}
$$

After expansion by means of the trigono metrical difference formulae, $\sin (A-B)=$ $\sin A \cos B-\cos A \sin B$ and $\cos (A-B)=$ $\cos A \cos B+\sin A \sin B$, and algebraic simplification these expressions reduce to

$$
\begin{aligned}
& \sigma=\frac{\sigma_{y}+\sigma_{x}}{2}+\frac{\sigma_{y}-\sigma_{x}}{2} \cos 2 \alpha+\tau_{x y} \sin 2 \alpha \\
& \tau=\frac{\sigma_{y}-\sigma_{x}}{2} \sin 2 \alpha-\tau_{2 y} \cos 2 \alpha
\end{aligned}
$$

which are identical with those derived above.
2. Derivation of Equation, $\sigma_{1}=\sigma_{3} \tan ^{2} \alpha$ From the definition of the tangent function, it is seen that the tangent of angle $\alpha$ in the right triangle $A P C$ of Figure 3 is

$$
\begin{aligned}
\tan \alpha & =\frac{P C}{A C}=\frac{P O^{\prime} \sin 2 \alpha}{A O^{\prime}-C O^{\prime}}=\frac{A O^{\prime} \sin 2 \alpha}{A O^{\prime}+A O^{\prime} \cos 2 \alpha} \\
& =\frac{\sin 2 \alpha}{1+\cos 2 \alpha}
\end{aligned}
$$

(canceling the common factor $A O^{\prime}$ )

$$
=\frac{M P / M O^{\prime}}{1-P O^{\prime} / M O^{\prime}}=\frac{M P}{M O^{\prime}-P O^{\prime}}
$$

(from right triangle MPO')
$=\sqrt{\frac{\left(\sigma_{1}+K \operatorname{ctn} \phi\right)\left(\sigma_{3}+K \operatorname{ctn} \phi\right)}{K \operatorname{ctn} \phi+\frac{\sigma_{1}+\sigma_{3}}{2}-\frac{\sigma_{1}-\sigma_{3}}{2}}}$
(using ${ }^{\text {"previously }}$ derived expressions for MP in Equations (8), (9) and (10))

$$
=\sqrt{\frac{\sigma_{1}+K \operatorname{ctn} \phi}{\sigma_{3}+K \operatorname{ctn} \phi}} \text { (by simplification) }
$$

Then

$$
\tan ^{2} \alpha=\frac{\sigma_{1}-K \tan 2 \alpha}{\sigma_{3}-K \tan 2 \alpha}
$$

(substituting $\operatorname{ctn} \phi=\tan (90-\phi)=-\tan 2 \alpha$ )

$$
\begin{aligned}
& =\frac{\sigma_{1}-K \frac{2 \tan \alpha}{1-\tan ^{2} \alpha}}{\sigma_{3}-K \frac{2 \tan \alpha}{1-\tan ^{2} \alpha}} \\
& \quad\left(\operatorname{since} \tan 2 \alpha=\frac{2 \tan \alpha}{1-\tan ^{2} \alpha}\right) \\
& =\frac{\left[\sigma_{1}\left(1-\tan ^{2} \alpha\right)-2 K \tan \alpha\right.}{\left[\sigma_{3}\left(1-\tan ^{2} \alpha\right)-2 K \tan \alpha\right.}
\end{aligned}
$$

(by multiplying numerator and denomerator by ( $1-\tan ^{2} \alpha$ ))
Then! $\sigma_{1}=\sigma_{3} \tan ^{2} \alpha+2 R \tan \alpha$ (solving for $\sigma_{1}$ and simplifying) or $\sigma_{1}=\sigma_{3} \tan ^{2} \alpha$ (when $K=0$ ).
3. Derivation of Equation (7)

- From right triangle $C P O^{\prime}$ of Figure 3,

$$
P O^{\prime}=\sqrt{\left(\frac{\sigma_{y}-\sigma_{x}}{2}\right)^{2}+\tau_{x y}^{2}}
$$

and, again, from right triangle $M P O^{\prime}$, and the definition of the sine function, one has

$$
\begin{aligned}
\sin \phi=\frac{P O^{\prime}}{M O^{\prime}} & =\frac{\sqrt{\left(\frac{\sigma_{y}-\sigma_{z}}{2}\right)^{2}+\tau^{2} x y}}{M O+O 0^{\prime}} \\
& =\frac{\sqrt{\left(\frac{\sigma_{y}-\sigma_{x}}{2}\right)^{2}+\tau_{x y}^{2}}}{K \operatorname{ctn} \phi+\frac{\sigma_{y}+\sigma_{x}}{2}}
\end{aligned}
$$

or,
$K \sin \phi \operatorname{ctn} \phi+\frac{\sigma_{y}+\sigma_{x}}{2} \sin \phi$
$=\sqrt{\left(\frac{\sigma_{y}-\sigma_{x}}{2}\right)^{2}+\tau_{x y}^{2}}$
which, by re-arrangement and reduction, becomes

$$
\begin{aligned}
& \sqrt{\left(\frac{\sigma_{y}-\sigma_{x}}{2}\right)^{2}+\tau_{x y}^{2}}- \\
& \frac{\sigma_{y}+\sigma_{x}}{2} \sin \phi-K \cos \phi=0
\end{aligned}
$$

Part 2

## the stress triangle diagram

As a preliminary to the statistical analysis of the properties of asphaltic concrete it was necessary to construct several hundred graphs from data taken in the triaxial compression test, showing the line of rupture, angles of shear and of internal friction, and other parametric constants of the Coulomb equation. Due to the considerable time required for the construction of these graphs by the Mohr diagram, a new method was developed in order to facilitate the construction and measurement of dimensions in the diagram.

Several graphical solutions for the analysis of stress relations at failure of a material under combined stress are in common use, but the method most widely used at present for general purposes is the graphical construction known as the Mohr Stress Circle diagram, developed by O. C. Mohr about 1868 (40). ${ }^{3}$ The superiority of the Mohr diagram over other methods lies in its simplicity of construction and in the relatively great amount of information it reveals concerning physical properties of the material under investigation. Still greater brevity and simplicity are possible of attainment, however, by means of a different, less complicated, construction in which the same results are secured with considerably less expenditure of time and effort. It is the purpose of Part 2 of this report to outline the method and apply it to

[^2]the analysis of a few typical sets of data taken in the triaxial compression test.
Procedure-The method consists in the construction of two right triangles from two simultaneous pairs of critical values of principal stress, ${ }^{4}$ having the major stress of one pair equal in magnitude to the minor stress of the other pair. All the magnitudes involved in analysis of the stress conditions at failure of the material under test which are shown in the Mohr diagram are then represented by the sides and angles of these two triangles. The construction may be described briefly in three steps as follows:

1. Two pairs of simultaneous principal stresses acting on a body at failure are selected (from the triaxial compression test, for instance) having the major stress of one pair

Compressive and tensile strengths, although not shown directly in the ordinary Mohr diagram, may be represented on the Triangle diagram by the simple operation of drawing a third right triangle with its right angle vertex at the intercept $K$ on the axis of shear stress and with its sides parallel to those of right triangle $\sigma_{3} P \sigma_{1}$.

In the actual construction of the Triangle diagram, as performed in this laboratory, the points $\sigma_{1}$ and $\sigma_{3}$ (Fig. 11) are located by pins on millimeter cross section paper. A 90 deg triangle is then inserted between the pins and a third pin passing through a hole at the vertex of the right angle of the triangle is traced upward along the perpendicular passing through the point $\sigma$ representing the principal stress common to the two pairs given until the


Figure 11. Stress Triangle Diagram
equal in magnitude to the minor stress of the other pair; and, from a point $O$ in a straight line (Fig. 11) two distances, $\sigma_{3}$ and $\sigma_{1}$, proportional to the arithmetic means of the two stresses comprising each pair are laid off to any convenient scale (41).
2. On the line $\overline{\sigma_{3} \sigma_{1}}$ connecting their terminals as a hypotenuse, a right triangle $\sigma_{3} P \sigma_{1}$ is constructed with its right angle vertex at the point $P$ on the perpendicular erected at the point $\sigma$ which represents the given common stress.
3. A second right triangle $I P A$ is next constructed having its right angle vertex coinciding with that of the first triangle, with its hypotenuse on the same base line and an acute angle at the mid point of the hypotenuse of the first triangle drawn.
${ }_{4}$ Symbols and definitions are given in Appendix A. In most cases the symbols selected are in accordance with ASTM Designation D-653-42T (25).
legs of the triangle touch the pins. The movable pin is inserted at the point $P$ of the graph, and lines are traced along the legs $P \sigma_{3}$ and $P \sigma_{1}$ of the triangle. The pin at $\sigma_{3}$ is then removed and inserted at the mid point $A$ of the hypotenuse of the triangle $\sigma_{3} P \sigma_{1}$ just drawn. With the right angle vertex of the 90 deg triangle still pinned at $P$, it is rotated about $P$ as a pivot until it lies to the left of and having its leg touching the pin at $A$. While in this position the legs are traced as before giving the right triangle IPA. Since the leg $P \sigma_{1}$ of the triangle $\sigma_{3} P \sigma_{1}$ is not required in the solution, it may be omitted from the construction.

As a further convenience in measuring the angles $\phi$ and $\alpha$ a small protractor, permanently attached to the triangle at its right angle vertex, is used to read the angles shown on the diagrams at $P$, the common vertex of triangles $\sigma_{3} P \sigma_{1}$ and $I P A$, while it remains in the final position used in the construction.

Construction of the triangle $\frac{T}{2} K \frac{C}{2}$ is most easily effected by inserting pins at $K, \sigma_{3}$, and $P$ of Figure 11, and placing the triangular ruler so that its legs just touch these pins, tracing the leg $K \frac{C}{2}$ of the required triangle and then with the right angle of the ruler at $K$ tracing its other leg $K \frac{T}{2}$. Segments $\frac{C}{2}$ and $\frac{T}{2}$ into which the hypotenuse of this triangle is divided by the altitude $K O$ are half the values of compressive and of tensile strengths, respectively, since, as a result of the con-
is the locus of all points, such as $P$, whose coordinates $\sigma$ and $\tau$ are values of the normal and tangential (or shear) components of stress acting on the plane of failure for the given critical values of principal stress. These variable coordinates $\tau$ and $\sigma$ are given in the diagram (Fig. 12) by the common altitude $P_{\sigma}$ of the two right triangles, and by the left hand one of the two segments $O_{\sigma}$ into which the common altitude divides the hypotenuse $O A$ of triangle $O P A$ (when $K=$ $0)$. In the general case, when $K \neq 0$, the normal component $\sigma$ is merely the distance from the origin to the foot of the common altitude $P_{\sigma}$ (Fig. 11).


Figure 12. Stress Triangle Diagram
struction, these distances represent the single forces of compression and of tension (i.e., the other principal force being zero) required to produce failure.

## Interpretation of the Stress Triangle Diagram-

 The acute angles $\alpha$ and $\beta$ of right triangle $\sigma_{3} P_{\sigma_{1}}$ are the minimum angles of inclination of the planes of failure to the major and minor principal planes, respectively, and the acute angles $\phi$ and $\gamma$ of right triangle IPA are the angles of internal friction of the material and the minimum angle betreen the two shear planes, respectively. Also, the leg $1 P$ of triangle IPA (produced through $P$ ) is the line of failure, and its intercepts on the axes of shear stress and of normal stress, respectively, are measures of the cohesion $K$ and intrinsic pressure, $I$, of the material under test.The line of failure so constructed, whose equation is the well known Coulomb relation

$$
\begin{equation*}
\tau=\sigma \tan \phi+K \ldots \tag{17}
\end{equation*}
$$

Proof of Correctness of the Triangle Construc-tion-The correctness of this construction will be demonstrated by showing that identical relations between the variables and parametric constants involved may be deduced from the geometry of the diagram, using properties of the triangle, and by analytical processes. For purposes of mathematical simplification a material having no cohesion will be assumed (Fig. 12).

1. Geometrical derivation: From right triangle $P_{\sigma} A$ in Figure 12 and the definitions of the trigonometrical sine and cosine functions, it is seen that

$$
P_{\sigma}=\tau=A P \sin \gamma
$$

and

$$
\begin{aligned}
O \sigma & =\sigma=O A-\sigma A \\
& =\left(\frac{\sigma_{1}+\sigma_{3}}{2}\right)-A P \cos \gamma
\end{aligned}
$$

but

$$
A P=\left(\frac{\sigma_{1}-\sigma_{3}}{2}\right) \text { (The mid-point }
$$

of the hypotenuse of a right triangle is equidistant from its vertices.)
$\sin \gamma=\sin 2 \alpha$ (The sine of an angle equals the sine of its supplement.)
and $\cos \gamma=-\cos 2 \alpha$ (The cosine of an angle is equal in magnitude but opposite in sign to the cosine of its supplement.)
With these substitutions the above equations become
and

$$
\begin{align*}
\tau= & \frac{\left(\sigma_{1}-\sigma_{3}\right)}{2} \sin 2 \alpha \ldots  \tag{18}\\
\sigma= & \frac{\left(\sigma_{1}+\sigma_{3}\right)}{2}+ \\
& \frac{\left(\sigma_{1}-\sigma_{3}\right)}{2} \cos 2 \alpha \tag{19}
\end{align*}
$$

2. Analytical derivation: In the diagram (Fig. 1) which represents an elementary prism bounded by the principal planes ACFD and $F C B E$, an oblique plane of failure $A B E D$, and two parallel planes $D E F$ and $A B C$, in a body under stress from combined axial and radial unit compressive critical stresses $\sigma_{1}$ and $\sigma_{3}$, respectively, it is evident, from the assumed condition of static equilibrium, that the total normal and tangential components of the reactive stress on the oblique plane $A B E D$ are given by the equations
$\sigma d s d z=\sigma_{1} d x d z \cos \alpha+\sigma_{3} d y d z \sin \alpha$
and
$\tau d s d z=\sigma_{1} d x d z \sin \alpha-\sigma_{3} d y d z \cos \alpha$
Dividing both sides of equations (20) and (21) by the area $d s d z$ and substituting $\cos \alpha$ for $\frac{d x}{d s}$ and $\sin \alpha$ for $\frac{d y}{d s}$, they reduce to

$$
\begin{align*}
\sigma & =\sigma_{1} \cos ^{2} \alpha+\sigma_{3} \sin ^{2} \alpha \\
& =\frac{\left(\sigma_{1}+\sigma_{3}\right)}{2}+\frac{\left(\sigma_{1}-\sigma_{d}\right)}{2} \cos 2 \alpha \ldots \ldots \tag{22}
\end{align*}
$$

and

$$
\begin{align*}
\tau & =\sigma_{1} \cos \alpha \sin \alpha-\sigma_{3} \cos \alpha \sin \alpha \\
& =\frac{\left(\sigma_{1}-\sigma_{3}\right)}{2} \sin 2 \alpha \ldots \ldots \ldots \ldots \tag{23}
\end{align*}
$$

Equations (22) and (23), above, are identical with equations (18) and (19) which were derived from the geometry of the Triangle
diagram. Thus the correctness of the stress triangle construction is established.

As a direct consequence of the geometrical theorem which states that the midpoint of the hypotenuse of a right triangle is equidistant from its vertices, it follows that angles $\alpha$ and $\beta$ are also measured by the angles $\sigma_{3} P A$ and $\sigma_{1} P A$ as indicated at the right angle vertex of triangle $\sigma_{3} P \sigma_{1}$ (Fig. 11). Also, since two angles are equal if their sides are perpendicular, each to each, the angle $\sigma P A$ is equal to the angle of internal friction $\phi$. This relocation of the angles of shear and of internal friction will facilitate their measurement, since by placing the center of a protractor at $P$, with its base line along $P A$, both $\alpha$ and $\phi$ (the angles usually used in a stress analysis) may be read directly from the scale with a single application of the protractor.

The linear dimensions $\sigma, \tau, \sigma_{1}, \sigma_{3}, K, I, C$ and $T$ are read directly from the graph, and the resultant $I P$ of the normal and tangential components of stress on the plane of failure may be measured, if desired for any purpose, with a millimeter scale on one leg of the triangular ruler.
The point on the line of failure corresponding to any other value of critical stress is readily located by finding its intersection with a straight line drawn through the point on the normal stress axis representing the given stress, at an angle $\alpha$ to the right hand direction, if the given stress is a minor principal stress or at an angle ( $180-\beta$ ) with this direction for a major principal stress. Also, if both major and minor principal critical stresses are given, the two points representing them, together with the corresponding point on the line of failure, form a right triangle whose sides are parallel to those of triangle $\sigma_{3} P \sigma_{1}$. Proof of this last statement derives from the implied initial assumption of a constant angle of shear characteristic of a given material whose properties are such that the stress relations are in conformity with Coulomb's equation.

Comparison of the Stress Triangle Construction uith the Mohr Stress Circle Diagram-It will be observed that no reference has been made in the method here outlined to the Mohr diagram, except incidentally by way of introduction; and that no use whatever has been made of the Mohr stress circle, or circles
of any kind, or of any of the ideas or nomenclature of the Mohr diagram or other construction. Moreover, properties of the triangle rather than properties of the circle are employed in the derivation of mathematical relations. It is essential, however, that the various methods should be in complete agreement as to the final results achieved. For this reason the Triangle diagram and the Mohr diagram (selected because it is a general method, representative of the, best contemporary methods) will be compared in some detail, in order to point out by contrast certain advantages of the Triangle construction:

1. In the first place, equations (22) and (23), which have been derived both by analytical methods, and, geometrically, from the Triangle diagram, are also derivable from the Mohr diagram (40) and (41).
2. In the second place, it may be shown either by measurement of dimensions in the two diagrams, or by a combination of the two diagrams on a single graph, that line $I P N$ in Figure 10 is a common tangent to two Mohr stress circles constructed from the original pairs of principal stresses whose arithmetic means are $\sigma_{1}$ and $\sigma_{3}$. Hence the line IPN, which represents the line of failure, is identical in the two diagrams and its intercepts on the stress axes and its angle of inclination $\phi$ to the axis of normal stress have identical meanings in the two diagrams.
3. On the other hand, while the two methods yield identical mathematical results in all cases, some of the standard equations of plastic equilibrium are more easily derived from the Triangle diagram, using properties of the triangle, than from the Mohr diagram by utilizing properties of the circle; and some other useful relations, apparently unknown or unused, as far as can be ascertained, are more readily obtained from the Triangle diagram for example.
(a) As an instance of the latter class of relations, it may be seen by inspection of right triangle $\sigma_{3} P \sigma_{1}$ (Fig. 11) that, as a result of the geometrical theorem which states that the altitude of a right triangle on its hypotenuse is the mean proportional between the segments into which the hypotenuse is divided, it follows that

$$
P_{\sigma}=\tau=\sqrt{\left(\sigma_{3} \sigma\right)\left(\sigma_{1} \sigma\right)}=\sqrt{\left(\sigma-\sigma_{3}\right)\left(\sigma_{1}-\sigma\right)}
$$

or, in other words, the shear stress corresponding to a normal stress common to two simultaneous pairs of critical values of principal stresses is equal to the geometric mean of the half differences between the major and minor stresses of each pair, i.e. of the radii of the two Mohr circles of failure constructed from the given stresses.
(b) From this corollary it also follows (See Triangle $\frac{T}{2} K \frac{C}{2}$ in Fig. 11) that cohesion is equal to one-half the geometric mean of the compressive and tensile strengths, or $K=$ $\sqrt{\frac{C}{2} \cdot \frac{T}{2}}=\frac{1}{2} \sqrt{C T}$ where $C$ and $T$ represent compressive and tensile strengths, respectively.
(c) Similar relations may be derived for Rankine's active and passive earth pressures occurring in Rankine's earth pressure theory (43), since these pressures are measured from a common point corresponding to the static pressure of a body of earth at rest. Stated as a mathematical equation, the relation is

$$
T_{R}=\frac{1}{2} \sqrt{D_{A} D_{P}}
$$

Where $D_{A}=$ The diameter of the circle of rupture representing the active Rankine state of earth pressure
$D_{P}=$ The diameter of the circle of rupture representing the passive Rankine state of earth pressure
And $\quad T_{k}=$ Shear resistance existing on a failure plane passing through a point of the body of earth when it is at rest
4. Another important advantage of the triangle construction over the Mohr diagram is its greater ease and speed of construction. This statement is best proved by an actual trial, but it is more or less apparent from the description. One example of the saving effected in the Triangle method is that it avoids the relatively complicated geometrical problem of constructing a common tangent to two circles. Accurate construction of the common tangent involves drawing an auxiliary circle with a radius equal to the difference between the radii of the given circles, drawing a tangent to this differential circle from a point, construction of perpendiculars, etc. (12). This procedure consumes considerable
time and further complicates the Mohr diagram.
5. This leads to another contrast between the two diagrams, viz., the difference in simplicity of form. The Triangle diagram contains only four straight lines, whereas the Mohr diagram requires the same number of lines in order to reveal the same amount of information, in addition to two Mohr stress circles plus other auxiliary circles and lines required in construction of the line of rupture.
6. Again, as performed in this laboratory, the derived dimensions are all read directly from the graph paper and from the one instrument used in the construction (a combination 90 deg triangle, protractor, and linear scale) with a single application. Besides an additional saving of time, the Triangle construction thus uses fewer tools than does the Mohr construction.

Adaptations of Stress Triangle Construction to Some Cases not Included in the Preliminary Assumptions-Two initial restrictions were placed on the materials and data involved in the Triangle diagram construction. They are as follows:

1. The physical properties of the material at failure are related in accordance with Coulomb's formula.
2. The available data consists of two simultaneous pairs of critical values of principal stress having one stress in common.

The first of these assumptions applies with equal weight to the Mohr diagram; for it is only on this hypothesis that a common tangent can be constructed, whose intercepts and angles of inclination with respect to the axes correctly represent physical properties of the material. In many cases both the physical properties and the dimensions representing them are variables during a test; and, if the properties are to be even approximately determined from the Mohr diagram, for some arbitrary combination of critical stresses, then it becomes necessary to construct and use a straight line common tangent. This approximation may also be effected with greater ease and speed by the Triangle diagram construction.

As to the second hypothesis it is just as easy, experimentally, to measure two pairs of principal stresses having a stress in common as without a common stress. But, if the
latter course has been followed in securing the data, one pair of stresses may be transformed to a pair having one of its stresses equal to one of the other pair by the simple process of linear interpolation from the table of data or a curve plotted from it. Of course, this procedure assumes a linear relation between the principal stresses over the range of the tabular differences used in the interpolation. That this assumption is approximately justified in practical cases is shown by the fact that curves plotted from series of pressures in several triaxial tests on asphaltic concrete specimens containing calcareous aggregates were found to be approximately straight, especially over the limited ranges used. However, it is usually possible to select two or more pairs of stresses from the series of readings ordinarily recorded in the triaxial experiment which fulfill the conditions of the construction.

It may be noted here that the results of triaxial tests on dry sands and asphaltic concrete made from flinty aggregates conform almost exactly to the Coulomb relation, while asphaltic concrete specimens made with calcareous aggregates usually have slightly curved envelopes or lines of failure of varying curvature and slope depending on the characteristics of the design used.

Although not of great practical importance, the construction of the individual locus (a circle) of points whose coordinates are values of normal and tangential components of stress acting on the general plane through an axis of a body can be accomplished even better by means of the Triangle construction than with the Mohr stress circle. With the legs of the triangular ruler held against the pins marking the given pair of major and minor stresses, a pencil inserted in the hole at the right angle vertex of the triangular ruler is made to trace the locus as the vertex is moved from one pin to the other.

While critical values of simultaneous principle stresses have been assumed for the sake of simplifying the discussion, the same construction applies to simultaneous principal stresses other than critical values, but the resulting envelope or common tangent will not be the line of failure, and its intercepts and angle of inclination to the axis of normal stress will not give the cohesion and intrinsic pressure of the material and its angle of internal friction. Besides possessing greater speed
and accuracy than the Mohr method, this method has the added advantage of dispensing with the calculation and location of a center for the arc. Also, by a repetition of this process, the succession of arcs may be drawn from a set of triaxial test data, corresponding to a series of Mohr stress circles, incident to the construction of the non-linear envelope of rupture for materials not conforming to Coulomb's relation.
Summary and Conclusions-An independent method of graphical stress analysis, equivalent in results obtained to the Mohr circle diagram and other methods in common use, but possessing greater ease and speed of construction and interpretation, as well as greater simplicity of form than the Mohr diagram, is developed and illustrated.

Comparison of this method with the Mohr circle construction also shows that, in addition to the economy of time effected, it possesses more simplicity of form, requires less tools for construction and measurement, and permits greater ease in derivation of certain fundamental relations in the theory of plastic equilibrium.

While the Stress Triangle diagram, as well as the Mohr Circle diagram, applies primarily and exactly to materials whose properties are related by Coulomb's formula, it is applicable approximately (and to the same extent as in the Mohr construction) to other road materials whose properties show a departure from the Coulomb relation. It is shown, however, that the construction and measurement of dimensions, even in this case, is expedited by the use of the Triangle construction, and that the resulting diagram is far less complicated than the corresponding Mohr diagram.

In case the available data does not show a common stress for any two simultaneous pairs, one set may be adjusted by interpolation so as to meet this requirement of the construction of the Triangle diagram.

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## APPENDIX A

## DEFINITIONS AND SYMBOLS OF TERMS

Principal stress: a normal unit stress acting on a given plane when the tangential component is zero.
Major principal stress ( $\sigma_{1}$ ): the greatest of three simultaneous mutually perpendicular principal stresses acting on an element of the stressed body.
Minor principal stress ( $\sigma_{3}$ ): the least of three simultaneous mutually perpendicular stresses acting on an element of the stressed body.
Intermediate principal stress $\left(\sigma_{\mathrm{z}}\right)$ : the third of the three simultaneous mutually perpendicular principal stresses acting on an element of the stressed body.

Principal plane: a plane on which only normał stresses act.
Major principal plane: the plane on which the major principal stress acts.
Minor principal plane: the plane on which the minor principal stress acts.
Plane of failure, rupture, or yield: that plane on which failure, or plastic yield occurs.
Critical principal stresses: values of the principal stresses acting at failure.
Critical normal stress ( $\sigma$ ): the normal component of stress acting on the plane of failure.
Critical tangential stress ( $\tau$ ): the tangential component of stress on the plane of failure.
Resultant stress on plane of failure ( $R$ ): the square root of the sum of the squares of the normal and tangential components of critical stress on the plane of failure.
General normal stresses ( $\sigma_{x}, \sigma_{y}$ ): the normal stress components on any two mutually perpendicular planes.
General tangential stresses ( $\tau_{x y}, \tau_{y x}$ ): the tangential stress components on the planes upon which $\sigma_{y}$ and $\sigma_{x}$ act.
Hydrostatic stress: the state of stress existing in a material in the liquid condition, in which all three principal stresses are equal and the shear stress is zero in all directions. Hydrostatic stress may be either compressive or tensile, depending on the direction of the applied force.
Line of failure or yield: the locus of points whose co-ordinates are the critical values of normal and tangential stress components on the plane of failure.

Cohesion (K) ${ }^{\text {a }}$ : that part of the tangential component of stress on the plane of failure which is assumed to be independent of the normal component of stress on this planerepresented graphically by the intercept of the line of failure on the axis of shear stress (Fig. 3 and 11).
Intrinsic pressure ( $I$ ): that internal force which binds the molecules of the material together-represented on the Mohr and stress triangle diagrams by the intercept of the line of failure on the axis of normal stress (Fig. 3 and 11). It is equal in magnitude to the hydrostatic tensile stress required to overcome true cohesion.
Angle of internal friction ( $\phi$ ): the angle between the line of failure and the axis of normal stress, when the line of failure is straight.
Angle of shear ( $\alpha$ ): the smaller of the two angles between the plane of failure and the major principal plane.
Compressive strength ( $C$ ): that single principal stress of compression required to produce failure.
$T$ ensile strength ( $T$ ): that single principal stress of tension required to produce failure.

[^3]
[^0]:    ${ }^{2}$ A list of symbols and definitions of terms is given in Appendix A. In most cases the symbols are in accordance with ASTM designation D-653-42T (25).

[^1]:    ${ }^{\text {a }}$ All values in the table are negative.
    $\mathrm{b}_{\mathrm{D}}=$ turns displacement; one turn of displacement pump handle is equivalent to $0.2-\mathrm{cu}$ in. change in volume of the liquid chamber or to $0.1-\mathrm{in}$. movement of the piston head.

[^2]:    ${ }^{3}$ Italicized figures in parentheses refer to references and footnotes at the end of the paper.

[^3]:    a The term "cohesion," as applied to the constant term in the Coulomb relation, is a misnomer since it is a composite of several factors, such as viscosity, surface tension, adhesion, and shape of component particles of the material, as well as the shear component of true cohesion as usually defined.

