the wall by minor principal stresses becomes oblique instead of horizontal. The farther the wall moves from the backfill—within a certain limited range, admittedly—the smaller is the lateral pressure on the wall since a part of it is absorbed (balanced) by horizontal shears.

CONCLUSIONS

1. The lateral pressure within an earth mass bounded by a vertical slope is balanced by horizontal shearing stresses. The bulk of the horizontal shears is concentrated between the slope and the first eventual crack.

2. In conjunction with horizontal shearing stresses, vertical shears develop; and as a result of their action, a part of the mass next to the slope is overloaded at the expense of the rest of the mass that is partly relieved of its weight.

3. The moment created by the vertical shears causes tension and fissuring at the upper part of the mass.

4. In an accurate analysis of the stress redistribution caused in the earth mass by the presence of a vertical slope, special attention should be paid to the stresses around its foot.

5. The lateral pressure gradually decreases toward a non-supported vertical slope or a translating retaining wall.

6. Extreme care should be recommended in the interpretation of test results on vertical slope or retaining wall models on hard bases.

ACKNOWLEDGEMENTS

The writer extends many thanks to Mr. Paul Baumann, of the Flood Control District, Los Angeles County, California, and to Dr. Henri Marcus of the Bureau of Yards and Docks, U. S. Navy, for their valuable suggestions concerning stability of slopes; and to Mr. George J. Tauxe, on the staff, University of California at Los Angeles, for reading over the draft of this paper.

SHEAR FAILURE IN ANISOTROPIC MATERIALS POSSESSING ANY VALUES OF COHESION AND ANGLE OF INTERNAL FRICTION

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SYNOPSIS

The radius of a Mohr's circle of failure is obtained in terms of the principal shear strengths existing on the principal planes at failure, induced by the stresses applied in plane deformation such as occurs in the triaxial compression test, for a material possessing any values of cohesion and sliding friction. The required radius is first obtained graphically from a modified Mohr stress circle plotted on the axis of shear stress, and analytical expressions are then developed in terms of cohesion, angle of internal friction, and a principal normal stress, for the radius and for the normal and tangential components of stress acting on the plane of failure. A number of special cases are deduced from the general solution and a Mohr circle of failure is constructed. It is shown that the formula developed applies to anisotropic materials possessing either or both components of shear resistance, i.e., cohesion and sliding or internal friction, and to isotropic materials, as a limiting special case.

The problem may be stated briefly as follows:

From given maximum and minimum values of the shear strength existing at failure (e.g., at the proportional limit) on two mutually perpendicular principal planes during the triaxial test (*1*)[1] and a principal normal stress on one of these planes, it is required to find

[1] Italicized figures in parentheses refer to the explanatory footnotes and list of references at the end of the paper.

the radius of the corresponding Mohr circle of failure (*2*) and the stress distribution on the plane of failure, together with the angles of shear and internal friction, and to construct the Mohr circle of failure. Besides the usual implicit assumption that the material is of such nature as to conform approximately with Coulomb's condition of failure (*3*) and Mohr's theory of rupture (*4*), it will also be assumed that the maximum and minimum values of shear strength are in the planes of the normal stresses inducing them, i.e., in the principal planes.[2]

This problem has been solved for the extreme special cases of a purely cohesive substance and of a substance with sliding friction, and zero cohesion, by Casagrande and Carillo (*5*); but, so far as is known, the general solution for a material possessing both cohesion and sliding friction has not been published.

In addition to the complete universality of the formula developed in this paper, it is believed that the method used is superior to that employed in the partial solution by Casagrande and Carillo in that it possesses greater simplicity and directness and does not require the trial and error procedure used by them in the derivations. It also appears that the method of construction of the Mohr Circle on the shear axis from maximum and minimum values of shear strength constitutes a new and useful extension of the Mohr Circle diagram.

MATHEMATICAL THEORY AND DERIVATIONS

On the assumption of the validity of Coulomb's condition of failure, the total shear strength or potential resistance to failure on the plane of failure for a material possessing cohesion and internal friction is defined by the well-known formula of Coulomb

$$\tau = \sigma \tan \phi + K \qquad (1)$$

in which: τ = the shear strength component in the plane of failure,

σ = unit normal stress acting perpendicular to the plane of failure,

ϕ = angle of internal friction

K = all tangential stresses in the plane of failure other than those due to sliding friction. It is generally called cohesion, and is assumed to be independent of the normal stress on the plane of failure.

The same relation with varying values of the parameters is also assumed to hold for all planes through a given point in the material, including the principal planes.

Now the law of distribution of shear strength on planes other than the principal planes will be assumed to be the same as that giving the stress distribution due to principal shear stresses, equivalent to the principal strengths, and represented graphically by a modified Mohr diagram constructed from these stresses (See Appendix 1 and Figure 1-a) in the following manner:

Lay off on the vertical or shear axis from an origin O distances OA and OB proportional to the given minimum and maximum principal shear strengths τ_3 and τ_1, respectively, (Fig. 1-a), and construct a circle passing through their terminals with center at $(\tau_3 + \tau_1)/2$. Draw the tangent OP.

Now it follows from the elementary theory of the Mohr Circle (*2*) and from Figure 1-a, that OC and CP represent, graphically, the contribution to the normal and tangential components of induced strength on the plane of failure of the given principal shear strengths in the principal planes, as resolved by the Mohr diagram method ordinarily used for obtaining the stress distribution on the plane of failure due to the principal normal stresses; and OP is the resultant of these components. But this resultant, or vector sum, represented graphically by the tangent OP, is equal numerically to the radius of the required Mohr Circle of failure, since it is equal to the maximum value of the shear stress on the plane of failure.

The analytical expression for this radius derives directly from the plane geometry theorem which states that the tangent from a point to a circle is the mean proportional between the entire secant through the same point and its external segment, thus,

$$\tau_3/OP = OP/\tau_1 \quad \text{or} \quad OP = \sqrt{\tau_1 \tau_3} \qquad (2)$$

It also follows from right triangle OPC in Figure 1-a and from the definitions of the sine

[2] The same terms and definitions, as well as fundamental assumptions, employed by Casagrande and Carillo (See Reference No. 5) are adopted in the present solution.

and cosine functions, that the normal and tangential components of shear strength induced in the plane of failure are given, respectively, by

$$CP = OP \sin \phi' = OP(O'P/O'O) = \sqrt{\tau_1 \tau_3}(\tau_1 - \tau_3)/(\tau_1 + \tau_3) \quad (3)$$

and

$$OC = OP \cos \phi' = OP(OP/OO') = OP^2/OO' = 2\tau_1\tau_3/(\tau_1 + \tau_3) \quad (4)$$

It may also be shown (*6*) from right triangle ACP and the definition of the tangent function that, after making appropriate trigonometric transformations, the angle of shear α between the plane of failure and the principal plane of maximum shear is defined by the equation

$$\tau_1 = \tau_3 \tan^2 \alpha \quad \text{or} \quad \tan \alpha = \sqrt{\tau_1/\tau_3} \quad (5)$$

It may be noted that, since general shear strength symbols τ_1 and τ_3 were used in the discussions and derivations of relations between dimensions of the tentative stress circle (Fig. 1-a) and the Mohr circle of rupture (Fig. 1-b), these relations will apply to all possible cases, by substitution of particular values for τ_1 and τ_3.

As already shown the tangent to the tentative Mohr circle constructed from the shear strengths represents the radius of the required Mohr circle of failure and is defined analytically by the geometrical mean of the principal shear strengths, thus

$$r = \sqrt{(\sigma_3 \tan \phi_3 + K_3)(\sigma_3 + 2r) \tan \phi_1 + K_1} \quad (6)$$

which, when solved for r and rearranged, gives

$$r = \sigma_3 \tan \phi_1 \tan \phi_3 + K_3 \tan \phi_1 + \sqrt{\begin{array}{l}(\sigma_3 \tan \phi_1 \tan \phi_3 + k_3 \tan \phi_1)^2 \\ + \sigma_3^2 \tan \phi_1 \tan \phi_3 + K_1 \sigma_3 \tan \phi_3 \\ + K_3 \sigma_3 \tan \phi_1 + K_1 K_3\end{array}} \quad (7)$$

As a check on the accuracy of the above relation and in order to illustrate its general utility, attention is called to the fact that substitution of $\phi_1 = \phi_3 = 0$ in equation (7) gives, for the specific case (a), the result that $r = \sqrt{K_1 K_3}$.

Also, if $K_1 = K_3 = 0$, one obtains, on making this substitution in (7), the special case (b), viz.,

$$r \doteq \sigma_3 \tan \phi_1 \tan \phi_3 (1 + \sqrt{1 + \operatorname{ctn} \phi_1 \operatorname{ctn} \phi_3}$$

These expressions are both seen to be identical with those derived by Casagrande and Carillo by a different method.

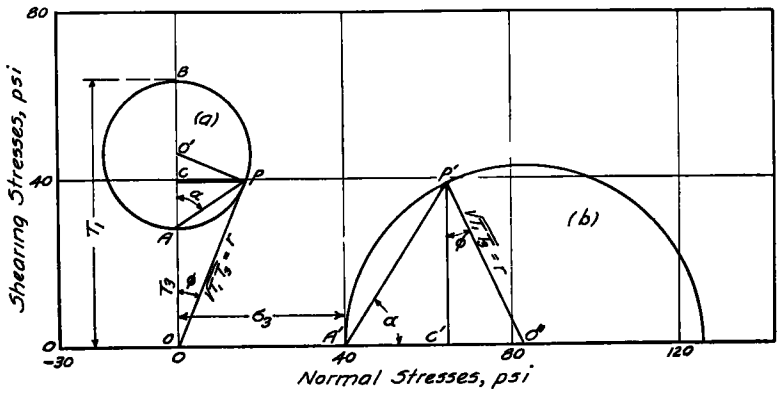

Figure 1. (a) Modified Mohr Stress Circle Constructed from Maximum and Minimum Shea Stresses at Failure in a Non-isotropic Material
(b) Mohr Circle of Rupture for the Same Material, Constructed from Tangent OP to the Mohr Stress Circle in (a) as radius and the Minor Principle Stress σ_3

CONSTRUCTION OF MOHR'S CIRCLE OF RUPTURE FOR AN ANISOTHROPIC MATERIAL

From the derived value of the radius of the Mohr rupture circle and a given value of one of the principal normal stresses[3], the

[3] Since shear stress in the plane of failure represented by OC in Figure 1 (a) is related to

corresponding Mohr's circle of rupture may now be constructed as follows:

Lay off on the axis of normal stress (Fig. 1-b) a distance proportional to the given minor principal stress σ_3 and draw a circle with a radius equal to $\sqrt{\tau_1\tau_3}$ and with center at $\sigma_3 + \sqrt{\tau_1\tau_3}$ through the terminal of σ_3. The circle so constructed is the required circle of rupture, which is the locus of points whose co-ordinates are the critical values of stress, existing at the time of failure, on any plane of failure passing through a given point of the body.

The critical stress components on the failure plane defined by

$$\alpha = \tan^{-1}\sqrt{\tau_1/\tau_3} \text{ arc } \tau = C'P' = OC = 2\tau_1\tau_3/(\tau_1 + \tau_3) \quad (8)$$

and

$$\sigma = OC' = \sigma_3 + 2\tau_3\sqrt{\tau_1\tau_3}/(\tau_1 + \tau_3) \quad (9)$$

which results from solving right triangle $C'P'O''$ for $C'O''$ and subtracting it from $r = \sqrt{\tau_1\tau_3}$ to get $A'C' = 2\tau_3\sqrt{\tau_1\tau_3}/(\tau_1 + \tau_3)$. The major, or maximum, principal (normal) stress is seen to be

$$\sigma_3 + 2\sqrt{\tau_1\tau_3}.$$

Substituting the following numerical values for the parametric constants in Coulomb's formula (assumed for the purpose of illustrating the method of computing the radius, angle of shear, and critical stress components for the critical circle) viz., $K_{min} = K_3 = 100$ psi, $K_{max} = K_1 = 225$ psi. and $\phi_1 = \phi_3 = 0$, and assuming the minor principal stress $\sigma_3 = 40$ psi, we obtain the following specific values for the dimensions in the Mohr circle of rupture:

$r = \sqrt{\tau_1\tau_3} = \sqrt{K_1K_3} = \sqrt{(100)(225)} = 150$ psi, with co-ordinates of its center at $\tau_c = 0$ and

$$\sigma_c = \sigma_3 + \sqrt{K_1K_3} = 40 + \sqrt{22500} = 190 \text{ psi.}$$

the normal stress by Coulomb's relation $\tau = \sigma \tan \phi + K$ the Mohr circle of rupture may be constructed *without* given values of either of the *principal stresses* for the special cases treated by Casagrande and Carillo, viz., $K = 0$ and $\phi = 0$.

The critical shear stress is

$$2\tau_1\tau_3/(\tau_1 + \tau_3) = 2K_1K_3/(K_1 + K_3) = 138 \text{ Psi}$$

and the critical normal stress is

$$\sigma_3 + 2\tau_3\sqrt{\tau_1\tau_3}/(\tau_1 + \tau_3) = 40 + \frac{200(150)}{325} = 132 \text{ psi.}$$

Also, $\alpha = \tan^{-1}\sqrt{\tau_1/\tau_3} = \tan^{-1}\sqrt{K_1/K_3} = \tan^{-1}\sqrt{225/100} = \tan^{-1} 1.5000 = 57°.2$

For the case represented by Figure 1, $\tau_3 = 29$ psi and $\tau_1 = 64$ psi so that $r = \sqrt{(29)(64)} = 43.1$ psi. The critical shear stress in the plane of failure is $\tau = \frac{2(29)(64)}{64 + 29} = 39.9$ psi. and the critical normal stress σ is $40 + \frac{2(29)\sqrt{(29)(64)}}{29 + 64} = 66.9$ psi; $\alpha = \tan^{-1}\sqrt{\frac{64}{29}} = 56°$ and $\phi = 2\alpha - 90° = 22°$.

Also, by substituting these values for σ, ϕ, and τ in Coulomb's relation one finds that the "cohesion" K is $39.9 - 66.9 \tan 22° = 12.9$ psi.

Reference to Figure 1-b shows these values calculated from the derived analytical formula are in approximate agreement with those obtained from the geometrical construction.

As a further graphical check on the constructions it is seen that OC and CP in Figure 2-a are approximately equivalent to $P'C'$ and $C'O''$ in Figure 1-b and that the angle of internal friction ϕ is approximately the same in the two circles.

ADDITONAL SPECIAL CASES

Other special cases easily derived by simple deduction from the general solution are:

Case (*c*) A material for which $K_1 = K_3 = 0$ and $\phi_1 = \phi_3 = \phi \neq 0$, in which case the radius of the circle of rupture reduces to

$$r = \sigma_3 \tan^2 \phi\,(1 + \csc \phi).$$

Case (*d*) A material for which $\phi_1 = \phi_3 = 0$, and $K_1 = K_3 = K \neq 0$, giving a Mohr Circle of rupture of radius $r = \sqrt{K^2} = K$.

Case (*e*) A material for which $K_1 = K_3 = K$ and $\phi_1 = \phi_3 = \phi \neq 0$. Substitution of

these values in equation (7) gives, after simplification,

$$r = (\sigma_3 \tan \phi + K)(\tan \phi + \sec \phi).$$

These three cases (c, d, and e) represent isotropic materials.

It is obvious that the Mohr Circles of rupture might just as easily have been constructed from a given value of the major principal stress, σ_1, instead of σ_3 by plotting the tentative stress circle from $\tau_1 = \sigma_1 \tan \phi + K_1$, and $\tau_3 = (\sigma_1 - 2r) \tan \phi_3 + K_3$, and then passing the Mohr circle of rupture, with center at $(\sigma_1 - r)$ through the terminal of σ_1. The Mohr diagram may also be constructed from given values of τ_1, τ_3, and K.

The graphical solution here developed for obtaining the value of the radius of the Mohr circle of failure, as well as the angles of shear and internal friction and the normal and tangential components of stress on the plane of failure, is obviously superior to the analytical method because of its far greater simplicity and speed of construction and evaluation of the quantities obtained in a stress analysis.

Incidentally, the graphical solution may also be effected for the case in which the given shear strengths are not in the principal planes but in two mutually perpendicular planes making an angle B with the principal planes by the following simple modification of the method here used:

1. Lay off on the axis of shear stress from the origin of a diagram such as that shown in Figure 1-(a) distances equal to the given shear strengths.
2. At the mid-point between their terminals construct an angle $2B$ with the shear stress axis.
3. Erect a perpendicular at the terminal of the major shear strength.
4. Construct a circle with its center at the mid-point of the shear strength terminals passing through the point of intersection of the perpendicular referred to in Step 3 and the terminal line of angle $2B$. This circle is the required tentative circle which represents the shear strength distribution on all planes through a given point of the material.
5. Draw a tangent from the origin to this circle for the radius of the required Mohr circle of failure and proceed as in the other case.

CONCLUSIONS

The foregoing simple procedure for determining the radius and other elements of Mohr's circle of failure from known values of the principal shear strengths may be summed up in the following brief and simple rule:

1. Graphically, the radius of the Mohr circle of failure is given by the length of the tangent from the origin of a Mohr diagram to a Mohr stress circle constructed by plotting the principal shear strengths on the shear stress axis.
2. Analytically, it is the geometrical mean of the given principal shear strengths.

Since the solution is valid for all finite values of τ_1 and τ_3, these rules are applicable to materials with induced anisotropy, possessing either or both components of shear resistance, i.e., cohesion and internal sliding friction, or to isotropic materials in which shear stress is equal in all directions about a point of the body under stress.

Inasmuch as shear stress in a given plane is a function of void ratio and consolidation, which in turn are functions of applied stress in the plane, it is reasonable to suppose that the degree of anistropy is changed under the unequal applied stresses occurring during the triaxial test and that a body which is initially isotropic may become anisotropic during the test. Hence, it appears that, while the present discussion, like that of Casagrande and Carillo previously referred to, is limited to a theoretical treatment of the subject, the results may not be devoid of practical value in soil mechanics, and other fields as stated in that paper, (7) and in the analysis of triaxial test results on some specimens of initially isotropic asphalt road materials which acquire anisotropy induced by unsymmetrical compaction during the test.

In concluding, the writers wish to acknowledge the aid given by Dr. Dana Young, Professor, University of Texas, and Mr. F. H. Scrivner, Senior Research Engineer, Texas Highway Department, in reading the manuscript and making helpful suggestions.

EXPLANATORY FOOTNOTES AND REFERENCES

(1) It can be shown by deduction from the general case of three dimensional stress that

since the intermediate principal stress equals the minor principal stress in the triaxial compression test, the plane diagram will correctly represent all combinations of stress. (Nadai, *Plasticity*, Chap. 7, pp. 39–47.)

(2) The Mohr Circle diagram, as ordinarily defined and used, is a graphical method, devised by O. C. Mohr, for determining the stress distribution on an oblique plane passing through a point of a body in equilibrium under stress from two (or three) known principal (normal) stresses acting on mutually perpendicular planes (principal planes) through the same point. If two given principal stresses be plotted, from a common origin on the axis of abscissae, in directions determined by their signs (opposite for compressive and tensile forces) and the conventions of analytic geometry, then the circle drawn through their terminals with center at the point representing their arithmetic mean, is called a Mohr stress circle; and it is the locus of points whose co-ordinates are values of the normal and tangential components of induced stress on all planes passing through the intersection of the principal planes. If the given principal stresses are the critical values existing at the time of failure of the material, the Mohr stress circle constructed from them is known as the Mohr circle of failure or of rupture.

For further details on the subject the reader is referred to engineering texts on statics by Timoshenko, Seely, and others, and to such articles as that by Hvorslev mentioned in the footnote which follows:

(3) and (4)
- (a) Plummer and Dore, Soils Mechanics and Foundations, Chap. 9.
- (b) M. Juul Hvorslev, Shearing Resistance of Remolded Cohesive Soils, *Proceedings* Soils and Foundations Conference of U. S. Engineer Department, 1938.
- (c) Timoshenko, Strength of Materials.

(5) Shear Failure of Anisotropic Materials, *Journal*, Boston Society of Civil Engineers, April 1944.

(6) L. E. McCarty: Applications of Mohr Circle and Stress Triangle Diagrams to Test Data Taken with the Hveem Stabilometer, *Proceedings*, Highway Research Board, Vol. 26, pp. 100–123, Section 2 of Derivations (1946).

(7) On page 75, the purpose of the article under reference (5) above is stated as follows: "The purpose of this paper is to present an extension of Mohr's theory for non-isotropic materials. The subject of this presentation was suggested by the behavior of soil samples, and the paper is primarily intended as a contribution in the field of soil mechanics, although it is expected that certain phenomena observed in the failure of such materials as steel and concrete, and the faulting of rocks, may also be explained in the light of the following considerations."

APPENDIX 1

ANALYTICAL DERIVATION OF EXPRESSIONS FOR THE SHEAR STRENGTH DISTRIBUTION ON PLANES OTHER THAN THE PRINCIPAL PLANES IN TERMS OF PRINCIPAL SHEAR STRENGTHS EXISTING IN THE PRINCIPAL PLANES AT FAILURE

Consider an elementary prism from a non-isotropic body bounded on its non parallel sides by the two principal planes and the surface of failure and held in static equilibrium by the potential critical shear stresses τ_1 and τ_3 equivalent to the given "shear strengths" acting at failure in the principal planes and resisting stresses τ_N and τ_S developed on the surface of failure (See figure 2).

From the definition of stress and the geometry of the figure it is seen that

$$\tau_N \, dsdz = \tau_1 \sin \alpha \, dxdz - \tau_3 \cos \alpha \, dydz \quad (1)$$

$$\tau_S \, dsdz = \tau_1 \cos \alpha \, dxdz + \tau_3 \sin \alpha \, dydz \quad (2)$$

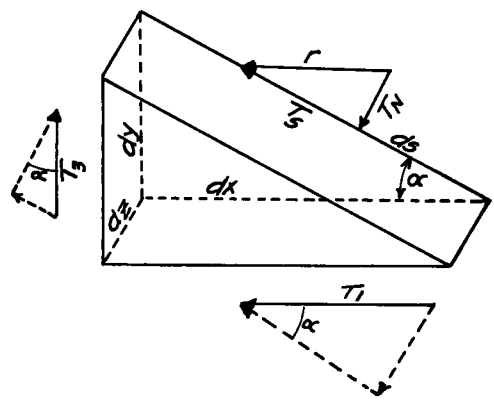

Figure 2

Dividing both members of these equations by the area $dsdz$, there results

$$\tau_N = \tau_1 \sin \alpha \cos \alpha - \tau_3 \sin \alpha \cos \alpha \quad (3)$$

$$\tau_S = \tau_1 \cos^2 \alpha + \tau_3 \sin^2 \alpha \quad (4)$$

for the normal and tangential critical shear strength components developed on the surface of failure inclined at an angle α (the angle of shear) to the major principal plane, τ_1 and τ_3 being drawn in a common direction for the purpose of comparing these expressions with similar results derived from the Mohr diagram in which the shear strengths are both measured in the positive direction.

On substitution of the trigonometrical identities $\sin \alpha \cos \alpha = \frac{\sin 2\alpha}{2}$, $\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}$ and $\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}$ equations (3) and (4) are transformed into

$$\tau_N = \frac{\tau_1 - \tau_3}{2} \sin 2\alpha \quad (5)$$

$$\tau_S = \frac{\tau_1 + \tau_3}{2} + \frac{\tau_1 - \tau_3}{2} \cos 2\alpha \quad (6)$$

These forms are easily recognized as the dimensions CP and OC in the tentative Mohr diagram Figure 1(a) and $C'O''$ and $C'P'$ in the conventional Mohr circle of failure shown in Figure 1(b), and their resultant

$$r = \sqrt{\tau_N^2 + \tau_S^2} = \sqrt{CP^2 + OC^2} = \sqrt{C'O''^2 + C'P'^2}$$

is seen to be equal to the maximum shear $C'P'$ (at $\alpha = 45°$) which is the required radius of the Mohr circle of failure.

Equations (3), (4), (5) and (6) express correctly the law of strength distribution on planes other than the principal planes.

Equivalence of $C'O''$ to CP and of $O''P'$ to OP in Figure 1-a and b depends on the equivalence between right triangles $O''C'P'$ and OCP which is established by the following construction of the Mohr circle of failure:

At the terminal A' of the given minor principal stress σ_3 construct an angle α equal to the angle of shear which is defined by equations (3), (4), (5) and (6) and by the construction in Figure 1-a. Draw a line $P'L$ parallel to the axis of normal stresses and at a distance from it equal to OC, which represents the critical shear stress in the plane of failure. From the intersection of this line with the terminal side of angle drop a perpendicular $P'C'$ to the axis of normal stresses. Next construct a circle thru A' and P' with its center O'' on the axis of normal stress. The circle so constructed is the Mohr Circle of failure. Draw $P'O''$, the radius r of the Mohr circle, thus forming the right triangle $C'P'O''$. Now $<C'O''P' = 180° - 2\alpha = <CO'P$. But $<CPO = <CO'P$ since their sides are mutually perpendicular. Then the right triangles $O''C'P'$ and OCP are equivalent because one leg and an acute angle of one equals a leg and acute angle of the other. Hence r equals the radius of the Mohr circle of failure $= O''P' = OP$, and $C'O'' = CP$ because they are corresponding sides of equivalent triangles.

DISCUSSION

E. S. Barber, *Associate Professor of Civil Engineering, University of Maryland*—This paper has a valuable contribution in Equation 7. The derivation, as presented, is obscure; for instance Figure 2 in the Appendix is not in equilibrium as stated. While a graphical method is given for constructing a circle of rupture assuming both principal strengths, these would generally not be known unless both principal stresses at failure were also known. It seems desirable to have a graphical solution of Equation 7; i.e. find the radius of the circle of rupture given one principal stress and ϕ_1, K_1, ϕ_3, and K_3. Following is the way I see it:

Given: $\tau_1 = \sigma_1 \tan \phi_1 + K_1$ and $\tau_3 = \sigma_3 \tan \phi_3 + K_3$ and τ for intermediate planes is assumed to vary linearly with σ as shown in Figure A. The circle of rupture is tangent to AC. Then $AB = \tau_3$ and $BC = \tau_1$, the radius OB is the altitude of a right triangle on its hypotenuse and equals $\sqrt{\overline{AB} \cdot \overline{BC}}$ or $r = \sqrt{\tau_1 \tau_3}$.

Substituting the given expressions for τ_1 and τ_3 Equations 6 and 7 and their corollaries are obtained.

One way to solve Equation 7 graphically (not Euclidian) is as follows: In Figure B, τ_3 is determined from σ_3, ϕ_3 and K_3; but τ_1 and the location of C and O are not known. A piece of tracing paper with two perpendicular lines, m and n, is placed over the

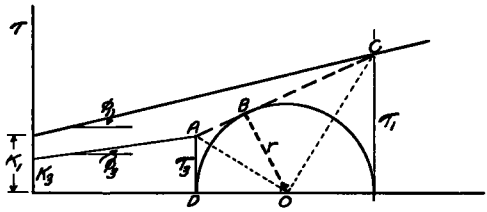

Figure A

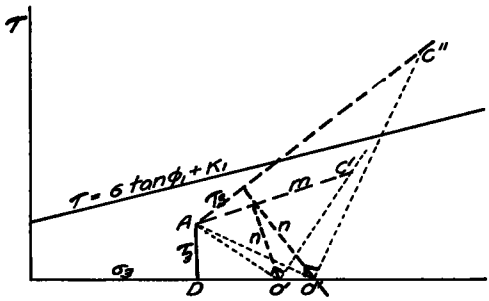

Figure B

figure; at a point on m at a distance τ_3 to the left of the intersection of m and n the paper is pinned through A. A right triangle placed against the pin with the right angle at the intersection O′ of n with the horizontal axis, cuts m at C′. C′ should be on $\tau_1 = \sigma_1 \tan \phi_1 + K_1$. This is accomplished by rotating the transparent paper while keeping the right angle at the intersection of n with the horizontal axis.

R. J. Hank and L. E. McCarty, *Closure*—The authors wish to express thanks to Mr. Barber for his constructive criticisms and suggestions. Inasmuch as we received Mr. Barber's discussion after this Volume of "Proceedings" had gone to press, time does not permit a formal and extended closure. However, certain brief explanations can be made.

The original paper included in an appendix two graphical derivations which were ommitted due to space limitations. These derivations were about the same as those suggested by Mr. Barber. The device suggested by Mr. Barber and illustrated in his Figure B is a little different from those used by the authors and it appears to be all right. Regarding Mr. Barber's reference to lack of equilibrium in Figure 2 of the appendix, it is believed that careful study of definitions and assumptions will justify use of this figure.

It is recommended that the article by Casagrande and Carillo (reference 5 in the appendix) be studied before the present derivations are read. The vectors $\tau_M \tau_S$ in Figure 2 are not drawn to scale which prevents testing graphically for equilibrium. Concerning Mr. Barber's statement that the two principle strengths "would generally not be known unless both principal stresses at failure were also known," it was the intention of the authors that these strengths would be determined experimentally in a direct shear test.