products rather than sums, or by the use of the total 24-hour traffic counts, peak-hour traffic counts, or individual traffic counts for each of the 24 hours, makes Grossman's index a much-more-reliable index than if the method of calculation resulted in large differences in the order of preference.

The idea behind the accident-exposure index is appealingly simple and appears to have considerable value for the intended purpose. Grossman realizes that an analysis of this type has its limitations and is useful only when applied in combination with many other criteria for the adequacy of an interchange layout.

There is, of course, the possibility that even though two layouts result in the same index, the crossings made in one layout may be far-more dangerous than those made in the other layout. It is believed, therefore, that the method can be applied most beneficially in the preliminary stages of design to select two or three layouts for a thorough and detailed investigation of such items as ramp and roadway capacity, safety features, and cost. As a final criterion for selecting the most satisfactory of several possible layouts, its use would be limited to those layouts in which the many other factors involved in a proper design have been found to be equally satisfactory.

Deformation Mechanism and Bearing Strength of Bituminous Pavements

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The mechanical behavior of bituminous pavements and their substructures is of importance in relation to the stresses acting on them. This paper deals with the deformation mechanism of such structures and the measurement of the bearing strength, defined as the maximum load per unit area which a bituminous pavement can carry without causing initial failure.

The deformation of bituminous pavements consists of an instantaneous and retarded elastic deformation followed by a plastic deformation. The mechanical behavior is primarily determined by the plastic deformation which is accompanied by hardening. As a result of the hardening process, the coefficient of plastic traction, which is stress over strain rate and is related to the viscosity, increases with increasing compressive stress and time to a maximum within a certain region of stress. At this point the shearing stress and shear are zero, and the maximum coefficient of plastic traction is an isotropic or volume viscosity, i.e., the material behaves like a solid. The principal stress corresponding to this maximum coefficient is the bearing strength. At greater stresses the coefficient of plastic traction decreases rapidly and the material is in the region of failure.

A bituminous pavement at rest is conceived as containing simultaneously particles in the disordered state and in the ordered state. The latter state refers to positions of minimum potential energy. Under stress the particles in the ordered state rarely escape their positions, while the remaining particles move from positions of disorder to those of order. At the maximum value of the coefficient of plastic traction the number of particles in the disordered state approaches zero. The change in free energy of activation in going from a disordered to an ordered state and the mass of a particle are also

maximums at this point. The process of hardening is comparable to a fusion of the disordered particles into a particle of larger mass.

The theory of the mechanical behavior of bituminous pavements applies also to the base course and subsoil, as shown by data from bearing-plate measurements.

● A BITUMINOUS road structure consists of a wearing course and a base course resting on the subsoil. Each layer of this structure must have a certain strength in order to carry the loads of modern motor vehicles, otherwise, failure will occur. A variety of methods for measuring this strength have been suggested in the literature. These methods are based on measuring the load at which the material fails in compression, tension, shear, or beam tests, or under extrusion through an orifice as in the Hubbard-Field stability test. Indentation and plate-bearing tests measure the load required to obtain a certain penetration or depression. In the triaxial test the vertical pressure is often determined at which the material fails under a given lateral pressure, or the lateral pressure is measured which is developed under a vertical pressure, as in the Hveem Stabilometer. Where the results can be expressed as load per unit area, these methods measure at best stresses. Stresses are associated with strains, which are generally ignored in these tests. It is generally agreed that the deformation of bituminous pavements is plastic, i.e., the strain is a function of time. Yet the time is not taken into consideration in any of these tests or appears only as a secondary factor in cases where the rate of loading or rate of deformation is kept constant.

A method for the measurement of the bearing strength of bituminous pavements must relate stress to strain and time and must conform to the following conditions: (1) the concept of bearing strength must be clearly defined; (2) the test procedure must permit the measurement of the bearing strength in a quantitative way; and (3) the method cannot be arbitrarily chosen but must be based on theoretical considerations which correspond to reality. None of the methods mentioned above satisfies these requirements. The majority of these methods are based on visual failure. There can be no doubt that before failure becomes visible, the material has gone through a stage of progressive weakening. The bearing strength is therefore defined here as the maximum load per unit area which a bituminous pavement can carry without causing initial failure. It is the purpose of this paper to submit a theory of the deformation mechanism of bituminous pavements which is based on an analysis of the relationship between stress, strain, and time obtained from the experiment, and which also permits the determination of the bearing strength.

The various layers of a bituminous road structure have similar compositions and consist of mixtures of mineral aggregate, a liquid, and air in the voids. The difference between these layers is that the liquid in the bituminous layer is asphalt, while in the other layers it is water. It can, therefore, be assumed that a theory of the deformation mechanism of bituminous pavements is also applicable to the base course and subsoil.

DEFORMATION MECHANISM OF BITUMINOUS PAVEMENTS

Types of Deformation

Road structures are subjected to compressive loads whether they are of a transient or static nature. The deformation of such pavements is best carried out under compressive loading; therefore, as will be shown later, the distribution of stresses in the material is rather complicated, and the compressive loads are preferably applied as uniaxial loads. With regard to the time of load application, it has to be borne in mind that at the moment of load application, solids as well as liquids behave like elastic bodies. This behavior can be readily demonstrated by hitting the surface of the water in the bathtub, for example, with the flat of the hand. The sensation of pain experienced is a manifestation of this rigidity. At this moment a water molecule is still surrounded by its original neighbors as in an elastic solid. It is only within a small time interval, the so-called relaxation time, that a molecule can free itself from its surroundings and move to a new position. The relaxation time is the ratio of viscosity to modulus of rigidity, $\bar{t} = \eta/G$. For a plastic material under simple compression, the modulus of rigidity is replaced by Young's modulus of elasticity and the viscosity by a coefficient of plastic traction, and the relaxation time becomes

$$\bar{t} = \mu/E \tag{1}$$

Since the stresses associated with μ and E are the same, and since the strain rate is plastic strain over time, the relaxation time is a measure of the time by which the plastic strain lags behind the elastic strain. With μ larger than E, the value of the relaxation time can be considerable, and a material under test can exhibit brittleness when the deformation is carried out during a time interval considerably smaller than the relaxation time. Since the relaxation time is not known for bituminous pavements, the condition of constant static load suggests itself for the investigation of the strain as a function of time. The strains obtained are generally small, and the condition of constant compressive load approaches, therefore, the condition of constant compressive stress.

Figure 1 represents the strain as a function of time at constant stress and temperature for a sheet asphalt mixture. The stress causes an immediate strain, OA, which is independent of time. Although from A on the strain increases with increasing time, it will be seen that the strain rate or strain per unit time decreases with increasing time. At B the load is removed, resulting in an immediate recovery BC. From C on the recovery is a function of time and finally ceases. The amount of nonrecoverable plastic strain corresponds to DE. The total strain consists, therefore, of the following three parts: (1) an instantaneous elastic strain independent of time, (2) a retarded elastic strain which is a function of time, and (3) a plastic strain whose rate decreases with time.

Plastic Deformation and Time-Hardening

With both elastic deformations being of a transient nature, the plastic deformation is most important for the mechanical behavior of bituminous pavements and will be discussed first. When the plastic deformation is due to plastic flow, the stress is a function of the strain rate at constant temperature, and at constant stress the strain increases linearly with time. When the strain rate decreases with time at constant stress and temperature, as with bituminous mixtures, the material under test becomes harder. The amount of hardening obtained is the result of the work done on the system, and the stress can be a function of the strain rate and strain (strain-hardening) or a function of strain rate and time (time-hardening).

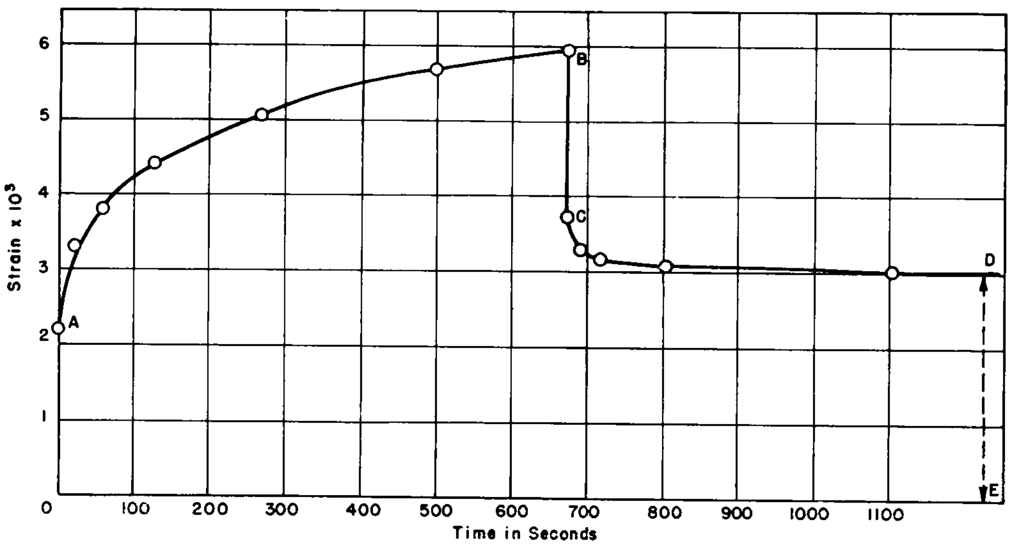

Figure 1. Deformation of a bituminous mixture as a function of time for loading and unloading.

The mechanical behavior of bituminous mixtures showed them to harden as a function of time. For this case the stress function is as follows (for details see Appendix A):

$$\sigma/\sigma_0 = [\dot{\epsilon}(t + 1)/\dot{\epsilon}_0]^b \qquad (2)$$

where σ and σ_0 are principal compressive stresses, $\dot{\epsilon} = d\epsilon/dt$ is the strain rate obtained at stress σ and time t. The time t refers to the total time from the moment of the first load application on to the time at stress σ. This time interval covers the plastic deformation only and not the retarded elastic deformation. It follows from Equation 2 that at constant stress

$$\dot{\epsilon}(t + 1) = \dot{\epsilon}_0(\sigma/\sigma_0)^{1/b} = \beta \qquad (3)$$

and

$$\dot{\epsilon} = \beta/(t + 1) \qquad (4)$$

i.e., at constant stress the strain rate is proportional to $1/(t + 1)$ and decreases with time. The parameter β is constant at constant stress and has the dimensions of strain. In view of Equation 3, Equation 2 may be written as follows:

$$\sigma/\sigma_0 = (\beta/\beta_0)^b \qquad (5)$$

The experiments were carried out by measuring the total strain as a function of time at constant stress. Since the plastic deformation is preceded by a retarded elastic deformation, there are two time scales involved which are not readily separated. For the evaluation of β the time scale is shifted in such a way that each load is applied at zero time. The retarded elastic deformation comes to a stop at total strain ϵ_0 and time t_m. From this point on the deformation is plastic, and the relationship between total strain and time is as follows:

$$\epsilon - \epsilon_0 = \beta \ln (t - t_m + 1) \qquad (6)$$

where ln is the natural logarithm. In the cases studied it was found that the plastic deformation runs concurrently with the retarded elastic deformation, in which case Equation 6 becomes

$$\epsilon - \epsilon_0 = \beta \ln (t/t_0) \qquad (7)$$

The choice between these two equations is determined by the experimental data. It is also found on occasion that the plastic deformation at constant stress is associated with two values of β.

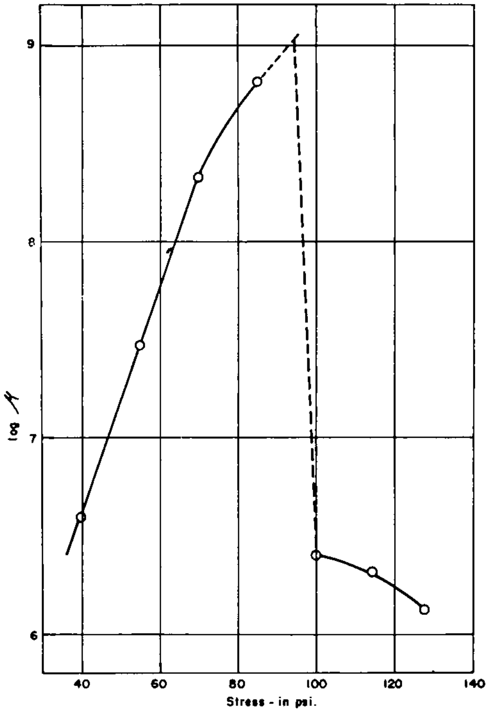

Figure 2. Relationship between log coefficient of plastic traction and stress.

The coefficient of plastic traction is

$$\mu = d\sigma/d\dot{\epsilon} = b\sigma/\dot{\epsilon} = b\sigma(t + 1)/\beta \qquad (8)$$

where t refers to the sum of the time intervals for the plastic deformations obtained at various stresses. This coefficient increases with increasing time at constant stress and increases with increasing stress and time to a maximum. At this point the bituminous mixture has received its maximum amount of hardening and behaves like a solid body. The stress associated with the maximum value of the coefficient represents the bearing strength at a given temperature. At stresses in excess of this strength, the material becomes weaker and finally fails. This is demonstrated in Figure 2 showing a plot of the logarithm of coefficient of plastic traction obtained at the end of each load application versus the stress for a mixture described in the experimental section.

The behavior of bituminous mixtures under load shows that the bearing strength is not an inherent property but is acquired as a

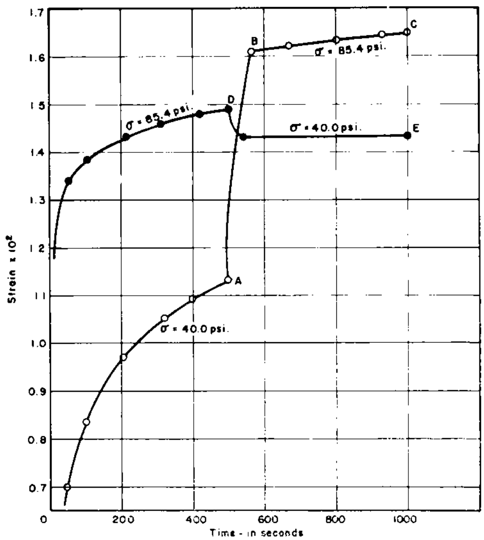

Figure 3. Effect of order of loading upon strain as a function of time.

TABLE 1

EFFECT OF CONSTANT STRAIN RATE AND STRESS RATE UPON THE MAGNITUDE OF STRESS FOR A GIVEN COEFFICIENT OF PLASTIC TRACTION

Constant Strain Rate		Constant Stress Rate	
Strain Rate	Stress	Stress Rate	Stress
sec.$^{-1}$	*psi.*	*psi./sec.*	*psi*
8.1956×10^{-8}	84.5	0	84.5
10^{-6}	1042	1	295.8
10^{-5}	10420	5	461.7
		10	559.2
		20	677.4

result of the work done on the mixture. This conclusion is borne out by Equation 2 representing the stress as a function of strain rate and time. The explicit inclusion of time in this equation indicates that the behavior depends upon the previous history of the material. As an illustration of this behavior, a sample of the above paving mixture was first subjected to a stress of 40 psi. for 500 seconds and then the stress was increased to 85.4 psi. for a second period of 500 seconds. A second sample was subjected to the reverse order of stress application, i.e. the first stress, 85.4 psi., was reduced to 40 psi. after 500 seconds. The strains obtained are plotted in Figure 3 as a function of time. At 40 psi. the curve follows OA for a period of 500 seconds, and an increase in stress to 85.4 psi. brings about a rapid increase in strain from A to B which is mainly elastic. From B to C the increase in strain as a function of time is relatively small. If the first stress is 85.4 psi, the strain along the curve to Point D increases at a greater rate than between Points B and C. Reducing the stress at Point D to 40 psi causes a small recovery in strain and the strain remains finally constant. The difference in the strains obtained by both procedures, CE, clearly demonstrates the dependence of the deformation upon the history of the material under test.

The importance of the time of load application has been discussed above in connection with the relaxation time. Since the coefficient of plastic traction increases with increasing stress and time, the relaxation time of bituminous mixtures increases also. Compression tests are often carried out under a constant rate of deformation or constant rate of loading, and it is of interest to evaluate the stress under these conditions, which would correspond to a given coefficient of plastic traction. Considering, for the sake of simplicity, constant strain rate and constant stress rate, then for the first case

$$\dot{\epsilon} = \text{constant} = b\sigma/\mu; \qquad \sigma = \text{const.}\,\mu/b \tag{8a}$$

For the second case, this stress is, as will be shown in Appendix A as follows:

$$\sigma = \left[\frac{\beta_0 c\mu\sigma_0{}^{1/b}}{b}\right]^{b/(1+2b)} \tag{8b}$$

where c represents the constant stress rate. The maximum coefficient observed was 6.45×10^8 psi. per sec. at a stress of 85.4 psi., which is close to the maximum bearing strength. The stresses corresponding to this coefficient for various constant strain rates and stress rates are given in Table 1. The data show the required stress to increase more rapidly with increasing constant strain rate than with increasing constant stress rate. This is not surprising, since the strain rates are small for bituminous mixtures under constant load. It follows therefore that for the condition of constant strain rate or rate of deformation, a bituminous mixture will fail before it has acquired its maximum strength. The data are also in qualitative agreement with the observation that the compressive strength increases with increasing constant rate of deformation or of loading.

Elastic Deformation

It has been shown that the plastic deformation is preceded by an instantaneous elastic deformation and a retarded elastic deformation. The latter is an elastic deformation superimposed upon a plastic deformation. The retarded elastic strain as a function of time at constant stress is as follows (Appendix B):

$$\epsilon' - \epsilon_0' = \alpha \ln (t + 1) \qquad (9)$$

where ϵ_0' is the retarded elastic strain at $t = 0$ and α is a constant with the dimension of strain. The similarity between Equations 6 and 9 suggests that the mechanisms are similar for the plastic and retarded elastic deformation except that for the latter the process is reversible on unloading.

It has been mentioned that often the plastic deformation starts immediately after the instantaneous elastic deformation and proceeds concurrently with the retarded elastic deformation. In this case the strain consists of the sum of the plastic and retarded elastic strain, and the parameter α' is a composite of a fraction of α proper and a fraction of β. The knowledge of this composite parameter serves to determine the instantaneous elastic strain. At the first load application this strain is $\epsilon_1 = \epsilon_0'$ which is obtained from Equation 9. The next incremental load produces a strain difference $\Delta\epsilon$ which is the difference between ϵ_0' for this load and the final total strain at the end of the previous load application, hence

$$\text{total instantaneous elastic strain} = \epsilon_1 + \textstyle\sum_i \Delta\epsilon \qquad (10)$$

Distribution of Stresses and Strains

A solid material under load is in a state of stress. Within a certain region of stress, the material behaves like an elastic body and the resulting strains are homogeneous and bear a linear relationship to the stresses. Beyond the elastic limit, the material may either fail by fracture or may deform plastically. Plastic deformation proceeds along lines of slip which are subject to shearing stresses.

The stress components of an element,

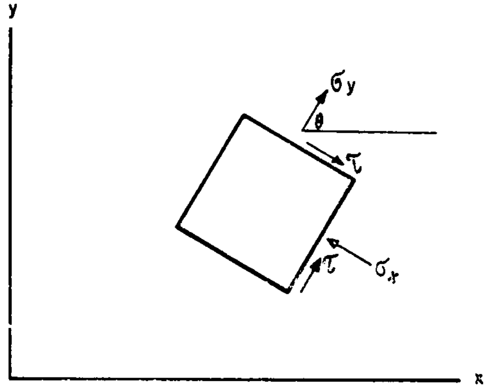

Figure 4. Distribution of stresses acting on an element in the YX plane.

which may represent a particle during plastic deformation, are shown in Figure 4. The direction of the algebraically larger principal stress σ_1 coincides with the y coordinate and the direction of the principal stress σ_2 with the x coordinate. The normal stresses are σ_y and σ_x and the shearing stresses are $\tau_{yx} = \tau_{xy} = \tau$. A particle can have any position so that its axis in the xy plane can be inclined to the horizontal at any angle between 0 and 180 deg. The angle between the direction of the normal stresses and their coordinates can be larger or smaller than 45 deg. and may therefore be denoted by $\theta = 45 \pm \varphi/2$. For these conditions there follows (*1*):

$$\begin{aligned}
\sigma_y &= \tfrac{1}{2}[\sigma_1 + \sigma_2 + (\sigma_1 - \sigma_2)\cos(90 \pm \varphi)] \\
&= \tfrac{1}{2}[\sigma_1 + \sigma_2 \mp (\sigma_1 - \sigma_2)\sin\varphi] \\
\sigma_x &= \tfrac{1}{2}[\sigma_1 + \sigma_2 - (\sigma_1 - \sigma_2)\cos(90 \pm \varphi)] \\
&= \tfrac{1}{2}[\sigma_1 + \sigma_2 \pm (\sigma_1 - \sigma_2)\sin\varphi] \\
\tau_{yx} &= \tau_{xy} = \tau = \tfrac{1}{2}(\sigma_1 - \sigma_2)\sin(90 \pm \varphi) \\
&= \tfrac{1}{2}(\sigma_1 - \sigma_2)\cos\varphi
\end{aligned} \qquad (11)$$

These equations hold in general and have to be adapted to the condition of the test. For compression σ_1 is generally taken negative and σ_2 positive or vice versa. It follows that at $\varphi = 90°$, $\tau = 0$, further $\sigma_y + \sigma_x = \sigma_1 + \sigma_2$, $\sigma_y - \sigma_x = \mp(\sigma_1 - \sigma_2)\sin\varphi$.

The angle φ represents the angle of internal friction. Substituting σ for σ_y as well as σ_x, it follows from Mohr's stress representation as a stress circle that

$$d\tau/d\sigma = \tan\varphi \qquad (12)$$

which holds for a constant as well as variable angle φ. With φ = constant, this term leads to Coulomb's equation:

$$\tau - \tau_0 = \pm\sigma \tan \varphi \qquad (13)$$

which indicates that slip cannot take place until the shearing stress τ exceeds the shear resistance τ_0, which is usually termed cohesion. When $\tau = \tau_0$, $\sigma \tan \varphi = 0$, a condition which is satisfied when both σ and φ are zero. For this case there follows from Equation 11 that $\sigma_1 = -\sigma_2 = \sigma_0 = \tau_0$ and τ_0 is equal to the yield value σ_0. Coulomb's equation refers to a plastic deformation which proceeds along slip lines making constant angles with the coordinates.

It has been shown that plastic deformation accompanied by hardening is not continuous in time, which suggests that at constant principal stress σ_1 the shearing and normal stresses and the principal stress σ_2 are not homogeneous in time. This condition is satisfied by a variable angle φ and a solution of Equation 12 is as follows:

$$\tau - \tau_u = \pm\sigma \tan \varphi \qquad (14)$$

where τ_u is a variable shear resistance varying with φ. Under constant principal stress σ_1 the strain rate decreases with time and the plastic deformation comes finally to a stop. At this point there is no slip and $\tau = \tau_u$, $\sigma \tan \varphi = 0$. This condition is satisfied with $\sigma = 0$, or

$$\sigma_1 + \sigma_2 \mp (\sigma_1 - \sigma_2) \sin \varphi = 0$$

It follows that

$$(\sigma_1 + \sigma_2)/(\sigma_1 - \sigma_2) = \pm \sin \varphi \qquad (15)$$

where the negative sign refers to an angle $\theta = 45 - \varphi/2$ and the plus sign to an angle $\theta = 45 + \varphi/2$. The ratio of the principal stresses are obtained from Equation 15 as follows:

$$\begin{aligned} -\sigma_1/\sigma_2 &= (1 - \sin \varphi)/(1 + \sin \varphi); \quad (\theta = 45 - \varphi/2) \\ -\sigma_1/\sigma_2 &= (1 + \sin \varphi)/(1 - \sin \varphi); \quad (\theta = 45 + \varphi/2) \end{aligned} \qquad (16)$$

indicating that for this condition the principal stresses are of opposite sign. For σ_1 being a compressive stress and negative, σ_2 becomes positive.

Equation 16 shows that σ_2 is larger than σ_1 for particles with an angle $\theta = 45 - \varphi/2$. With increasing stress the particle rotates accompanied by an increase in the angle θ and decrease in the value of σ_2. At $\theta = 45$ deg., $\varphi = 0$, and $\sigma_1 = \sigma_2$. With $\theta > 45$ deg., σ_1 is larger than σ_2 and σ_2 finally reaches a value of zero at $\theta = 90$ deg., $\varphi = 90$ deg. At this point the shearing and normal stresses and the minor principal stress are zero, and the system behaves like a solid material under simple compression, i.e., the only stress acting on the system is $\sigma_1 = \sigma_m$. At this stress the maximum amount of hardening is obtained, and the yield value has increased from an initial value σ_0 to the final value σ_m.

At stresses in excess of σ_m slip can start again accompanied by a decrease in the angle φ and can finally lead to failure.

The principal compressive strain is conventionally expressed as the ratio of the change in height to the original height. Since a variation in height depends upon the actual height, it is more accurate to express the strain as natural strain. For a cylinder which under compression changes its height from h_0 to h and its radius from r_0 to r, the principal strains are

$$\epsilon_1 = \ln (h_0/h); \ \epsilon_2 = \ln (r/r_0) = \nu\epsilon_1 \qquad (17)$$

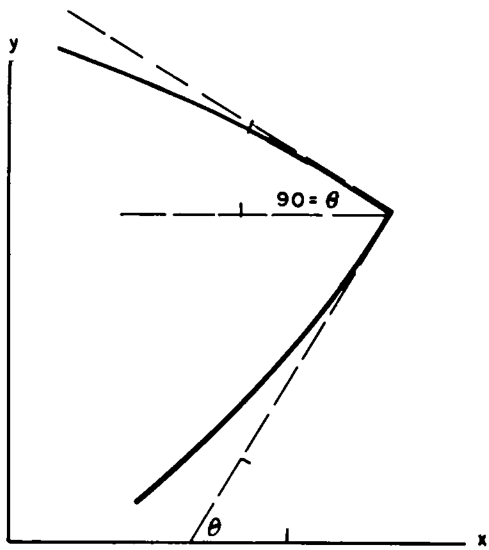

Figure 5. Slip lines due to rotation of element.

where ν = Poisson's ratio = ratio of lateral to vertical strain, and ln is the natural logarithm. When the natural strains are small they are practically equal to the conventional strains.

In the plastic region the resultant of the two principal strains is shear. As the planes perpendicular to the normal stresses σ_y and σ_x rotate, they describe two curves representing two sets of slip lines as shown in Figure 5. A comparison with Figure 4 shows that the tangents to the slip lines form in one case an angle $\theta = 45 \pm \varphi/2$ with the horizontal and in the other case an angle of $90 - \theta = 90 - (45 \mp \varphi/2) = 45 \mp \varphi/2$. The tangents of these angles are the shears as follows:

$$\begin{aligned} -dy/dx &= -dh/dr = \tan(45 \pm \varphi/2) \\ &= \sin(90 \pm \varphi)/[1 + \cos(90 \pm \varphi)] \\ dy/dx &= dh/dr = \tan(45 \mp \varphi/2) \\ &= \sin(90 \mp \varphi)/[1 + \cos(90 \mp \varphi)] \end{aligned} \tag{18}$$

When $\varphi = 90°$, $\sin(90 \pm \varphi) = 0$, hence there is no shear in agreement with the state of stress at $\varphi = 90$ deg. described above.

Kinetics of Plastic Deformation

It has been shown that, in the case of plastic deformation accompanied by time-hardening, the coefficient of plastic traction and the yield value are at a maximum when the shearing stress and shear approach zero values. Since this effect is the result of a rotation of the slip planes, it follows that the plastic deformation causes a change in the internal structure of the material under test. A bituminous mixture, compacted in the ordinary way, can be conceived as containing at rest a number of particles acting as units of flow whose axes form any angle between 0 and 180 deg. with the horizontal. The particles with zero angles are in regions of greater density and are attached to their neighbors by strong cohesive forces. The remaining particles are, on account of their greater angles of position, in regions of lower density and weaker cohesive forces. The arrangement is similar to that in ordinary solids where the particles in regions of greater density are in a state of order and the particles in regions of lower density in a state of disorder. For reasons of simplicity, this terminology will be also applied to bituminous mixtures. The essential difference between the ordered and disordered state is that the particles in the ordered state are in positions of minimum potential energy, whereas the particles in the disordered state are considered to be distributed over various levels of greater potential energy.

In a stress region which has the lower yield value as upper limit, the system is deformed elastically. The particles change slightly their positions, but each particle retains its neighbors. At stresses in excess of the yield value, the particles in the ordered state rarely leave their positions, since on account of their zero angle, the shearing stress associated with them is zero according to Equation 11. The particles in the disordered state are subjected to shear and move to new positions. Such a particle can move over a barrier of potential energy only if under the influence of interaction with its neighbors it acquires a kinetic energy equal to the magnitude of the barrier. The energy barrier of smallest magnitude is associated with the lower yield value. Since the strain rate decreases with increasing time and stress in the region of hardening, some of the particles in the disordered state, after having passed the energy barrier, lose their kinetic energy again under the influence of their new neighbors and move to positions of lower potential energy or, in favorable cases, to positions of minimum potential energy, i.e., to positions of order.

In order that a particle may move from one position to another, it is necessary that space be provided. It is agreed that all matter, solid or liquid, contains a number of vacancies which represent the unoccupied sites of the system. In bituminous mixtures these vacancies are provided by the voids. For flow to occur, the energy barriers or the cohesive forces between the particles must be overcome by an externally applied force, and for simple liquids the viscosity is a measure of these cohesive forces. Since plastic deformation in the hardening range is caused by motion of the particles in the disordered state which are associated with weaker cohesive forces, the coefficient of plastic traction is a measure of these weaker forces. Further, this coefficient varies with stress and time and, therefore, with the number of particles in the disordered state. This suggests that the coefficient depends not only on the magnitude of the weak cohesive forces but also on the number of particles associated with them. A variation in the number of particles in the disordered state brings about a variation in the number of vacancies, hence the coefficient of plastic traction is a function of the number of particles in the disordered state and of the vacancies, viz.,

$$\mu = \mu(q, \phi) \tag{19}$$

where q is the fraction of the particles in the disordered state and ϕ the fraction of vacancies.

A formulation of the energies associated with plastic deformation can be based on the two accessible quantities, stress and coefficient of plastic traction. The forces acting on a particle of mass m moving with a velocity v are as follows:

$$f - (f_1 + f_2 + B) = f - \psi v = m dv/dt \tag{20}$$

where mdv/dt is mass times acceleration and is a force due to inertia. The term f is the external force acting on the particle, f_1 is the frictional force due to the shear resistance to flow, and f_2 is the frictional force due to a resistance to a change in volume and is associated with the isotropic or volume viscosity. Finally, B represents body forces which consist of the negligible force due to gravitational attraction and of the more important cohesive forces. The last three forces act against f and may be combined into one term ψv, where ψ is the sum of frictional and body forces per unit velocity. Since the effective stress for plastic deformation is $b\sigma$, $\psi v = bf$.

It will be shown in Appendix C that the solution of Equation 19 is as follows for b = constant:

$$\mu = q\mu/b + [g(\phi_0) - g(\phi)]\mu_0/g(\phi) \tag{21}$$

where μ_0 is a coefficient of plastic traction at a state very near to that of rest and $g(\phi)$ is a function of the fraction of vacancies. This equation shows the coefficient μ to consist of two terms, one referring to the motion of the particles in the disordered state and the other, being a function of the change in the vacancies or change in volume, is the isotropic or volume viscosity. It will be seen that $g(\phi)$ decreases with decreasing values of q in the region of hardening, and the process is accompanied by an increase in density. In the region of failure the reverse holds true. It also follows from Equation 21 that at $q = 0$

$$\mu_m = [g(\phi_0) - g(\phi_m)]\mu_0/g(\phi_m) \tag{22}$$

i.e., the coefficient of plastic traction is at a maximum and is equal to the volume viscosity. At this point there is no shear, which is in agreement with the condition of zero values for the shearing stress and shear as shown before.

Rearranging Equation 21 gives

$$\begin{aligned} b = q_0 &= g(\phi_0) \\ &= q + [g(\phi_0) - g(\phi)]\mu_0/\mu g(\phi) \end{aligned} \tag{23}$$

which defines the parameter b in terms of q and $g(\phi)$ and gives it a physical meaning.

Multiplying Equation 21 by the plastic strain rate $\dot{\epsilon}$ and substituting $1 - (1 - q)$ for q gives

$$\begin{aligned} b\sigma = \sigma &+ [g(\phi_0) - g(\phi)]\mu_0\dot{\epsilon}/g(\phi) \\ &- (1 - q)\sigma \end{aligned} \tag{24}$$

This equation represents the combination of frictional and body forces per unit area mentioned in connection with Equation 20. The first and second term are the frictional forces due to a resistance to shear and change in volume, and since $1 - q$ represents the fraction of particles in the ordered state, the last term is a measure of the stronger cohesive forces.

On multiplying Equation 20 by $d\lambda$, the distance moved by a particle, and considering that $d\lambda/dt = v$, there results the following expression for the work done by the system in going from a state of rest to a state of stress:

$$\begin{aligned} W = (f - f_0)\lambda &= \Delta(f_1 + f_2 + B)\lambda + 0.5\, mv^2 \\ &= b(f - f_0)\lambda + 0.5\, mv^2 \end{aligned}$$

Multiplying and dividing by a, the area of a particle on which the force acts, gives with $f/a = \sigma$, $a\lambda = v_f$, the volume of flow,

$$W = (\sigma - \sigma_0)v_f = b(\sigma - \sigma_0)v_f + 0.5\, mv^2 \tag{25}$$

where σ_0 is the initial yield value or a stress so close to it that q and $g(\phi)$ can be considered to be constant. Substituting $b\sigma$ from Equation 24 leads to

$$\begin{aligned} W = (\sigma - \sigma_0)v_f &= (\sigma - \sigma_0)v_f \\ &- \{(1 - q)\sigma - (1 - q_0)\sigma_0 - [g(\phi_0) \\ &- g(\phi)]\mu_0\dot{\epsilon}/g(\phi)\}v_f + 0.5\, mv^2 \end{aligned} \tag{26}$$

The second term on the right side is the work related to the stronger cohesive forces and to the change in volume, and is therefore a change in potential energy, hence

$$W = W - \Delta(PV) + 0.5\,mv^2 = W - \Delta(PV) + \Delta E_k \tag{26}$$

and the change in potential energy is equal to the change in kinetic energy. This result is in agreement with the postulate that a particle can pass the barrier of potential energy only if it acquires a kinetic energy equal to the magnitude of the barrier. The change in internal energy for the system is as follows:

$$\Delta U = W + T\Delta S = \Delta F - \Delta(PV) + T\Delta S$$

where U = internal energy, T = absolute temperature, S = entropy, F = Gibbs' free energy, P = pressure, V = volume, PV = potential energy. It follows from this and Equation 26 that:

$$\Delta F = W + \Delta(PV) = W + \Delta E_k$$

The free energy change is obtained by treating plastic deformation as a rate process. The particles in the disordered state move in the direction of the force applied and the vacancies in the opposite direction. Each process is carried out with the same rate, and a condition of equilibrium is established. This is shown by rearranging Equation 21, keeping in mind that $b = q_0 = g(\phi_0)$ and by replacing μ and μ_0 by $E'\bar{t}$ and $E_0'\bar{t}_0$, the products of relaxation time and modulus of elasticity according to Equation 1. The moduli are considered to be variable, since with increasing hardening and densification, the modulus is also expected to increase. The result is then

$$(q_0 - q)/\bar{t}_0 = [g(\phi_0) - g(\phi)]g(\phi_0)E_0'/g(\phi)E'\bar{t} \tag{27}$$

This equation shows that the change with time in the concentration of the particles in the disordered state is equal to a function of the change in the concentration of the vacancies with time. The specific rate constants are

$$k' = 1/\bar{t}_0 = E_0'/\mu_0\,, \quad k'' = E_0'/E'\bar{t} = E_0'/\mu$$

and the equilibrium constant is

$$K = k'/k'' = \mu/\mu_0 = \exp\,(\Delta F/kT) \tag{28}$$

where ΔF is the above free energy change for a particle going from one state to another, k = Boltzmann's constant = 1.38×10^{-16} ergs per degree Kelvin, and exp = exponential. Equation 28 shows that ΔF is positive when $\mu > \mu_0$, and vice versa. It follows from Equation 26 that

$$\Delta F = W + \Delta(PV) = W + \Delta E_k = (2 - b)(\sigma - \sigma_0)v_f \tag{29}$$

$$\Delta F = [2(\sigma - \sigma_0) - (b\sigma - b_0\sigma_0)]v_f$$

depending upon whether one or two values of b are associated with μ and μ_0. Since the value of ΔF is obtained from Equation 28, the value of the kinetic energy and therefore mass of a particle can be calculated from Equation 29.

It can also be shown that the process of ordering is not restricted to the plastic deformation but starts already with the onset of the retarded elastic deformation.

Cohesion of Bituminous Mixtures

Mineral aggregate, when dry, forms a loose mass with no coherence of the particles. When an asphalt is added and the mixture compacted, a certain force must be applied to separate the particles. This force is a measure of cohesion and is related to the yield value, which increases as a function of stress and time from a minimum to a maximum. In view of the irregular surface of mineral particles, the cohesion is determined by the number of points at which the mineral particles contact each other. In a compacted bituminous mixture where the particles are distributed at random, the number of these contact points is relatively small. With increasing orientation under stress, the number of contact points, and therefore the cohesive force, increases.

In view of the surface irregularities of mineral aggregate the cohesive force cannot be directly assessed. An insight into the magnitude of this force is obtained by considering two flat surfaces held together by a liquid. It will be shown in Appendix D that this force is as follows:

$$f = 2S^2w/\rho g a^2 \tag{30}$$

where S = surface tension of the liquid, w = width of the surfaces, ρ = density of the liquid, g = gravitational constant and a = distance between the two surfaces or thickness of the liquid layer. When a is very small, the force reaches large values.

The cohesive force is in general attributed to the high viscosity of the asphalt present in the bituminous mixture, yet Equation 30 contributes this force only to the surface tension and film thickness. This effect has been demonstrated by Griffith (*2*) in an experiment where a hardened steel ball was fitted to a carefully made hole in a plate. The clearance between the ball and the hole permitted the ball to fall easily through the hole. However, when a drop of water was placed in the annular space between the ball and the plate, the ball stuck, and could only be forced through by applying considerable pressure.

The viscous effect is nil in comparison to the cohesive force and appears only as a time factor affecting the rate with which plates can be separated. Since the modulus of rigidity is considerably larger than the viscosity, the relaxation time of asphalts is of the order of a fraction of a second, according to Equation 1, and is of minor importance. Equation 30 suggests, therefore, a method for measuring the force with which two surfaces are held together by a layer of asphalt of given thickness. Such a method consists in placing a given amount of asphalt between two flat surfaces and subjecting it to a high pressure at elevated temperature to obtain uniform distribution of the asphalt. At a given constant temperature, the force is measured at which the plates separate under a constant low rate of loading. From the amount of asphalt used and measurement of the area covered by the asphalt, the force per unit area for a given film thickness can be determined.

Experiments carried out by the author showed that the load per unit area necessary to separate two surfaces of granite held together by a 60-penetration asphalt increased with increasing film thickness to a maximum and then decreased. This result is in contradiction with Equation 30 and suggests that the film thickness is restricted to a certain boundary. Inspection of the asphalt after rupture showed that there was no visible flow at all but the asphalt was cracked up to the film thickness at the maximum load. At greater film thicknesses flow occurred in the asphalt layer.

Although asphalt is a liquid, it behaved like a solid in these experiments. Marcelin (*3*) showed that when a liquid is compressed between two parallel plates, there is a critical film thickness which cannot be reduced by application of increased pressure. For a number of liquids varying widely in viscosity, he found the critical film thickness to vary with the viscosity according to

$$\eta = 2a^2 \times 10^{11} \qquad (31)$$

where a is the critical film thickness in cm. and η is the viscosity in poises. This film was found to be remarkably resistant to pressures up to 100 kg./cm.² and to become disrupted at greater pressures.

Although viscosity increases with increasing molecular weight, Equation 31 shows the critical film thickness to be considerably larger than molecular dimensions. For example the critical film thickness of water at a viscosity of 0.01 poise is 22.36×10^{-8} cm., whereas the height of a water molecule is of the order of 1.9×10^{-8} cm. For asphalts the critical film thickness is considerably larger. Hubert (*4*) presented data relating the force necessary to rupture a film as a function of film thickness for various asphalts contained between polished silica and limestone surfaces at a constant rate of loading of 100 g. per sec. The data show an increase in this force with increasing film thickness to a maximum, as mentioned above. Figure 6 represents such data for a fluxed Trinidad asphalt and its petrolenes. The maximum force and film thickness are given in Table 2 for various asphalts.

TABLE 2
MAXIMUM COHESIVE FORCE AS FUNCTION OF AMOUNT OF ASPHALT

Asphalt	Penetration at 77 F.	Type of Surface	Amount of Asphalt	Cohesive Force
			mg./cm.²	*psi.*
Galicia	60	Silica	1.250	233
Bedford	100	Silica	0.389	426
Trinidad Lake blended with Shale Oil	75	Silica	0.725	632
Trinidad Lake blended with red. Trinidad Crude	60	Silica	0.825	711
Ditto minus Asphaltenes	—	Silica	0.376	647
Mexican	42	Silica	0.556	671
Mexican	42	Limestone	1.080	739

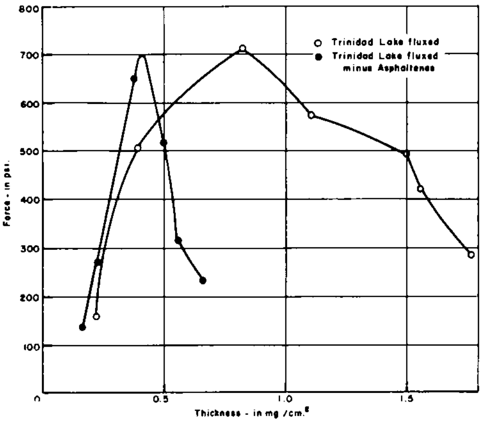

Figure 6. Force required to break film versus thickness of film.

Figure 6 is an illustration of the known fact that the strength of bituminous mixtures increases with increasing asphalt content to a maximum and then decreases again. Since the density of asphalts is close to 1, the critical amount in mg. per cm.^{2} corresponds to the critical film thickness in millimeters. Table 2 shows this film thickness as well as the maximum cohesive force to vary with the source or with the viscosity and surface tension according to Equations 31 and 30.

TABLE 3
STRAIN AND TIME AS FUNCTIONS OF STRESS FOR A BITUMINOUS MIXTURE UNDER COMPRESSION AT 60 F.

Region of Superimposed Strains		Plastic Region	
Stress = 40.01 psi.			
Strain × 10²	Time in sec.	Strain × 10²	Time in sec.
0.5474	13.1	0.7176	90.5
0.6256	29.7	0.7958	130.6
0.6808	52.2	0.8901	203.0
		0.9453	262.0
		0.97757	306.0
		0.9982	339.0
Stress = 55.2 psi.			
1.0994	12.8	1.2098	163.0
1.1224	25.8	1.2512	229.0
1.1431	45.8	1.2972	322.0
1.1661	80.6	1.3317	414.0
		1.3685	560.0
		1.4099	750.0
Stress = 70.4 psi.			
1.5203	15.7	1.5732	127.5
1.5433	42.2	1.6330	250.0
1.5663	116.5	1.6606	342.0
		1.6813	423.0
		1.6951	508.0
		1.8446	3070.0
Stress = 85.4 psi.			
1.9550	22.1	2.0010	142.0
1.9780	82.5	2.0470	217.0
		2.0562	354.0
		2.1229	1010.0
		2.2011	3466.0

Region of Superimposed Strains		Plastic Region	
Stress = 100.3 psi.			
Strain × 10²	Time in sec.	Strain × 10²	Time in sec.
2.3138	12.4	2.3782	126.0
2.3276	33.0	2.4127	203.5
2.3460	80.0	2.4288	241.0
		2.5024	380.0
		2.5599	460.0
		2.7393	701.0
Stress = 114.6 psi.			
3.3620	18.9	3.4402	71.4
3.3735	27.4	3.4655	84.0
3.3850	49.0	3.5521	135.0
		3.5921	161.5
		3.6426	206.0
		3.7001	242.0
		3.8243	316.0
		3.9370	391.0
		4.0405	457.0
		4.1532	530.0
		4.2659	592.0
		4.3809	655.0
		4.4936	717.0
Stress = 127.6 psi.			
4.6086	14.5	4.8248	91.5
4.6299	22.6	5.0525	188.0
		5.2825	277.0
		5.5125	357.0
		5.9748	520.0
		6.4371	661.0

EXPERIMENTAL PART

Determination of the Bearing Strength of Bituminous Mixtures

It has been shown that when the compressive stress is homogeneous, all other stresses are not homogeneous. It is essential, therefore, that bituminous mixtures be moulded so that the height as well as density are as uniform as possible in order to ensure a uniform load distribution over the surface of the sample. To reduce the friction between the surfaces of the sample and of the platens of the compression machine, the sample surfaces are coated with a paste prepared by stirring starch into boiling water. The amount of starch used should be such that the mixture sets to a gel at room temperature.

A sample of a bituminous mixture so prepared is subjected to a compressive pressure smaller than the yield value for several minutes to permit the surface of the sample to adapt itself to the surface of the piston carrying the load. The load is then increased and the decrease in height measured as a function of time. When the rate of deformation approaches zero, the next load increment is added.

Table 3 gives the strains as a function of stress and time for a mixture containing 80 percent sand passing No. 40 retained on No. 50 mesh sieve, 10 percent of limestone dust and 10 percent Mid-Continent asphalt of 100/120 penetration. This mixture was compacted from top and bottom under a pressure of 1,000 psi. in a mould having a diameter of 2 inches and was aged 48 hours at room temperature before testing. The yield value of such mixtures is not readily measured and the stress was therefore determined at which the mixture began to yield plastically. This stress was 8.4 psi. at the test temperature of 60 F. It will be seen from Table 3 that the increase in strain obtained during a constant

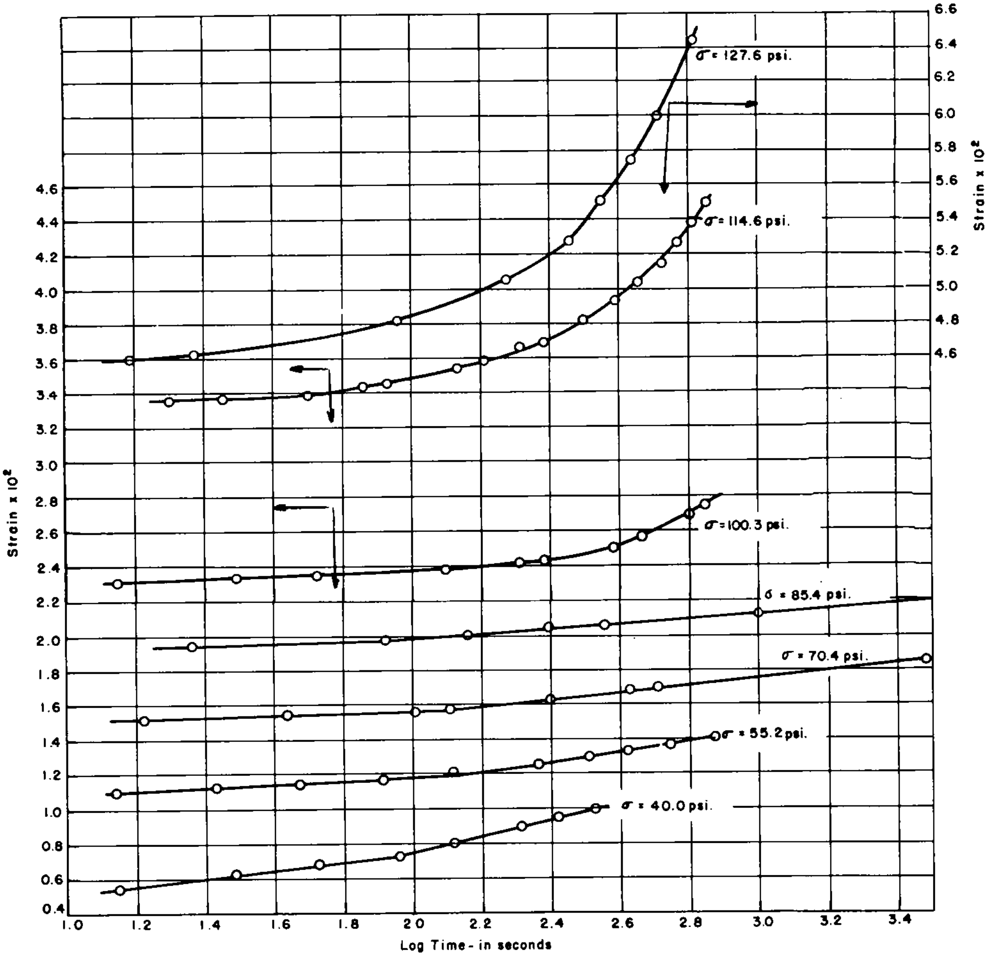

Figure 7. Relationship between strain and log time at various constant stresses.

load increment is very small, hence the variation in stress is also small, and the stress is expressed as load per mean actual area for each load.

The strain as a function of the logarithm of time is plotted in Figure 7 for each stress. It is seen that there are two linear relationships at the lower stresses. The first part with the smaller slope corresponds to the retarded elastic deformation in agreement with Equations 9 and 7. At the greatest stresses the plastic strain is linear with the logarithm of time for a short interval only and then becomes curvilinear. The latter part of the strain is a linear function of time, and the plastic strain rate becomes constant. A plot of the logarithm of stress versus the logarithm of β, the slope of the plastic strain-to-time relationship, forms a straight line within a certain region of stress in agreement with Equation 5. The same holds also for the relations between log stress and log α', α' being the slope of the combined retarded elastic strain-time relationship mentioned before. These plots are shown in Figure 8. For the first part the slope of both relations is negative and then becomes positive. The point of intersection, which corresponds to a stress of 94.4 psi. for the plastic as well as retarded elastic deformation, represents the maximum bearing strength of the bituminous mixture.

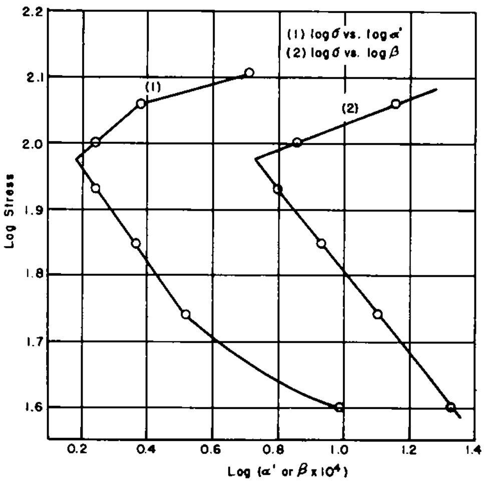

Figure 8. Log stress versus log strain parameter for bituminous mixture.

The pertinent data for each stress are given in Table 4. They represent the values of α' and β, the constant strain rates where observed, the coefficient of plastic traction, ΔF, the change in free energy of activation, m, the mass of a particle acting as a unit of flow, the instantaneous strain and the corresponding Young's modulus. The coefficients of plastic traction are those obtained at the end of a load application, except the first one, which represents a value extrapolated to the beginning of the plastic deformation. For the calculation of the mass of a particle, the velocity must be known. Assuming the velocity to be linear, then $v = -dh/dt$, and since the plastic strain rate is $\dot{\epsilon} = d\epsilon/dt = -dh/ht$, the velocity is $v = h\dot{\epsilon}$. The values of the mass thus obtained are probably somewhat too large. They are, however, of the right order of magnitude.

It will be seen that the coefficient of plastic traction, the free energy change, the mass of a particle, and Young's modulus increase with increasing stress to maximum values in the region of hardening. At stresses in excess of the bearing strength, these values decrease with increasing stress, and this region is associated with a loosening of the structure and progressive weakening. The results are in agreement with the theory presented.

Effect of Asphalt Content and Temperature upon the Bearing Strength of Bituminous Mixtures

It has been shown that the force which is required to separate two parallel plates depends upon the thickness of the layer of asphalt used as adhesive. Since the bearing strength is a measure of the maximum cohesive force acting between the two mineral particles, the bearing strength should vary with the asphalt content. This expectation is confirmed by the experimental results given in Table 5, which are augmented by Hubbard-Field stability values obtained on the same samples. The data show the maximum values for the bearing strength and Hubbard-Field stability to coincide with the same optimum asphalt content. The Hubbard-Field stability test appears therefore to be useful for the determination of the optimum asphalt content of a given bituminous mixture. In view of what has been said of the effect of constant rate of deformation on the strength, no correlation exists between Hubbard-Field stability and bearing strength.

TABLE 4
PARAMETERS RELATED TO THE DEFORMATION MECHANISM OF A BITUMINOUS MIXTURE AT 60 F.

No.	Stress in psi,	$\alpha' \times 10^4$ Equation 9	$\beta \times 10^4$ Equation 7	Constant Strain Rate $\times 10^6$	μ in psi-sec Equation 8	ΔF in ergs $\times 10^{13}$ Equation 28	m in g Equation 29	Instantaneous Elastic Strain $\times 10^3$	Young's Modulus in psi
1	8.4	—	—	—	1.964×10^2	—	—	—	—
2	40.01	9.660	21.320	—	3.95×10^6	3.951	0.00686	2.9684	13480
3	55.2	3.285	12.650	—	2.94×10^7	4.754	0.0774	3.2012	17240
4	70.4	2.298	8.540	—	2.12×10^8	5.542	9.13	3.6702	19180
5	85.5	1.748	6.250	—	6.45×10^8	5.985	62.2	4.2306	20180
6	100.3	1.730	7.143	7.38	2.47×10^6	3.765	0.00797	4.9192	20390
7	114.6	2.389	14.295	16.14	1.29×10^6	3.506	0.00101	5.8512	19580
8	127.6	5.066	—	27.78	8.34×10^5	3.332	0.000536	7.0534	18090

Stress No. 2–5: $d \log\sigma/d \log \alpha' = -0.692 = b'$
Stress No. 2–5: $d \log\sigma/d \log \beta = -0.619 = b$
Stress No. 6-8: $d \log\sigma/d \log \dot{\epsilon} = 0.1816 = b$

TABLE 5

EFFECT OF ASPHALT CONTENT UPON THE BEARING STRENGTH AND HUBBARD-FIELD STABILITY AT 60 F.

	Asphalt Content in g. per 90 g. of Mineral Aggregate				
	8	9	10	11	12
Bearing strength in psi	72.2	85.0	94.4	83.2	65.6
Hubbard Field Stability in lb	5800	6250	6500	6100	5650

TABLE 6

BEARING STRENGTH OF BITUMINOUS MIXTURES AS FUNCTIONS OF TEMPERATURE

Asphalt	Bearing Strength in psi at			
	32 F	60 F	77 F.	100 F.
Mid-Continent 100/120 Pen	104.2	94.4	65.0	42.2
Mid-Continent 85 Pen	108.8	101.0	67.0	43.5

TABLE 7

EFFECT OF HEIGHT ON THE BEARING STRENGTH AT 60 F.

Height of Sample	Ratio: Height to Diameter	Bearing Strength
in.		*psi.*
4.0	2.0	95.6
1.96	0.98	92.9
0.91	0.455	94.4
0.20	0.10	127.0

TABLE 8

BEARING STRENGTH AS FUNCTION OF TOTAL AREA TO LOADED AREA AND CONFINEMENT

Total Surface Area and Condition of Testing	Loaded Area	Bearing Strength
sq. in.	*sq. in.*	*psi.*
3.141 unconfined	3.141	94.4
12.564 unconfined	3.141	97.1
12.564 confined in mould	3.141	95.5

The effect of temperature on the value of the bearing strength is shown in Table 6 for two bituminous mixtures containing Mid-Continent asphalts of 100/120 and 85 penetration at the optimum asphalt content.

The decrease in strength with increasing temperature is relatively small. For the same temperature interval the viscosity of the asphalts varies between approximately 10^9 and 10^5 poises. Hence the viscosity of the asphalt bears no relation to the bearing strength, and the latter is mainly affected by the surface tension of the asphalt as discussed before. This is corroborated by data obtained on sand asphalt mixtures containing the optimum amount of various asphalts. These asphalts had a penetration of 90–92 penetration at 77 F., their surface tensions at 60 F. were 27, 32.4, and 36 dynes per cm.^{2}, and the bearing strengths of the mixtures at 60 F. were 97, 112, and 133 psi., respectively.

Bearing Strength of Bituminous Mixtures in Relation to the Size of Samples

The effect of a variation in height of the sample on the bearing strength is demonstrated in Table 7 for the same mixture prepared with the 100/120-penetration asphalt. The data show that the bearing strength is not affected by variations in the height of the samples between 0.91 and 4.0 inches, while at a height of 0.2 inches the strength is greater by 34 percent. In the latter case, the height approaches a critical thickness which is determined by the size of the largest mineral particle present.

Housel (*5*) developed a theory of bearing strength which is assumed to be the combined effect of two stress reactions, viz., the perimeter shear and the developed pressure which acts against the column of material under the loaded area. In view of this concept, bearing strength measurements were carried out on the same bituminous mixture which was moulded into cylinders 1.14 inches in height and 4 inches in diameter. These cylinders were centrally loaded and the ratio of total to loaded area was 4 to 1. The measurement was carried out on one cylinder while contained in the mould. The results are given in Table 8.

The results indicate that neither confinement nor increase in the ratio of total to loaded area affect the bearing strength. Any developed pressure, acting against the material under the loaded area, should have increased the bearing strength of the confined sample. As the material hardens under the loaded area with increasing stress and time, the zones of deformation become broader with increasing angle of internal friction. The slip planes disappear gradually under the loaded area and move to regions under the unloaded area which have not been deformed. Since, in the case of confinement, the diameter of the sample remained constant, there occurred an upward movement of the bituminous mixture between the loaded area and the wall of the mould.

Bearing Strength in Relation to Composition of Bituminous Mixture

The data presented so far were obtained for a simple sand-asphalt mixture. The following data refer to mixtures, which were prepared by adding stone to this sand asphalt mixture, by increasing its filter content, and by grading the mineral aggregate.

The mixtures containing stone were prepared by adding crushed limestone, passing the ¼-inch and retained on the ⅛-inch sieve, in increasing amounts to the mixture of 80 g. sand passing No. 40 retained on No. 50 mesh screen, 10 percent limestone filler and 10 percent of 100/120-pen. Mid-Continent asphalt. The asphalt content of the final mixtures was adjusted according to their stone content (see Table 9 for results).

It will be seen from the data that the bearing strength is hardly affected up to a stone content of 40 percent by weight. Within this concentration range the stones apparently can rotate freely under stress. At greater concentrations the rotation of a stone is hindered by its neighbors and the bearing strength increases with increasing stone content to a maximum at 65 percent. At stone contents in excess of 65 percent the strength decreases, which appears to be due to an increase in void content as evidenced by the open structure of the mixture containing 70 percent of stones.

The sand-asphalt mixtures with varying amounts of filler were prepared from the same sand, limestone dust, and asphalt at the optimum asphalt content in proportions shown in Table 10.

The results show the bearing strength to increase with the filler content and to reach a maximum at about 25 percent of filler. A further increase in filler content seems to reduce the strength.

The considerable increase in bearing strength, experienced by the incorporation of the optimum amount of crushed stone or optimum amount of filler in a simple sand-asphalt mixture, suggests that the bearing strength of a properly graded asphaltic-concrete pavement should be of such a magnitude that the pavement could carry any type of traffic. For the study of this effect, a mixture was prepared with an aggregate of the following composition: 65 percent crushed limestone passing ¼-inch and retained on ⅛-inch screen, 22 percent siliceous sand passing No. 10 retained on No. 40 screen, 3 percent siliceous sand passing No. 40 retained on No. 80 screen, 4 percent siliceous sand passing No. 80 retained on No. 200 screen and 6 percent limestone dust passing the No. 200 screen. The optimum asphalt content was 5.6 g. per 100 g. of aggregate. A sample of this bituminous concrete mixture 4.5 inches in height, was found to have a bearing strength of 389 psi. at 77 F.

TABLE 9
BEARING STRENGTH AT 60 F. AS FUNCTION OF STONE CONTENT OF BITUMINOUS MIXTURE

In Weight	In Vol.	Asphalt Content In Weight	Bearing Strength
%	%	%	*psi.*
0.0	0.0	10.00	94.4
30.0	25.85	7.54	96.0
40.0	35.04	6.72	97.5
46.0	40.70	6.22	110.6
50.0	44.50	5.90	124.8
55.0	49.40	5.49	142.8
60.0	54.33	5.08	165.3
65.0	59.42	4.67	194.2
70.0	62.46	4.26	160.3

TABLE 10
EFFECT OF FILLER CONTENT ON THE BEARING STRENGTH OF SAND-ASPHALT MIXTURES AT 60 F.

Sand	Asphalt	Filler	Bearing Strength
%	%	%	*psi.*
80.0	10.0	10	94.4
74.0	11.0	15	123.4
69.0	11.0	20	148.0
63.5	11.5	25	174.2
57.5	12.5	30	161.0

Bearing Strength Measurements with Bearing Plates

The theory of plastic deformation, as outlined for uniaxial compression, can also be applied to the deformation of road structures under a loaded bearing plate. Since the loaded area is constant, the principal compressive stress is

$$L/A = L/\pi r^2 \tag{32}$$

where L = load and r = radius of a circular bearing plate.

Making use of the mathematical device suggested by Trefftz (*6*), it can be shown that the elastic strain under the center of the bearing plate is as follows:

$$\epsilon = 2z/\pi r(1 - \nu^2) \tag{33}$$

TABLE 11
DEFORMATION OF SOIL UNDER BEARING PLATE AS FUNCTION OF STRESS AND TIME

No.	Stress in psi	Strain Difference × 10^2					$\beta \times 10^2$
		20 Min.	35 Min.	40 Min.	50 Min.	60 Min.	
1	62.50	1.8807	—	—	—	2.0687	0.1711
2	83.33	2.6330	—	3.0090	—	3.1971	0.5134
3	104.17	3.5732	—	4.2879	—	4.8896	1.1980
4	125.00	6.5823	7.8988	—	8.5570	9.2152	2.3961
5	145.83	9.0272	11.0951	—	12.4120	13.1645	3.7653
6	166.66	12.7885	15.2335	—	16.7377	17.4900	4.2787
7	187.50	16.3618	19.7468	—	22.0040	23.1322	6.1617

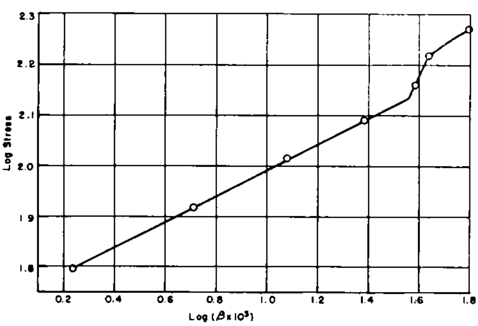

Figure 9. Log stress versus log β for natural soil.

where z is the central depression and ν is Poisson's ratio.

This expression for the strain explains Housel's observation (*5*) that a plot of the load per unit area versus the ratio of perimeter to area of bearing plates varying in diameter gives a straight line for a given value of the deflection under the plate. This behavior obviously applies only to the elastic region for which

$$\sigma = E\epsilon$$

Multiplying and dividing Equation 33 by πr gives on substitution

$$\sigma = L/A = (Ez/\pi(1 - \nu^2))(2\pi r/\pi r^2)$$

which shows the load per unit area to increase linearly for a given value of z with $2\pi r/\pi r^2$, the perimeter per area ratio.

In the plastic region, the coefficient is $\mu = b\sigma/\dot{\epsilon}$, and since b contains a term related to a volume change, Equation 33 can be simplified by inserting a value of 0.5 for Poisson's ratio. The plastic strain becomes then

$$\epsilon = 8z/3\pi r \quad (34)$$

To demonstrate the mechanical behavior of a natural soil under load, the field data are used which were presented by Housel and taken from Figure 5 of his paper. In this investigation the settlement of a soil was measured as a function of time while a constant load was applied on a circular bearing plate having an area of 4 sq. ft. The load was kept constant for 60 minutes and was then increased by increments of 3,000 lb. The settlement at zero time was reported as zero in the paper quoted. Therefore, these settlements actually represent settlement differences. The data are given in Table 11, the strain differences being expressed in accordance with Equation 34. The strain at constant load was found to be a linear function of the logarithm of time obeying Equation 7.

$$\epsilon - \epsilon_0 = \beta \ln (t/t_0)$$

indicating that for this soil the plastic deformation started at the moment of load application.

The data show the strains and the values of β to be larger than those given for the sand asphalt mixture. Figure 9 represents the relationship between log stress and log β, which is linear in the lower stress region in agreement with Equation 5.

$$\sigma/\sigma_0 = (\beta/\beta_0)^b$$

The value of b is 0.2626 in the region of the first four stresses and increases to 1.05 for the next two stresses. It appears from the low and positive value of the first b, that this soil was in a poor state of consolidation which improved under loading. The bearing strength of this soil seems to be acquired at the second last load which is 166 psi.

A load applied to a bituminous pavement is transmitted to the base course and subsoil. Each layer of the road structure undergoes a deformation which is a different function of the load applied and its duration for each layer. To evaluate the bearing strength of a road structure, bearing-plate tests should be carried out in such a manner that the deformation of each layer can be measured. Such measurements have been made by Lancaster and Driscoll (*7*) which consisted in applying a constant load on a bearing plate, 30 inches in diameter, until the rate of deflection was 0.001 inches per 15 seconds and then releasing it.

This procedure was carried out for three load increments, after which the load was applied continuously so as to produce a constant rate of deflection of 0.5 inches per minute until a maximum deflection of 2 inches was obtained. By means of a special device it was possible to simultaneously measure the deflections at the top of the bituminous pavement, the top of the base course and at the top of the subsoil. The data (taken from Figure 7 of their paper) are presented in Table 12 as strains observed at the stresses acting on the surface of the bituminous pavement. Since this part of the test begins with zero load, all strains have been corrected for zero strain at zero load.

It will be seen from the data that the strains of the pavement and base course are approximately the same between 50 and 300 psi., indicating that the deformation mechanism of these layers is approximately the same. The corresponding strains of the subsoil are smaller.

The rate of deformation was 0.5 inches per minute, corresponding to a constant strain rate of 1.4147×10^{-3} sec^{-1}. This strain rate is so large in comparison to those observed for the bituminous mixture under constant load that the strains in Table 12 can be assumed to belong to the combined retarded elastic and plastic region. It will be shown in Appendix A that for the condition of constant strain rate the relationship between stress and strain in the region of hardening is as follows:

$$\sigma/\sigma_0 = (\epsilon/\epsilon_0)^b \qquad (35)$$

and the logarithm of stress should be a linear function of the logarithm of strain. The data are graphically presented in this manner in Figure 10 for the three parts of the road structure. The stresses acting on the base course and subsoil are not known. They are, however, proportional to the load applied on the surface of the pavement. The relationships between log unit load and log strain are linear up to a certain load, beyond which the structures failed, as indicated by a rapid increase in strain for a small increment in unit load. The point of inflection appears at the same load for each layer; the strength is 225 psi. for the pavement and appears at the compressive stresses corresponding to the surface load of 225 psi. for the base course and subsoil. In

TABLE 12

DEFORMATION OF ROAD STRUCTURE AT CONSTANT STRAIN RATE

Load Developed on Surface of Pavement	Pavement	Base Course	Subsoil
psi.	*Strain* $\times 10^2$	*Strain* $\times 10^2$	*Strain* $\times 10^2$
50	0.88766	0.94972	1.0570
100	2.0160	2.1007	—
150	3.2667	3.4210	2.8925
200	4.8521	4.8671	3.6992
250	7.9438	7.9138	6.2401
300	15.076	14.441	12.107
313	18.436	—	—
318	25.959	19.931	17.750
320	—	23.692	21.511
316	—	27.454	25.272

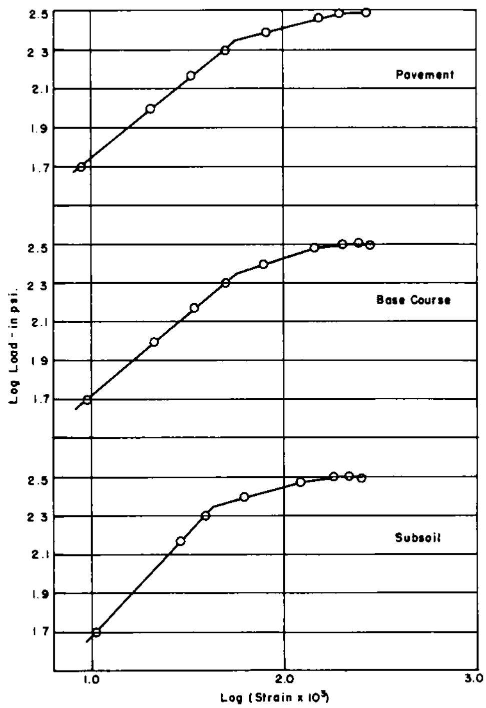

Figure 10. Log load per unit area versus log strain for road structure loaded at constant rate of deformation.

view of the similarity in the deformation mechanisms of the pavement and base course, the values of b are also practically the same, viz., 0.816 and 0.848, while the value of b for the subsoil is 1.067.

It has been shown in connection with the data presented in Table 1, that a material when loaded under the condition of constant strain rate, will fail before it has acquired its maximum strength. The same argument also applies to the above road structure. The data show however that this structure could carry loads up to 225 psi. without causing failure.

APPLICATION OF THE THEORY OF BEARING STRENGTH TO THE DESIGN OF ROAD STRUCTURES

The theory presented here leads to the determination of bearing strength. As a material constant for a given temperature, the bearing strength is independent of the conditions of the test procedure, provided that sufficient time is allowed for the establishment of equilibrium. With this provision, the bearing strength for each layer of a road structure may be determined in the laboratory.

A load applied to the surface of a road is distributed through the material in such a way that the vertical stress decreases with increasing vertical distance from the loaded area. With a knowledge of the bearing strength of the subsoil, the required thickness of pavement and base course can be calculated and must be such that the transmitted vertical stress corresponds at least to the bearing strength of the subsoil in its weakest condition. For the computation of the transmitted stresses the tables compiled by Barber (*8*) appear to be useful.

The required strength of a bituminous pavement is determined by the stresses exerted by stationary and moving wheels and by thermal stresses resulting from the change in volume with changing temperature. The bearing strength is determined on a uniformly loaded material. This condition is not satisfied by pneumatic tires. Bradbury (*9*) in an investigation of the wheel load distribution in concrete pavements, reported that the contact pressure of a pneumatic tire on the road surface is of the same order of magnitude as the inflation pressure. He quoted 90 psi. as the maximum inflation pressure and discussed circumstances which might lead to a contact pressure exceeding the inflation pressure by a maximum of 25 percent. Marwick and Starks (*1*), on the other hand, found that for bituminous pavements the vertical stress is approximately 1.5 times the inflation pressure. Thus, for a stationary wheel with an inflation pressure of 90 psi., the bearing strength of the pavement should be 135 psi. The data given in this paper show that such a strength can be obtained for any layer of the road.

For moving wheel loads, it appears from the studies of Marwick and Starks (*10*) that the contact pressure of a pneumatic wheel is independent of the speed of the vehicle. This holds only for level surfaces, but imperfections in the contour of the surface increase the contact pressure by an amount which varies with the nature of the impact. It appears that a shock coefficient of two sufficiently corrects the contact pressure for the effect of impact. For this condition the stress exerted by a moving wheel with an inflation pressure of 90 psi. would be of the order of 270 psi. Being of a transient nature, this stress will cause a very small strain, the magnitude of which further decreases with increasing speed. It follows, therefore, that a bituminous pavement with a bearing strength less than 270 psi. will not fail under a moving wheel with added impact.

Thermal stresses arise from a change in volume with changing temperature. It follows from thermodynamics that

$$dp = \alpha K dT - K dV/V \qquad (36)$$

where p = pressure, α = cubical coefficient of expansion, K = bulk modulus, T = temperature and V = volume. This equation cannot be readily applied; however, it shows the pressure to be a maximum at constant volume, $dV = 0$. Keeping α and K constant, since they are rather insensitive to temperature, it follows for this condition that

$$\Delta p/\Delta T = \alpha K = \alpha E/3(1 - 2\nu) \qquad (37)$$

and the change in pressure with changing temperature, or the increase in pressure per degree is equal to the product of coefficient of expansion and bulk modulus. The latter is related to Young's modulus, E, as shown in Equation 37 with ν = Poisson's ratio. For the sand asphalt mixture described in this paper, the maximum modulus of elasticity was 21,000 psi., $\nu = 0.214$ and $\alpha = 6 \times 10^{-5}$ per deg. Fahrenheit, hence the increase in internal pressure is 0.735 psi. per deg. F. In comparison, the values for portland-cement concrete are $E = 3{,}500{,}000$ psi., $\nu = 0.33$, $\alpha = 14.4 \times 10^{-6}$ per deg. F. as reported by Teller and Sutherland (*11*), and the increase in pressure is 50.4 psi. per deg. F. These values are maximums, and are considerably smaller under actual road conditions. They show, however, that thermal stresses are negligibly small for bituminous pavements and can become critical for portland-cement pavements.

Road structures can be damaged by impact forces when the energy transmitted to the surface is of a vibrational nature. The disintegrating effect of vibrations due to moving traffic is greatly minimized if the road is highly resilient. High resilience is related to a low modulus of elasticity. Since the propagation of the speed of dynamic waves increases with that of the moving traffic, it would appear that increased speed reduces the effect of vibratory disturbances. On the other hand, the road surface can vibrate inharmoniously with the supporting medium, resulting in a sharp plane of discontinuity between the two layers which may cause failure. However, Ramspeck (*12*) showed that, if the elastic properties of the various layers are similar, the road structure will respond as a unit to vibratory motions. Such a behavior can be expected with roads paved with bituminous pavements, since their modulus of elasticity is of the same order as that of the base course and subsoil.

It was shown that bearing strength is not an inherent property of a road material but is acquired as a result of the orientation of the particles which is obtained under prolonged exposure of the material to stresses. Consequently, a freshly laid bituminous pavement is not yet in a state corresponding to that of the bearing strength. The actual load-carrying capacity, i.e., the load which can be carried without causing plastic deformation, is a function of the amount of work done on each layer by rolling during construction. The load-bearing capacity increases with the amount of rolling only if the stress transmitted to the surface bears some relation to the bearing strength of the material at the temperature of rolling. Being of a transient nature this stress can be a multiple of the bearing strength. Under this condition the layer is consolidated, the particles become oriented in the direction of rolling, and the layer is in the region of hardening. If the ratio of the transmitted stress to the bearing strength is large enough so that the layer is in the region of failure, rolling will cause surface cracks which are often observed. The effect of rolling on the increase in the load-bearing capacity and increasing orientation of the particles has its cause in the fact that the strain, obtained after the instantaneous deformation, is the sum of a plastic strain and retarded elastic strain. The plastic strain does not recover on unloading, and since the roller is in contact with the surface during a certain time, repetitive rolling of a given area is equivalent in effect to static loading for a time period equal to the sum of each contact time. Traffic subsequent to the completion of the bituminous road has the same effect, which explains the known increase in strength and density with increasing traffic.

It has been shown that asphaltic-concrete mixtures can be prepared with a bearing strength of the order of 400 psi. Such bituminous pavements can carry loads of great intensity, provided that the substructure has a correspondingly high bearing strength, otherwise the pavement will fail through overstraining. It sometimes is observed that bituminous pavements of high strength laid on proper bases show occasional cracks. This type of failure has its origin in structural flaws such as microcracks. A vertical stress applied to the surface above the flaw results in a stress concentration at the sharp edges of the microcrack. Griffith (*2*) showed that for an elliptic flaw under a stress σ perpendicular to the long axis of the ellipsoid, the stress at the sharp edges is as follows:

$$\sigma_e = \sigma(2a + b)/b \tag{38}$$

where a and b are the long and short axes. Thus for a ratio, $a/b = 10$, the stress concentration factor is 21. Under a load of 40 psi. applied to the surface of the pavement, the stress at the sharp edges would be 840 psi., large enough to crack the pavement along the long axis of the microcrack.

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APPENDIX A

Plastic Deformation and Time-Hardening

It has been shown by experiment that bituminous mixtures under load become harder. The hardening process is the result of the work done on the system and is often referred to as work-hardening. In this case the stress is not only a function of the strain rate but also of another factor which may be the strain (*13*) and

$$\sigma = \sigma(\dot{\epsilon}, \epsilon) \qquad \text{(A-1)}$$

where $\dot{\epsilon} = d\epsilon/dt$ is the strain rate. Differentiating this equation gives

$$d\sigma = (\partial\sigma/\partial\dot{\epsilon})_{\epsilon}\, d\dot{\epsilon} + (\partial\sigma/\partial\epsilon)_{\dot{\epsilon}}\, d\epsilon \qquad \text{(A-2)}$$

where $\partial\sigma/\partial\dot{\epsilon}$ measures the change in stress with changing strain rate at constant strain, and $\partial\sigma/\partial\epsilon$ measures the change in stress with changing strain at constant strain rate. The term $\partial\sigma/\partial\dot{\epsilon}$ has the same dimensions as viscosity and may be termed coefficient of plastic traction. The stress as a function of strain rate is a curve in the case of plastic deformation and the coefficient is determined by the tangent to a point of this curve, hence

$$\partial\sigma/\partial\dot{\epsilon} = b\sigma/\dot{\epsilon} \qquad \text{(A-3)}$$

where $b\sigma$ is the difference between the stress and the intercept of the tangent on the stress coordinate. It will be noticed that $b\sigma$ is different from the actual stress σ, and may be called effective stress.

The term $\partial\sigma/\partial\epsilon$ has the dimensions of a modulus of elasticity. The solution of Equation A-2 depends upon whether or not $\partial\sigma/\partial\epsilon$ is constant or variable at constant stress. With $\partial\sigma/\partial\epsilon$ being variable, this term may in the simplest case be expressed as follows:

$$\partial\sigma/\partial\epsilon = b\sigma/\epsilon \qquad \text{(A-4)}$$

Introducing Equations A-3 and A-4 into Equation A-2 leads on integration between σ and σ_0, $\dot{\epsilon}$ and $\dot{\epsilon}_0$, ϵ and ϵ_0, to

$$\sigma/\sigma_0 = (\dot{\epsilon}\epsilon/\dot{\epsilon}_0\epsilon_0)^b \qquad \text{(A-5)}$$

This equation shows that at constant stress the product of strain rate and strain is constant. Since $\dot{\epsilon} = d\epsilon/dt$, integration of this equation between ϵ and ϵ_0, t and 0 at constant stress gives

$$\epsilon^2 - \epsilon_0^2 = 2\dot{\epsilon}_0\epsilon_0(\sigma/\sigma_0)^{1/b}t \qquad \text{(A-6)}$$

which shows that the square of the plastic strain is a linear function of time independent of the stress. The strain-time curves for each stress applied are therefore parallel to each other. The same holds true for any power factor of the strain other than 2.

With $\partial\sigma/\partial\epsilon$ being constant at constant stress, Equation A-2 becomes for constant stress

$$0 = b\sigma\, d\dot{\epsilon}/\dot{\epsilon} + (\partial\sigma/\partial\epsilon)\, d\epsilon$$

Dividing by $b\sigma\, dt$ and writing $1/\beta$ for $\partial\sigma/b\sigma\,\partial\epsilon$ gives

$$-d\dot{\epsilon}/\dot{\epsilon}\, dt = d\epsilon/\beta\, dt = \dot{\epsilon}/\beta$$

Integrating between $\dot{\epsilon}$ and $\dot{\epsilon}_0$, t and t_0 gives

$$1/\dot{\epsilon} - 1/\dot{\epsilon}_0 = (t - t_0)/\beta$$

from which follows that $1/\dot{\epsilon}_0 = t_0/\beta$, hence

$$\dot{\epsilon} = \beta/t; \qquad \dot{\epsilon}t = \beta = b\sigma\partial\epsilon/\partial\sigma$$

It follows that

$$\partial\sigma/\partial\epsilon = b\sigma/\dot{\epsilon}t = b\sigma/\beta$$

which is constant at constant stress but varies with the stress. Introducing this term into Equation A-2 gives

$$d\sigma = (b\sigma/\dot{\epsilon})\, d\dot{\epsilon} + (b\sigma/\dot{\epsilon}t)\, d\epsilon$$

Since $\dot\epsilon = d\epsilon/dt$, this equation becomes

$$d\sigma = (b\sigma/\dot\epsilon)\, d\dot\epsilon + (b\sigma/t)\, dt$$

Integration between σ and σ_0, $\dot\epsilon$ and $\dot\epsilon_0$, t and 0 leads to

$$\sigma/\sigma_0 = [\dot\epsilon(t + 1)/\dot\epsilon_0]^b \qquad \text{(A-7)}$$

In this case the product of strain rate and time is constant at constant stress and this equation represents time-hardening. With $\dot\epsilon(t + 1) = \beta$ this equation can be written as follows:

$$\sigma/\sigma_0 = (\beta/\beta_0)^b \qquad \text{(A-8)}$$

The strain rate is

$$\dot\epsilon = d\epsilon/dt = (\sigma/\sigma_0)^{1/b}\dot\epsilon_0/(t + 1) = \beta/(t + 1)$$

Integration between ϵ and ϵ_0, t and 0, at constant stress gives

$$\epsilon - \epsilon_0 = \beta \ln (t + 1) \qquad \text{(A-9)}$$

where ln is the natural logarithm and t refers to the time interval of plastic deformation only. This equation shows that at constant stress the plastic strain increases linearly with the logarithm of time and that the parameter β varies with the stress. This condition is a criterion of time-hardening. Any other constant product of strain rate and a function of time at constant stress, such as $\dot\epsilon(t + 1)^c$ for example, would on integration make $\epsilon - \epsilon_0$ at constant stress proportional to $(t + 1)^{1-C}$. This relationship is similar to Equation A-6 and represents therefore strain-hardening.

The amount of hardening which the system receives as a function of stress and time, depends upon the value of b and is most pronounced with negative values of b or values larger than 1.

Experiments are often carried out by loading in such a way that the rate of deformation or the rate of loading are kept constant. For the evaluation of these conditions, Equation A-7 may be written as follows:

$$\sigma/\sigma_0 = (\dot\epsilon t/\dot\epsilon_0 t_0)^b \qquad \text{(A-10)}$$

Considering first the simpler case of constant strain rate, the stress becomes a function of time only. Since $\dot\epsilon = d\epsilon/dt =$ constant, it follows that $\epsilon =$ const. t, hence

$$\sigma/\sigma_0 = (\epsilon/\epsilon_0)^b \qquad \text{(A-11)}$$

For the conditions of constant rate of deformation $-dh/dt =$ constant, hence $\dot\epsilon = d\epsilon/dt = -dh/hdt =$ constant $/h$ and $\dot\epsilon_0 =$ constant $/h'$, where h' and h are the heights observed at stress σ_0 and σ. Further with $-dh/dt =$ constant, it follows that $h_0 - h =$ const. t, and $h_0 - h' =$ const. t_0. Introducing these terms into Equation A-10 gives

$$\sigma/\sigma_0 = [h'(h_0 - h)/h(h_0 - h']^b \qquad \text{(A-12)}$$

When $h_0 - h$ is small in comparison to h, so that $(h_0 - h)/h \approx \epsilon$ Equation A-12 approaches the condition of Equation A-11. It has been shown that bituminous mixtures are so sensitive to the condition of constant rate of deformation that failure will occur long before these mixtures have acquired their maximum strength. Such a test condition does therefore give not much information about the mechanical behavior.

For the case of constant stress rate, σ/t is constant, therefore multiplying Equation A-10 by t_0, dividing by t and substituting β_0 for $\dot\epsilon_0 t_0$ gives

$$1 = (\dot\epsilon/\beta_0)^b t_0/t^{(1-b)}$$

from which follows

$$\dot\epsilon = \beta_0 t^{(1-b)/b}/t_0^{1/b} = \dot\epsilon_0(t/t_0)^{(1-b)/b} \qquad \text{(A-13)}$$

Since $\sigma/t =$ constant $= c$, this equation may be written as follows:

$$\dot\epsilon = \dot\epsilon_0(\sigma/\sigma_0)^{(1-b)/b}$$

For a negative value of b, this equation becomes

$$\dot\epsilon = \dot\epsilon_0(\sigma_0/\sigma)^{(1+b)/b} = (\beta_0 c/\sigma_0)(\sigma_0/\sigma)^{(1+b)/b}$$

which has been used for Equation 8b in the main part of the report. Taking β_0, t_0 and the constant resulting from integration together into one constant C, there results on integration of Equation A-13 between ϵ and 0, t and 0, the following expression for the strain as a function of time or stress:

$$\epsilon = Ct^{1/b} = C(\sigma/c)^{1/b} \qquad \text{(A-14)}$$

which is similar to Equation A-11.

For the condition of constant rate of loading $L/t = c$ Equation A-10 becomes:

$$\frac{\sigma t_0}{\sigma_0 t} = \frac{LA_0 t_0}{L_0 A t} = \frac{A_0}{A} = \left(\frac{\dot\epsilon t}{\dot\epsilon_0 t_0}\right)^b \frac{t_0}{t}$$

The ratio of the areas, A_0/A, is according to Equation 17 in the main text as follows:

$$A_0/A = (r_0/r)^2 = (h/h_0)^{2\nu}$$

where ν is Poisson's ratio. Since $ln\ (h_0/h) = \epsilon$, $(h/h_0)^{2\nu} = \exp\ (-2\nu\epsilon)$. Introducing this term into above equation and rearranging gives

$$\exp\ (2\nu\epsilon/b)\ d\epsilon/dt = \dot{\epsilon}_0(t/t_0)^{(1-b)/b}$$

Integration between ϵ and 0, t and 0, leads to

$$\exp\ (2\nu\epsilon/b) = (2\nu\dot{\epsilon}_0/t_0^{(1-b)/b})t^{1/b} + 1 \quad \text{(A-15)}$$

This equation cannot be readily applied to experimental data. However, when $2\nu\dot{\epsilon}_0/t_0^{(1-b)/b}$ is of such a magnitude that 1 can be neglected at large values of t, then

$$\epsilon \approx (b/2\nu)[\ln 2\nu\dot{\epsilon}_0 - (1-b)/b \ln t_0] + (1/2\nu) \ln t \quad \text{(A-16)}$$

or

$$\epsilon \approx \text{constant} + (1/2\nu) \ln t$$

$$\approx \text{constant}' + (1/2\nu) \ln L$$

and the strain is approximately linear with the natural logarithm of time or of load.

APPENDIX B

Elastic Deformation

It has been shown that the plastic deformation is preceded by an instantaneous elastic deformation and a retarded elastic deformation. The latter is an elastic deformation superimposed upon a plastic deformation. The strains accompanying these deformations are the same. The stress, however, consists of an elastic and plastic part:

$$\sigma = \sigma' + \sigma'' = E'\epsilon' + \sigma'' \quad \text{(B-1)}$$

where σ is the total stress, σ' and σ'' are the elastic and plastic stresses, and E', and ϵ' are the modulus of elasticity and retarded elastic strain for this type of deformation. At the moment of application a constant stress is carried wholly by the plastic part and $\sigma = \sigma''$. Subsequently the elastic deformation takes over part of the stress, and this deformation increases until the maximum elastic strain is obtained. At this point the plastic stress is zero, and $\sigma = \sigma'$. Differentiating Equation B-1 with respect to time gives

$$d\sigma/dt = E'd\epsilon'/dt + d\sigma''/dt = E'\dot{\epsilon}' + d\sigma''/dt - \mu'd\dot{\epsilon}$$

With $d\sigma'' = \mu'\ d\epsilon'$ this equation becomes for $\sigma = \text{constant}$

$$E'\dot{\epsilon}' = -\mu'd\dot{\epsilon}'/dt \quad \text{(B-2)}$$

For the solution of this equation, it is necessary to know the variation of μ' with varying stress σ'' and strain rate $\dot{\epsilon}'$. This information is obtained from the experiment. In the case of bituminous mixtures, it was found that the plastic part of the retarded elastic deformation obeyed the following relationship

$$\dot{\epsilon} = \dot{\epsilon}_0 \exp\ (\sigma - \sigma_0)/b\sigma_0 \quad \text{(B-3)}$$

expressing the strain rate as an exponential function of the stress. The equation requires the substitution of the proper stresses for the retarded elastic deformation. According to Equation B-1, the plastic stress is

$$\sigma'' = \sigma - E'\epsilon',$$

and σ_0 in Equation B-3 corresponding to the strain rate $\dot{\epsilon}_0$ becomes σ hence

$$\dot{\epsilon}' = \dot{\epsilon}_0' \exp\ (\sigma - E'\epsilon' - \sigma)/b\sigma = \dot{\epsilon}_0' \exp\ (-E'\epsilon'/b\sigma) \quad \text{(B-4)}$$

Taking logarithms and differentiating with respect to $\dot{\epsilon}'$ gives the following expression for the coefficient μ'

$$\mu' = d\sigma''/d\dot{\epsilon}' = -E'd\epsilon'/d\dot{\epsilon}' = -d\sigma'/d\dot{\epsilon}' = b\sigma/\dot{\epsilon} \quad \text{(B-5)}$$

Introducing this term into Equation B-2 gives on rearranging:

$$-d\dot{\epsilon}'/\dot{\epsilon}'^2 = E'dt/b\sigma = dt/\alpha \quad \text{(B-6)}$$

where $b\sigma/E' = \alpha$ is a constant at constant stress with the dimension of strain. Integration between $\dot{\epsilon}'$ and $\dot{\epsilon}_0'$, t and t_0, gives

$$1/\dot{\epsilon}' - 1/\dot{\epsilon}_0' = (t - t_0)/\alpha \quad \text{(B-7)}$$

It follows immediately that $1/\dot{\epsilon}_0' = t_0/\alpha$ hence

$$\dot{\epsilon}' = \alpha/t \quad \text{(B-8)}$$

Integrating again between ϵ' and ϵ_0', t and 0 leads to

$$\epsilon' - \epsilon_0' = \alpha \ln\ (t + 1) \quad \text{(B-9)}$$

which is similar to Equation A-9 expressing the plastic strain as a function of time for time-hardening.

APPENDIX C

Kinetics of Plastic Deformation

A compacted bituminous mixture can be conceived as containing a number of particles acting as units of flow whose axes form any angle between 0 and 180 deg. with the horizontal. The particles with zero angles are in regions of greater density and are attached to their neighbors by strong cohesive forces. The remaining particles are, on account of their angle of position being larger than zero, in regions of lower density and weaker cohesive forces. This arrangement is similar to that in ordinary solids, where the particles in regions of greater density are in a state of order and the particles in regions of lower density in a state of disorder. For reasons of simplicity, this terminology will be applied in the following. Such systems can be visualized as containing N sites or cells which are occupied by the particles. Part of these particles, qN, occupies the disordered sites where q is the degree of disorder or fraction of the particles in the disordered state, while the remainder, $(1 - q)N$, occupies the ordered sites. The sites of order and disorder are assumed to be distributed at random so that the system is as isotropic as a simple liquid from a macroscopic point of view. The essential difference between the ordered and disordered state in bituminous mixtures is that the particles in the ordered state are in positions of minimum potential energy, whereas the particles in the disordered state are considered to be distributed over various levels of greater potential energy.

In a region of stress which has the lower yield value as the upper limit, the system is deformed elastically. The particles change slightly their positions but each particle retains its neighbors. At stresses in excess of this yield value, the particles in the ordered state rarely leave their positions, since, on account of their zero angle of position, the shearing stress associated with them is zero, according to Equation 11, in the main text. The particles in the disordered state are subjected to shear and move to new positions. Such a particle can move over a barrier of potential energy only if under the influence of interaction with its neighbors it acquires a kinetic energy equal to the magnitude of the barrier. The barrier of smallest magnitude is the one associated with the lower yield value. Since the strain rate decreases with increasing stress and time in the region of hardening, some of the particles in the disordered state, after having passed the energy barrier, lose their kinetic energy again under the influence of their new neighbors and move to positions of lower potential energy, i.e., to positions of order.

In order that a particle may move from one position to another, it is necessary that space be provided. It is agreed that all matter, solid or liquid, contains a number of vacancies or holes, which represent the unoccupied sites of the system. In bituminous mixtures these vacancies are provided by the voids. Let V be the volume of the system, V_s the effective volume of the particles, n the number of the vacancies, v_h the average volume of a vacancy, then

$$nv_h = V - V_s$$

The volume of a particle is a multiple of the volume of a vacancy, $v_s = cv_h$. With the particles occupying a volume $V_s = Ncv_h$, the ratio of the number of vacancies to that of the particles becomes

$$n/N = c(V - V_s)/V_s = \phi \qquad \text{(C-1)}$$

For flow to occur the energy barrier or the cohesive forces between the particles must be overcome by an externally applied force, and for simple liquids the viscosity is a measure of these cohesive forces. Since plastic deformation in the hardening range is caused by the motion of the particles in the disordered state which are associated with weaker cohesive forces, the coefficient of plastic traction is a measure of these weaker forces. Further, this coefficient varies with stress and time and, therefore, with the number of particles in the disordered state. This suggests that the coefficient depends not only on the magnitude of the weak cohesive forces but also on the number of particles associated with these forces. A variation in the number of particles in the disordered state brings about a variation in the number of vacancies, hence the coefficient of plastic traction is a function of the number of particles in the disordered state and the number of vacancies, viz.,

$$\mu = \mu(q, \phi) \qquad \text{(C-2)}$$

For an understanding of the process of plastic deformation, the energies associated with it must be known. A formulation of these energies can be based on the two accessible quantities, stress and coefficient of plastic traction. For this purpose use is made of the complete Stokes-Navier differential equations of motion. They cannot be directly solved when subject to boundary conditions of such complexity as encountered with plastic flow. However, by considering a particle of average mass m moving in one direction with a velocity v, the following simplification is obtained:

$$f - (f_1 + f_2 + B) = m\, dv/dt \qquad \text{(C-3)}$$

where $m\, dv/dt$ is mass times acceleration and is a force due to inertia. The term f is the force acting on the particle, f_1 is the frictional force due to a shear resistance to flow, and f_2 is the frictional force due to a resistance to a change in volume and is associated with the isotropic or volume viscosity. Finally B represents body forces which consist of the negligible force due to gravitational attraction and of the more important cohesive forces. The last three forces act against f and may be combined into one term ψv, where ψ is the sum of frictional and body forces per unit velocity. There results then

$$f - \psi v = m\, dv/dt \qquad \text{(C-4)}$$

Multiplying by $d\lambda$ the distance travelled by a particle gives with $d\lambda/dt = v$, the following expression for the work done as a function of the kinetic energy

$$f\, d\lambda = \psi v\, d\lambda + mv\, dv = v\psi\, d\lambda + dE_k \qquad \text{(C-5)}$$

For the solution of Equation C-4, the case of a Newtonian liquid may be considered first, for which ψ is constant and proportional to to the viscosity. The result may then be applied to plastic flow by analogy. Integration between v and 0, t and 0, at f = constant gives

$$\ln f - \ln(f - \psi v) = \psi t/m$$

from which follows

$$\psi v = f[1 - \exp(-\psi t/m)] \qquad \text{(C-6)}$$

The exponential term tends towards a value of zero the more rapidly the larger the value of ψ or the smaller the value of m. Thus, for a simple liquid, the velocity of a particle increases rapidly to a constant value, hence

$$f = \psi v$$

i.e., the force acting on a particle is equal to the frictional force. This is in agreement with the idea that any work (force times distance) done on a simple liquid in the steady state is work done against the viscous resistance to flow, and the change in kinetic energy is zero according to Equation C-5.

In the case of plastic deformation, ψ is variable and with $b\sigma$ being the effective stress ψv becomes equal to bf, hence

$$f - \psi v = (1 - b)f = m\, dv/dt \qquad \text{(C-7)}$$

$$f\, d\lambda = bf\, d\lambda + mv\, dv = bf\, d\lambda + dE_k \qquad \text{(C-8)}$$

Here the work consists in part of the work done against the viscous and other resistances, $bf\, d\lambda$, and the remainder appears as kinetic energy. It follows also on comparing Equations C-7 and C-3 that b contains parameters related to the frictional forces due to shear and volume viscosity and due to the body force.

For the evaluation of these parameters, Equation C-2 is differentiated as follows:

$$d\mu = (\partial\mu/\partial q)_\phi\, dq + (\partial\mu/\partial\phi)_q\, d\phi \qquad \text{(C-9)}$$

The coefficient of plastic traction at constant ϕ depends upon the number of particles in the disordered state and the stress acting on these particles may be assumed to be equal to the product of q and stress, $q\sigma$ in the simplest case. For other cases this stress may be expressed as the product of σ and a function of q, $\sigma f(q)$. For the sake of simplicity, the term q will be used since the following derivations lead to the same result irrespective of the type of the function of q. With b being constant, the coefficient of plastic traction at constant ϕ becomes then

$$\mu_\phi = q\sigma/\dot{\epsilon} = q\mu/b \qquad \text{(C-10)}$$

The coefficient at constant q varies with the variation in the number of vacancies in such a manner that this coefficient decreases with an increase in volume and vice versa. This condition is satisfied if the coefficient at constant q and constant temperature is inversely proportional to a function of the volume of the vacancies, $V_s f(\phi/c) = V_s g(\phi)$

$$\mu_q = C/V_s g(\phi) \qquad \text{(C-11)}$$

where C is a proportionality factor. Equation C-11 also satisfies Equation C-9, which indicates that a positive dq must be associated with a negative $d(\phi)$, and vice versa. To eliminate the proportionality factor, the coefficient is referred to a coefficient of plastic traction μ_0 at a state so close to that of rest that q_0 and $g(\phi)$ can be considered to be constant, $\mu_0 = C/V_s g(\phi_0)$. With these stipulations, the coefficient at constant q is as follows:

$$\mu_q = \mu_0 g(\phi_0)/g(\phi) \tag{C-12}$$

Combining the proper derivatives of Equations C-10 and C-12 with Equation C-9 gives on integration between μ and μ_0, q and q_0, $g(\phi)$ and $g(\phi_0)$, keeping b constant

$$\mu = q\mu/b - [q_0/b - g(\phi_0)/g(\phi)]\mu_0 \tag{C-13}$$

Differentiation of this equation with respect to μ gives

$$1 = q/b + (\mu/b)dq/d\mu - [\mu_0 g(\phi_0)/(g(\phi))^2]\, d\phi/d\mu$$

from which follows that at $\mu = \mu_0$, $g(\phi) = g(\phi_0)$ and

$$b = q_0$$

Introducing this term into Equation C-13 gives on rearranging

$$b = q_0 = g(\phi)\left[\frac{q - q_0}{g(\phi) - g(\phi_0)}\right]\frac{\mu}{\mu_0}$$

Taking logarithms and expanding into a series gives

$$\ln b = \ln g(\phi) + \ln \mu/\mu_0 + q - q_0 - 1$$
$$- \tfrac{1}{2}(q - q_0 - 1)^{1/2} + \cdots$$
$$- [g(\phi) - g(\phi_0) - 1] + \cdots$$

from which follows that at $\mu = \mu_0$, $\ln b = \ln g(\phi_0)$, hence

$$b = q_0 = g(\phi_0)$$
$$= q - \frac{g(\phi_0)}{g(\phi)}[g(\phi) - g(\phi_0)]\frac{\mu_0}{\mu} \tag{C-14}$$

This equation defines the parameter b in terms of q and $g(\phi)$.

Equation C-13 may then be written as follows:

$$\mu = b\sigma/\dot{\epsilon} = q\mu/b + [g(\phi_0) - g(\phi)]\mu_0/g(\phi) \tag{C-15}$$

indicating that the coefficient of plastic traction is a composite of two terms, the first one refers to the motion of the particles in the disordered state, and the second represents the volume viscosity. During plastic deformation accompanied by time-hardening, the particles in the disordered state shift to positions of order. Hence, the value of q decreases to zero at a stress σ_m, and at this point the coefficient of plastic traction is equal to the volume viscosity, i.e., at stress σ_m there is no slip which coincides with a value of the angle of internal friction of 90 deg., as discussed in the main text in connection with the distribution of stresses. It follows also from Equation C-15 that as q decreases with increasing μ, $g(\phi)$ also decreases and the process of time hardening is accompanied by a decrease in the number of vacancies present and therefore by a densification of the bituminous mixture under test. At stresses in excess of σ_m, q can increase again and the deformation can lead to failure.

Multiplying Equation C-15 by $\dot{\epsilon}$ and substituting $1 - (1 - q)$ for q gives

$$\mu\dot{\epsilon} = b\sigma = \sigma + [g(\phi_0) - g(\phi)]\mu_0\dot{\epsilon}/g(\phi) - (1 - q)\sigma \tag{C-16}$$

This equation represents the sum of the frictional and body forces per unit area mentioned in Equation C-3. The first and the second term are the frictional forces due to a resistance to shear and to a change in volume, and since $1 - q$ is the degree of order, the last term represents the strong cohesive forces.

These forces are related to the energies associated with plastic deformation. The change in internal energy in going from a state of rest to a state of stress is as follows:

$$\Delta U = W + T\Delta S = \Delta F - \Delta(PV) + T\Delta S$$

where U = internal energy, W = work done, T = absolute temperature, S = entropy, F = Gibbs' free energy, P = pressure, V = volume, PV = potential energy. It follows that

$$\Delta F = W + \Delta(PV) \tag{C-17}$$

The work done is represented by Equation C-8. Multiplying and dividing this equation by a, the area of a particle on which the force acts gives with $f/a = \sigma$, $a\lambda = v_f$, the volume of flow, the following expression for the work done in going from a state of rest to a state of stress

$$W = (\sigma - \sigma_0)v_f = b(\sigma - \sigma_0)v_f + \tfrac{1}{2}mv^2$$
$$= b(\sigma - \sigma_0)v_f + \Delta E_k \tag{C-18}$$

where σ_0 refers to the yield value or to a stress so close to it that q and $g(\phi)$ can be considered to be constant. The velocity of a particle at this stress is either zero or so small that its kinetic energy is negligible. Substituting in the above equation the value of $b\sigma$ from Equation C-16 gives

$$W = (\sigma - \sigma_0)v_f = (\sigma - \sigma_0)v_f - \{(1 - q)\sigma - (1 - q_0)\sigma_0 - [g(\phi_0) - g(\phi)]\mu_0\dot{\epsilon}/g(\phi)\}v_f + \Delta E_k \quad \text{(C-19)}$$

The second expression on the right side is the work related to the stronger cohesive forces and to the change in volume and is therefore a change in potential energy, hence

$$W = W - \Delta(PV) + \Delta E_k$$

and

$$\Delta(PV) = \Delta E_k \quad \text{(C-20)}$$

This result is in agreement with the postulate that a particle can pass the barrier of potential energy only if it acquires a kinetic energy equal to the magnitude of the barrier. It follows from Equation C-17 and C-20 that

$$\Delta F = W + \Delta(PV) = W + \Delta E_k$$

and from C-18

$$\Delta F = (\sigma - \sigma_0)v_f + (1 - b)(\sigma - \sigma_0)v_f = (2 - b)(\sigma - \sigma_0)v_f \quad \text{(C-21)}$$

In cases where the plastic deformation passes through various regions with different values of b, Equation C-21 becomes

$$\Delta F = [2(\sigma - \sigma_0) - (b\sigma - b_0\sigma_0)]v_f \quad \text{(C-22)}$$

The free energy change is obtained as a function of the coefficient of plastic traction by considering plastic deformation as a rate process.

The shift of particles from positions of disorder to those of order is a kinetic process. The particles move in the direction of the force applied and the vacancies in the opposite direction. Each process is carried out with the same rate and a condition of equilibrium is established. This is shown by rearranging Equation C-15 as follows, taking into consideration that $b = q_0 = g(\phi_0)$:

$$(q_0 - q)/\mu_0 = [g(\phi_0) - g(\phi)]g(\phi_0)/g(\phi)\mu$$

The coefficients of plastic traction can be replaced by $\mu_0 = E_0\bar{t}_0$ and $\mu = E\bar{t}$ according to Equation 1 of the main text. The moduli of elasticity are assumed to be variable on account of the structural and volume changes taking place in a bituminous mixture under stress. The above equation becomes then

$$(q_0 - q)/\bar{t}_0 = [g(\phi_0) - g(\phi)]g(\phi_0)E_0/g(\phi)E\bar{t} \quad \text{(C-23)}$$

Since $q_0 - q = (1 - q) - (1 - q_0)$, this equation shows that the rate of decrease in the concentration of the particles in the disordered state or rate of increase in the concentration of the particles in the ordered state is equal to the variation of a function of the concentration of the vacancies with time. The specific rate constants are

$$k' = 1/\bar{t}_0 = E_0/\mu_0 ; \quad k'' = E_0/E\bar{t} = E_0/\mu \quad \text{(C-24)}$$

It is interesting to note that k' is independent of the stress, whereas k'' varies with the stress. This suggests that plastic deformation is primarily brought about by motion of the vacancies. The equilibrium constant is

$$K = k'/k'' = \mu/\mu_0 = \exp(\Delta F/kT) \quad \text{(C-25)}$$

where ΔF is the change in free energy of activation for a particle going from one state to another and its accompanying change in volume, k = Boltzmann's constant and T = absolute temperature. This equation in combination with Equations C-21 or C-22 serves for the calculation of the kinetic energy and mass of a particle.

It has been shown that the retarded elastic strain is a similar function of time at constant stress as the plastic deformation which suggests that the processes involved are also similar for both types of deformation. According to Equation B-5, the coefficient of plastic traction associated with the retarded elastic deformation is as follows:

$$\mu' = b\sigma/\dot{\epsilon}'$$

and shows that this coefficient is independent of the plastic stress and varies only with the strain rate. It can, therefore, be concluded that the parameter b is variable in such a manner that $b'\sigma' = b\sigma$ and $b' = b\sigma/\sigma'$. It can be shown (*14*) that for such systems $g(\phi_0) = g(\phi)$, $b' = q$, $b = q_0$ hence $E'\epsilon' = \sigma' = b\sigma/b' =$

$b\sigma/q$. Introducing this term into Equation B-9 gives

$$(\epsilon' - \epsilon_0')/\alpha = E'(\epsilon' - \epsilon_0')/b\sigma \\ = 1/q - 1/q_0 = \ln (t + 1) \qquad \text{(C-26)}$$

Hence, for the retarded elastic deformation at constant total stress, $1/q$ increases or q decreases with increasing strain and time. Therefore, the ordering of the particles is not restricted to the plastic deformation but starts with the onset of the retarded elastic deformation.

APPENDIX D

Cohesion of Bituminous Mixtures

The geometrical shape and irregular surfaces of mineral particles make a quantitative description of the attractive forces resulting in cohesion difficult. For reasons of simplification two parallel plates are considered to be immersed vertically in a liquid which wets the plates (Fig. A). When a, the distance between the two plates, is small, the liquid rises and forms a surface at the contact with air, of which one radius of curvature is $a/2$ and the other is infinitely large. The capillary pressure is then

$$p = S(1/a/2 + 1/\infty) = 2S/a \qquad \text{(D-1)}$$

where S = surface tension in dynes per cm. The hydrostatic pressure of the liquid column in dynes per sq. cm. is equal to the product of height, h, density, ρ, and the gravitational constant g, viz., $h\rho g$. Since the capillary pressure and hydrostatic pressure are in equilibrium, they must be equal and

$$2S/a = h\rho g; \quad h = 2S/\rho g a \qquad \text{(D-2)}$$

The plates are considered to move freely and the liquid not only rises between the plates but also pulls them toward each other. If the width of the plates is w, the rise of the liquid dh reduces the area of the two plates above the meniscus by $2\,wdh$ and the surface of the meniscus by $\pi wda/2$. Hence the change in free energy is

$$-dF = -Sw(2dh + \pi da/2) \qquad \text{(D-3)}$$

The rise of the liquid increases its potential energy in the gravitational field. As the center of gravity of the liquid rises by $dh/2$, when

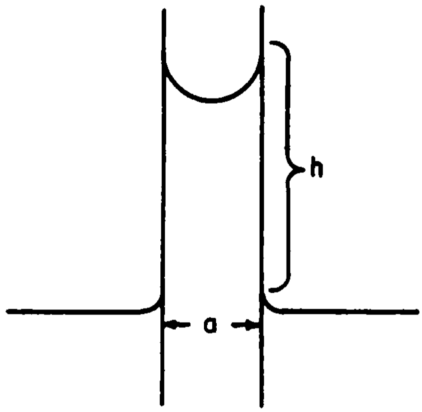

Figure A. Capillary attraction between two plates.

the meniscus rises by dh, the change in potential energy becomes

$$dE_{pot} = 2Sw\,dh/2 = Sw\,dh \qquad \text{(D-4)}$$

The sum of both energy changes is the change in the work done and

$$-dF + dE_{pot} = dW \\ = -Sw\,dh - Sw\,da/2 \qquad \text{(D-5)}$$

Differentiating Equation D-2 gives

$$dh = \frac{-2S}{ga^2}\,da$$

which on introduction into Equation D-5 results in

$$dW = \frac{2S^2w}{ga}\,da - \frac{\pi}{2}\,Sw\,da$$

Dividing by da gives the force with which the plates are pulled together or that which is necessary to pull them apart by a distance da:

$$\frac{dW}{da} = \frac{2S^2w}{\rho g a^2} - \frac{\pi}{2}\,Sw$$

Since a is considered to be small, $0.5\ \pi Sw$ can be neglected without introducing a serious error. The capillary force then becomes

$$f = 2S^2w/\rho g a^2 \qquad \text{(D-6)}$$

APPENDIX E

Nomenclature

a = area of a particle; thickness of a film
b = parameter related to plastic flow

c = constant
exp = exponential
f = force
g = gravitational constant
$g(\phi)$ = function of the ratio of number of vacancies to number of particles
h = height
k = Boltzmann's constant
k', k'' = specific rate constant
ln = natural logarithm = 2.303 log
m = mass of a particle
n = number of vacancies per unit volume
p = pressure
q = fraction of particles in the disordered state
r = radius
t = time
$\bar{t}$ = relaxation time
v = velocity
v with subscripts = volume
w = width
z = depression under bearing plate
A = area
B = body force
C = constant
E = Young's modulus of elasticity
E_k = kinetic energy
F = Gibbs' free energy
G = modulus of rigidity
K = bulk modulus; equilibrium rate constant
L = load
N = number of particles per unit volume
P = pressure
S = entropy; surface tension
T = absolute temperature in degree Kelvin
U = internal energy
V = volume
W = work
α = parameter related to retarded elastic deformation; coefficient of cubical expansion
β = parameter related to plastic deformation
ϵ = principal strain
$\dot{\epsilon}$ = $d\epsilon/dt$ = strain rate
η = coefficient of viscosity
θ = angle
λ = distance moved by a particle
μ = coefficient of plastic traction
ν = Poisson's ratio
ρ = density
σ = principal or normal stress
τ = shearing stress
φ = angle of internal friction
Δ = difference
Σ = sum
ϕ = ratio of number of vacancies to number of particles
ψ = sum of frictional and body forces per unit velocity.