7. LAMBE, T. W., 1952, "Differential Thermal Analysis," Highway Research Board Proceedings *31:* 621-641.
8. LAMBE, T. W. AND MARTIN, R. T., 1953, "Composition and Engineering Properties of Soil," Highway Research Board Proceedings *32:* 576-590.
9. LAMBE, T. W., 1954 "The Improvement of Soil Properties with Dispersants," presented at Boston Society of Civil Engineers, Boston, Massachusetts.
10. LAMBE, T. W. AND MARTIN, R. T., 1954, "Composition and Engineering Properties of Soil (II)," Highway Research Board Proceedings *33:* 515-532.
11. MARSHALL, C. E. ET AL., 1942, Soil Science *54:* 149-158.
12. MARTIN, R. T., 1954, "Ethylene Glycol Retention by Clays," (in press) Journal of Soil Science Society of America.
13. MARTIN, R. T., 1954, "Reference Chlorite Characterization for Chlorite Identification in Soil Clays," presented before National Clay Minerals Conference, Houston, Texas.
14. ROSS, C. S. AND HENDRICKS, S. B., 1945, "Minerals of the Montmorillonite Group," U. S. Department of the Interior Geological Survey Professional Paper, 205B.
15. "The Behavior of Soils Under Dynamic Loadings, 2. Interim Report on Wave Propagation and Strain-Rate Effect," M.I.T. Soil Mechanics Laboratory, July, 1953.
16. "Final Report, Soil Solidification Research—Vol. 2, Fundamental Properties, Clay—Water Systems," Cornell University, Ithaca, New York, September, 1951.
17. "Preliminary Reports Reference Clay Minerals—American Petroleum Institute Project 49," Columbia University, New York, New York, 1951.

Distributed Loads on Elastic Foundations: The Uniform Circular Load

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FORMULAS are presented for computing the stresses, strains, and deflections produced at a point within a semi-infinite, homogeneous, elastically isotropic body by a uniform circular load applied to its surface. The expressions presented are obtained principally by extending and developing the work of A. E. H. Love. Reference is made also to an earlier work by Kwan-ichi Terazawa who obtained a solution by a method different from Love's. Some discussion is also included on the relation between the two methods. It is suggested that the formulas may be useful in developing theoretical concepts relating to the design of airfield pavements.

● THIS paper presents the results of certain mathematical studies (*1*) made in connection with projects for improving the present methods for the design of airfield pavements. In computing the theoretical values of the stresses, strains, and deflections produced in the pavement or subgrade by the loads on the airplane wheels, it was found convenient to assume the tire contact area as circular and to employ formulas for a uniform circular load from the theory of elasticity. Where the point at which such values are to be computed is taken directly beneath the center of the circular area, the computations may be made from relatively simple formulas. However, for points not under the center, the formulas are much more complex and in some cases had not been completely developed.

Values at such "offset" points can be obtained graphically by means of charts developed by Professor N. M. Newmark (*4*, *5*) but to obtain a more complete understanding of the pattern of stress distribution beneath the plane's wheels, it was considered desirable to employ mathematical functions. The most complete treatment of the uniform circular load is that of A. E. H. Love (*2*) and it is with his work that this paper largely concerns itself. Some reference will, however, also be made to an earlier work by Kwan-ichi Terazawa (*6*).

Love's paper is well known and has been widely quoted in the various treatises dealing with the theory of elasticity, but the formulas presented therein are too complex for computational use by the average engineer and, in some cases, are not completely developed. So far as is known, no statement of Love's findings has heretofore been presented in more detailed form. Love presents stress formulas in the form of partial derivatives using cylindrical coordinates for the case of the uniform circular load on a semi-infinite, homogeneous, isotropic, elastic body. These are reproduced here in column B of table 1.

In his discussion of the uniform circular load, Love refers to paragraph 1, page 379, where formulas are presented for the general case of a uniform load distributed over any area. These formulas are expressed in rectangular coordinates and are reproduced here in column A of table 1. Love expresses the various stresses and deflections in terms of the first and second partial derivatives of χ (the Boussinesq three-dimensional logarithmic potential) and V (Newton's potential of a surface distribution).

These two functions are given by the double integrals

$$\chi = p \iint \log \left[z + \sqrt{(x-x')^2 + (y-y')^2 + z^2}\right] dx'\, dy'$$

$$V = p \iint \frac{dx'\, dy'}{\sqrt{(x-x')^2 + (y-y')^2 + z^2}}$$

where x, y, and z are taken as the coordinates of the point within the solid at which the stress or deflection is to be expressed and x', y', and 0 as the coordinates of the point on the surface at which the elemental pressure $pdx'dy'$ is applied.

It can be shown by differentiating under the integral sign that V equals $\partial\chi/\partial z$. Values for the first and second partial derivatives of χ and V with respect to $\rho(\partial\chi/\partial\rho$, $\partial^2\chi/\partial\rho^2$, $\partial V/\partial\rho$, and $\partial^2 V/\partial\rho^2)$ can be obtained by applying Green's theorem, (*7*, pages 191–192) integrating by parts, and substituting an identity for the value of the trigonometric function of the numerator.

The resulting parts are integrated separately with some of the terms being the standard elliptic integral forms. Values for the first and second derivatives of V with respect to $z(\partial V/\partial z$ and $\partial^2 V/\partial z^2)$ can be found from the values of the derivatives of χ and V with respect to ρ in conjunction with the relations:

$$\nabla^2 \chi = 0; \qquad \nabla^2 V = 0; \qquad \frac{\partial\chi}{\partial z} = V$$

where ∇^2 denotes the operation:

$$\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}$$

The methods employed by Love in evaluating the preceding derivatives of χ and V with respect to ρ and z are not adaptable to the integration of $\partial\chi/\partial z$ and this step had to be accomplished by other means. Love states on page 398 that "an expression for V as an elliptic integral is known"; however, he does not present it in his paper. A value for $\partial V/\partial z$ as a single integral, obtained from the relation $\nabla^2 \quad = 0$, was integrated to give the value for V. This is shown on line 23, table 2. This value was later confirmed by transformations from Terazawa's terms.

It will be noted from the terms in columns A and B of table 1 that Love employs Lame's elastic constants λ and μ. These have been translated to terms containing the more familiar Poisson's ratio ν and modulus of elasticity (E_m). The translated terms are shown in column C of table 1 and the substitutions made in each case are shown in the notations listed in table 4.

The derivative terms required for the expression of the various stresses, strains, and deflections, are presented in column D of table 2. This is simply a convenient tabulation of the terms employed by Love. In columns E, F, G, and H of table 2, the values of the derivatives are given in terms of elliptic integrals for four cases: ρ less than r, ρ more than r, ρ equal to r, and ρ equal to 0. The numbers

TABLE 1
BASIC DIFFERENTIAL EQUATIONS FROM THE WORK OF A. E. H. LOVE

Line Number	COLUMN A		COLUMN B		COLUMN C	
	Love's Terms	Load distributed over any area Love page 379, formulas 1.1 (5 and 7)	Love's Terms	Load distributed over a circular area Love page 390, formulas 3.1 (4 and 5)	WES Terms	Terms obtained from those shown in Column B by substituting constants of elasticity shown on Table 4
1	$\widehat{xx}$	$\frac{1}{2\pi}\left[\frac{\lambda}{(\lambda+\mu)}\frac{\partial V}{\partial z}-\frac{\mu}{\lambda+\mu}\frac{\partial^2\chi}{\partial x^2}-z\frac{\partial^2 V}{\partial x^2}\right]$	$\widehat{\rho\rho}$	$\frac{1}{2\pi}\left[\frac{\lambda}{(\lambda+\mu)}\frac{\partial V}{\partial z}-\frac{\mu}{(\lambda+\mu)}\frac{\partial^2\chi}{\partial \rho^2}-z\frac{\partial^2 V}{\partial \rho^2}\right]$	σ_ρ	$\frac{1}{2\pi}\left[2\nu\frac{\partial V}{\partial z}-(1-2\nu)\frac{\partial^2\chi}{\partial\rho^2}-z\frac{\partial^2 V}{\partial\rho^2}\right]$
2	$\widehat{yy}$	$\frac{1}{2\pi}\left[\frac{\lambda}{(\lambda+\mu)}\frac{\partial V}{\partial z}-\frac{\mu}{\lambda+\mu}\frac{\partial^2\chi}{\partial y^2}-z\frac{\partial^2 V}{\partial y^2}\right]$	$\widehat{\omega\omega}$	$\frac{1}{2\pi}\left[\frac{\lambda}{(\lambda+\mu)}\frac{\partial V}{\partial z}-\frac{\mu}{(\lambda+\mu)}\frac{1}{\rho}\frac{\partial\chi}{\partial \rho}-z\frac{1}{\rho}\frac{\partial V}{\partial \rho}\right]$	σ_θ	$\frac{1}{2\pi}\left[2\nu\frac{\partial V}{\partial z}-(1-2\nu)\frac{1}{\rho}\frac{\partial\chi}{\partial\rho}-z\frac{1}{\rho}\frac{\partial V}{\partial\rho}\right]$
3	$\widehat{zz}$	$\frac{1}{2\pi}\left[\frac{\partial V}{\partial z}-z\frac{\partial^2 V}{\partial z^2}\right]$	$\widehat{zz}$	$\frac{1}{2\pi}\left[\frac{\partial V}{\partial z}-z\frac{\partial^2 V}{\partial z^2}\right]$	σ_z	$\frac{1}{2\pi}\left[\frac{\partial V}{\partial z}-z\frac{\partial^2 V}{\partial z^2}\right]$
4	$\widehat{zx}$	$-\frac{1}{2\pi}z\frac{\partial^2 V}{\partial z\partial x}$	$\widehat{\rho z}$	$-\frac{z}{2\pi}\frac{\partial^2 V}{\partial\rho\partial z}$	$\tau_{\rho z}$	$-\frac{z}{2\pi}\frac{\partial^2 V}{\partial\rho\partial z}$
5	u	$-\frac{1}{4\pi}\left[\frac{1}{\lambda+\mu}\frac{\partial\chi}{\partial x}+\frac{z}{\mu}\frac{\partial V}{\partial x}\right]$	u_ρ	$-\frac{1}{4\pi}\left[\frac{1}{\lambda+\mu}\frac{\partial\chi}{\partial\rho}+\frac{z}{\mu}\frac{\partial V}{\partial\rho}\right]$	w_ρ	$-\frac{(1+\nu)}{2\pi E_m}\left[(1-2\nu)\frac{\partial\chi}{\partial\rho}+z\frac{\partial V}{\partial\rho}\right]$
6	v	$-\frac{1}{4\pi}\left[\frac{1}{\lambda+\mu}\frac{\partial\chi}{\partial y}+\frac{z}{\mu}\frac{\partial V}{\partial y}\right]$			w_θ	0
7	w	$\frac{1}{4\pi\mu}\left[\frac{\lambda+2\mu}{\lambda+\mu}V-z\frac{\partial V}{\partial z}\right]$	u_z	$\frac{1}{4\pi\mu}\left[\frac{\lambda+2\mu}{\lambda+\mu}V-z\frac{\partial V}{\partial z}\right]$	w_z	$+\frac{(1+\nu)}{2\pi E_m}\left[2(1-\nu)V-z\frac{\partial V}{\partial z}\right]$

ADDITIONAL TERMS

Line Number	COLUMN A_1		COLUMN B_1	COLUMN C_1
	WES Terms	Formulas from theory of elasticity (Timoshenko) and integrals for force summations	Terms obtained from Column A_1 by substituting constants of elasticity shown on Table 4 and terms of Column A	Terms obtained from Column A_1 or B_1 (ν = 0.5)
8	ϵ_ρ	$\frac{\sigma_\rho - \nu(\sigma_\theta + \sigma_z)}{E_m}$	$-\frac{(1+\nu)}{2\pi E_m}\left[(1-2\nu)\frac{\partial^2 X}{\partial\rho^2} + z\frac{\partial^2 V}{\partial\rho^2}\right]$	$-\frac{3z}{4\pi E_m}\frac{\partial^2 V}{\partial\rho^2}$
9	ϵ_θ	$\frac{\sigma_\theta - \nu(\sigma_\rho + \sigma_z)}{E_m}$	$-\frac{(1+\nu)}{2\pi E_m}\left[(1-2\nu)\frac{1}{\rho}\frac{\partial X}{\partial\rho} + z\frac{1}{\rho}\frac{\partial V}{\partial\rho}\right]$	$-\frac{3z}{4\pi E_m}\frac{1}{\rho}\frac{\partial V}{\partial\rho}$
10	ϵ_z	$\frac{\sigma_z - \nu(\sigma_\rho + \sigma_\theta)}{E_m}$	$\frac{(1+\nu)}{2\pi E_m}\left[(1-2\nu)\frac{\partial V}{\partial z} - z\frac{\partial^2 V}{\partial z^2}\right]$	$-\frac{3z}{4\pi E_m}\frac{\partial^2 V}{\partial z^2}$
11	e	$\epsilon_\rho + \epsilon_\theta + \epsilon_z$	$\frac{(1+\nu)(1-2\nu)}{\pi E_m}\frac{\partial V}{\partial z}$	0
12	$\Theta = \sigma_{vol}$	$\sigma_\rho + \sigma_\theta + \sigma_z$	$\frac{1}{\pi}(1+\nu)\frac{\partial V}{\partial z}$	$\frac{3}{2\pi}\frac{\partial V}{\partial z}$
13	$2\pi\int_0^{\rho_1}\rho\sigma_z d\rho$	$\int_0^{\rho_1}\rho\left[\frac{\partial V}{\partial z} - z\frac{\partial^2 V}{\partial z^2}\right]d\rho$		$\left[\rho z\frac{\partial V}{\partial\rho} - \rho\frac{\partial X}{\partial\rho}\right]_0^{\rho_1}$
14	$2\pi\int_{z_1}^{z_2}\rho\tau_{\rho z}dz$	$-\rho\int_{z_1}^{z_2} z\frac{\partial^2 V}{\partial\rho\partial z}dz$		$\left[-\rho z\frac{\partial V}{\partial\rho} + \rho\frac{\partial X}{\partial\rho}\right]_{z_1}^{z_2}$
15	$\int_0^{\rho_1}\tau_{z\rho}d\rho$	$-\frac{z}{2\pi}\int_0^{\rho_1}\frac{\partial^2 V}{\partial\rho\partial z}d\rho$		$-\frac{z}{2\pi}\left[\frac{\partial V}{\partial z}\right]_0^{\rho_1}$

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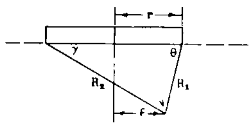
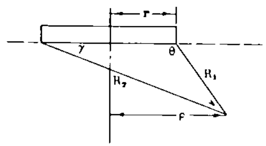
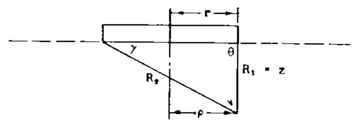
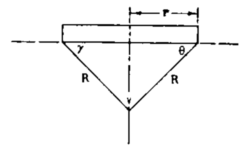
TABLE 2
EVALUATION OF DERIVATIVES

Line Number	COLUMN D	COLUMN E		COLUMN F		COLUMN G	COLUMN H
	Derivatives	Love's Formula No.	Case I $\rho < r$	Love's Formula No.	Case II $\rho > r$	Case III $\rho = r$	Case IV $\rho = 0$
16	$\frac{1}{\rho}\frac{\partial \chi}{\partial \rho}$	3.24-1	$p\left[\pi + \frac{r^2-\rho^2}{\rho^2}J + \frac{z}{\rho^2 R_2}\left\{R_2^2E' - (2R_1R_2b - z^2)K'\right\}\right]$	3.24-2	$p\left[\frac{\pi r^2}{\rho^2} - \frac{r^2-\rho^2}{\rho^2}J + \frac{z}{\rho^2 R_2}\left\{R_2^2E' - (2R_1R_2b - z^2)K'\right\}\right]$	$p\left[\pi + \frac{zR_2}{r^2}E' - \frac{zR_2}{r^2}K'\right]$	$p\pi\left[1 - \frac{z}{R}\right]$
17	$\frac{\partial^2 \chi}{\partial \rho^2}$	3.3-5	$p\left[\pi - \frac{r^2+\rho^2}{\rho^2}J - \frac{zR_2}{\rho^2}E' + \frac{z(2r^2+z^2)}{R_2\rho^2}K'\right]$	3.3-6	$p\left[-\frac{\pi r^2}{\rho^2} + \frac{r^2+\rho^2}{\rho^2}J - \frac{zR_2}{\rho^2}E' + \frac{z(2r^2+z^2)}{R_2\rho^2}K'\right]$	$\frac{zp}{R_2r^2}\left[(2r^2+z^2)K' - R_2^2E'\right]$	$p\pi\left[1 - \frac{z}{R}\right]$
18	$\frac{\partial V}{\partial z} = -\left(\frac{1}{\rho}\frac{\partial \chi}{\partial \rho} + \frac{\partial^2 \chi}{\partial \rho^2}\right)$	3.4-1	$-2p\left[\pi - J - \frac{z}{R_2}K'\right]$	3.4-2	$-2p\left[J - \frac{z}{R_2}K'\right]$	$p\left[\frac{2z}{R_2}K' - \pi\right]$	$-2p\pi\left[1 - \frac{z}{R}\right]$
19	$\frac{\partial^2 V}{\partial z^2} = -\left(\frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{\partial^2 V}{\partial \rho^2}\right)$	3.5-4	$2p\left[\frac{K'}{R_2} - \frac{\cos V}{R_1}E'\right] = p\left[\frac{2K'}{R_2} - \frac{R_2}{\rho^2}\left\{1 - ab\right\}E'\right]$		Same as Case I.	$p\left[\frac{2}{R_2}K' - \frac{2}{R_2}E'\right]$	$2p\pi\frac{r^2}{R^3}$
20	$\frac{1}{\rho}\frac{\partial V}{\partial \rho}$	3.5-1	$-p\frac{R_2}{\rho^2}\left[(1+k^2)K' - 2E'\right] = -p\frac{2}{\rho^2}\left\{R_1bK' - R_2E'\right\}$		Same as Case I.	$-p\frac{2}{r^2R_2}\left[(2r^2+z^2)K' - R_2^2E'\right]$	$-p\pi\frac{r^2}{R^3}$
21	$\frac{\partial^2 V}{\partial \rho^2}$	3.5-2	$\frac{p}{\rho^2R_2}\left[2(r^2+z^2)K' - R_2^2(1+ab)E'\right]$		Same as Case I.	$p\left[\frac{2(z^2+r^2)K'}{r^2R_2} - \frac{6r^2+2z^2}{R_2r^2}E'\right]$	$-p\pi\frac{r^2}{R^3}$
22	$\frac{\partial^2 V}{\partial \rho \partial z}$	3.5-3	$p\frac{z}{R_2\rho}\left[\left\{1 + \frac{1}{k^2}\right\}E' - 2K'\right]$		Same as Case I.	$\frac{2p}{zR_2r}\left[(2r^2+z^2)E' - z^2K'\right]$	0
23	$V = \frac{\partial \chi}{\partial z}$		$-2p\left[z(\pi - J) - \frac{1}{R_2}\left\{R_2^2E' + (r^2-\rho^2)K'\right\}\right]$		$-2p\left[zJ - \frac{1}{R_2}\left\{R_2^2E' + (r^2-\rho^2)K'\right\}\right]$	$p\left[-\pi z + 2R_2E'\right]$	$-2p\pi(z - R)$

ϕ	θ	$180 - \theta$	θ	θ
				

Note: In every case the point at which the stress is to be computed is taken beneath the ρ-axis.

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to the left of columns E and F refer to the formula numbers in Love's paper.

Formulas for the stresses and deflections have been obtained by substituting the values for the various derivatives shown in columns E, F, G, and H of table 2 in the expressions shown in column C of table 1. Additional formulas have also been developed for the strains and for certain summations and are shown in column C_1 of table 1. These formulas (shown in table 3) are given only for a value of Poisson's ratio of 0.5. This value assumes that the stressed elastic body has undergone no change in volume and was selected because it yields the simplest expressions. However, formulas for other values of Poisson's ratio can be obtained by making suitable substitutions in the expressions shown in table 1.

Love also shows that all the stresses produced by a uniform circular load at a point Q may be expressed in terms of the solid angle subtended at Q by the circular area. The expressions which Love gives for the solid angle (page 397) can be obtained from the expressions for the partial derivative of V with respect to $z(\partial V/\partial z)$. This expression is shown on line 18 of table 2.

Several methods were employed in checking the accuracy of the final expressions. Simplest of these is that of checking the previously developed formulas for special cases. Where the point is taken beneath the center of the circular area ($\rho = 0$), the formulas for σ_z and for $\sigma_\rho = \sigma_\theta$ agree with those obtained by Love (*2*, page 415) and by Timoshenko (*9*, pages 335 and 336). These terms are listed in column M of table 3.

It can also be shown that where the point is taken beneath the edge of the circular area ($\rho = r$), the formulas yield the terms listed in column L of table 3. These were developed by the authors by integration of the Boussinesq point load equations over the area of the circle and are given in Reference 3.

A more comprehensive check of the derived formulas was gained through conversion of results published in 1916 by Terazawa (*6*).[1] Terazawa gives a solution to the problem of a uniformly loaded circular area on the boundary of a semi-infinite body by employing Bessel's functions. His results have been converted to expressions involving elliptic functions and as such are found to agree with those developed from Love's work.

Terazawa also developed the Newtonian potential of a surface distribution as an elliptic integral. This function appears in the formula for the vertical deflection. By using the transformations described in the historical note shown below, the Newtonian potential was expressed in terms which agree with those derived by the authors in interpreting Love's paper. It can also be shown (see figure 1) that the formulas yield numerical results which agree with those obtained from the Newmark charts.

In presenting the formulas for direct computation of the stresses, strains, and deflections produced by a uniform circular load, it is intended to emphasize the usefulness of such formulas for analytical purposes rather than to suggest that the values obtained by their use are more accurate or can be computed more quickly than those obtained graphically from charts. Charts for such computations can be constructed to any desired degree of accuracy and any saving of time is open to question. However, as is shown in the following paragraphs, a number of analytical studies can be accomplished with the formulas which would be impossible without them.

Principal stresses and angles of principal planes. Formulas for the direct computation of the principal stresses and of the angles of the principal planes may be developed by combining the formulas for the normal and shearing stresses. The general three-dimensional

[1] In introducing his paper, Love states that there are two methods used in solving the problem he is presenting. One is called the "potential method" (his method); the other he calls the Bessel's function method, and is the work of Kwanichi Terazawa. The stress function for the potential method is

$$= \frac{z}{2\pi} \chi - \frac{\nu}{\pi} \int \chi \, dz$$

while that for the Bessel's function method applied to the case of a uniform circular load is

$$\phi = \frac{2pV}{\pi r} \int_0^\infty e^{-kz} J_1(k, r)\, J_0(k, \rho)\, \frac{dk}{k^2} - \frac{zp}{\pi a} \cdot \int_0^\infty e^{-kz} J_1(k, r)\, J_0(k, \rho)\, \frac{dk}{k^2}$$

By using methods developed by H. Nagaoko (Phil. Mag. VI: 6:1903), Terazawa developed functions identical with those obtained by Love after Love had applied Green's theorem. From this point Love and Terazawa deviate: Love develops the terms in the elliptic functions K', E', and $\Pi(\phi, n, k)$; Terazawa develops the terms in Weierstrass and theta functions. The works of Whittaker and Watson (*10*) were used to develop transformations which express Terazawa's results in the terms employed by Love. These results agree in detail with those of Love.

TABLE 3
FORMULAS: POISSON'S RATIO, $\nu = 0.5$

Line Number	Stress or Angle	COLUMN L Case I ($\rho < r$), II($\rho > r$), III($\rho = r$) Values obtained by substituting terms shown in Columns E, F, and G in expressions of Column C or C_1		COLUMN M Case IV ($\rho = 0$) Values obtained by substituting terms shown in Column H in expressions of Column C or C_1	
24	$\frac{\sigma_\rho}{p}$	$B + A\left[2an - (1 + ab)m\right]$	(26)	$1 - \frac{3}{2}\cos\frac{v}{2} + \frac{1}{2}\cos^3\frac{v}{2}$	(16)
25	$\frac{\sigma_\theta}{p} = \frac{\sigma_2}{p}$	$B - A\left[(3b - a)n - 2m\right]$	(25)	$1 - \frac{3}{2}\cos\frac{v}{2} + \frac{1}{2}\cos^3\frac{v}{2}$	(16)
26	$\frac{\sigma_z}{p}$	$B - \frac{z \cos v}{\pi R_1} E'$	(24)	$1 - \frac{z^3}{R^3} = 1 - \cos^3\frac{v}{2}$	(15)
27	$\frac{\sigma_{vol}}{p}$	$3\left[B - \frac{z}{R_2\pi}K'\right]$	(27)	$3\left[1 - \frac{z}{R}\right]$	(17)
28	$\frac{\sigma_1}{p}$	$B - A\left[(1 - a)n + (1 - b)m\right]$	(30)	$1 - \cos^3\frac{v}{2}$	(15)
29	$\frac{\sigma_3}{p}$	$B - A\left[(1 + b)m - (1 + a)n\right]$	(31)	$1 - \frac{3}{2}\cos\frac{v}{2} + \frac{1}{2}\cos^3\frac{v}{2}$	(16)
30	$\frac{\tau_{max}}{p}$	$A\ (bm - n)$	(32)	$\frac{2\cos\frac{v}{2}\sin^2\frac{v}{2}}{4}$	(19)
31	$\frac{\tau_{\rho z}}{p}$	$\frac{z^2}{\pi\rho R_1 R_2}\ (bm - n)$	(29)	0	
32	tan 2β	$-\frac{2z\rho}{R_1 R_2 a}$		0	
33	β	$180^\circ - \left[\frac{\theta - \gamma}{2}\right]$		180°	

Line Number	Deflection Strain or Integral	COLUMN L Case I ($\rho < r$), II($\rho > r$), III($\rho = r$) Values obtained by substituting terms shown in Columns E, F, and G in expressions of Column C or C_1		COLUMN M Case IV ($\rho = 0$) Values obtained by substituting terms shown in Column H in expressions of Column C or C_1	
34	$\frac{w_\rho}{p}$	$\frac{3z}{2\pi\rho E_m}\ (bn - m)$		0	
35	$\frac{w_\theta}{p}$	0		0	
36	$\frac{w_z}{p}$	$\frac{3}{2\pi E_m}\ (m - n\cos v)$	(35)	$\frac{3r^2}{2RE_m}$	(33)
37	$\frac{\epsilon_\rho}{p}$	$\frac{3z}{4\pi\rho^2 E_m}\left[(a + b)n - (1 + ab)m\right]$		$-\frac{3zr^2}{4R^3E_m}$	
38	$\frac{\epsilon_\theta}{p}$	$\frac{3z}{2\pi\rho^2 E_m}\ (m - bn)$		$-\frac{3zr^2}{4R^3E_m}$	
39	$\frac{\epsilon_z}{p}$	$\frac{3z}{2\pi R_1 R_2 E_m}\ (n - m\cos v)$	(36)	$\frac{3zr^2}{2R^3E_m}$	(34)
40	$\int_0^{\rho_1} \tau_{z\rho} d\rho$	$z\rho\left[-B + \frac{zK'}{R_2\pi} + (1 - \frac{z}{R})\right]$ $(\rho = \infty)$ $zp\left[1 - \frac{z}{R}\right]$		0	
41	$\int_0^{\rho_1} \sigma_z \rho d\rho$	$\frac{p}{2}\left[\frac{r^2 + \rho^2 - \lvert r^2 - \rho^2\rvert}{2} + \frac{\lvert r^2 - \rho^2\rvert}{\pi}J + \frac{z}{\pi R_2}(z^2K' - R_2^2E')\right]$		0	
42	$\int_{z_1}^{z_2} \tau_{\rho z} \rho dz$	$\frac{p}{2}\left[\frac{r^2 + \rho^2 - \lvert r^2 - \rho^2\rvert}{2} + \frac{\lvert r^2 - \rho^2\rvert}{\pi}J + \frac{z}{\pi R_2}(z^2K' - R_2^2E')\right]_{z_1}^{z_2}$		0	

Note: The numbers shown in parentheses at the right of Columns L and M are the numbers of formulas given on pages A10 through A19 of reference 3.

82851d

TABLE 4
NOTATIONS

Relations between Love's Terms and Those Used by WES		
Love's Terms		WES Terms
Rectangular Coordinates	Cylindrical Coordinates	Cylindrical Coordinates
x	$\rho \cos \omega$	ρ
y	$\rho \sin \omega$	θ
z	z	z
x'	$\rho' \cos \omega'$	$r \cos \theta_1$
y'	$\rho' \sin \omega'$	$r \sin \theta_1$
r_1	r_1	R_1
r_2	r_2	R_2
–	a	r
$\frac{\partial^2 V}{\partial x^2}$	$\frac{\partial^2 V}{\partial \rho^2}$	$\frac{\partial^2 V}{\partial \rho^2}$
$\frac{\partial^2 \chi}{\partial x^2}$	$\frac{\partial^2 \chi}{\partial \rho^2}$	$\frac{\partial^2 \chi}{\partial \rho^2}$
$\frac{\partial^2 V}{\partial y^2}$	$\frac{1}{\rho}\frac{\partial V}{\partial \rho}$	$\frac{1}{\rho}\frac{\partial V}{\partial \rho}$
$\frac{\partial^2 \chi}{\partial y^2}$	$\frac{1}{\rho}\frac{\partial \chi}{\partial \rho}$	$\frac{1}{\rho}\frac{\partial \chi}{\partial \rho}$

Constants of Theory of Elasticity	
Lamé's Terms Used by Love	WES Terms
$\frac{\lambda}{2(\lambda + \mu)}$	ν
$\frac{\mu}{\lambda + \mu}$	$1 - 2\nu$
$\frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$	E_m
$\frac{1}{\lambda + \mu}$	$\frac{2(1 + \nu)(1 - 2\nu)}{E_m}$

Notations for Table 3	
A	$\frac{z}{2\pi\rho^2}$
B $\rho < r$	$1 - J/\pi$
B $\rho > r$	J/π
B $\rho = r$	$1/2$
m	R_2E'
n	R_1K'

Notation and Values of Factors for Certain Values of ρ			
Factor	General Case $\rho = \rho$	$\rho = r$	$\rho = 0$
R	$\sqrt{z^2 + r^2}$	R	R
R_1	$\sqrt{z^2 + (r - \rho)^2}$	z	R
R_2	$\sqrt{z^2 + (r + \rho)^2}$	$\sqrt{z^2 + 4r^2}$	R
k	$\frac{R_1}{R_2}$	$\frac{z}{\sqrt{z^2 + 4r^2}}$	1
k'	$\sqrt{1 - k^2}$	$\frac{2r}{\sqrt{z^2 + 4r^2}}$	0
E	$\int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2\theta}\, d\theta$	E	1
K	$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}$	K	∞
E'	$\int_0^{\frac{\pi}{2}} \sqrt{1 - k'^2 \sin^2\theta}\, d\theta$	E'	$\frac{\pi}{2}$
K'	$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k'^2 \sin^2\theta}}$	K'	$\frac{\pi}{2}$
$E_{(k,\phi)}$	$\int_0^{\phi} \sqrt{1 - k^2 \sin^2\theta}\, d\theta$	E	$\frac{z}{R}$
$F_{(k,\phi)}$	$\int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}$	K	$\text{arc tanh} \frac{z}{R}$
J	$K'E_{(k,\phi)} - (K' - E')F_{(k,\phi)}$	$\frac{\pi}{2}$	$\frac{\pi z}{2R}$
$\sin \phi$	$\frac{z}{R_1}$	1	$\frac{z}{R}$
$\sin v$	$\frac{2rz}{R_1R_2}$	$\frac{2r}{R_2}$	$\frac{2rz}{R^2}$
$\cos v$	$\frac{z^2 - r^2 + \rho^2}{R_1R_2}$	$\frac{z}{R_2}$	$\frac{z^2 - r^2}{R^2}$
$\cos v$	$\frac{R_1R_2}{2\rho^2}(1 - ab)$		
a	$\frac{z^2 + r^2 - \rho^2}{R_1R_2}$	$\frac{z}{R_2}$	1
a	$\cos(\theta - \gamma)$		
a	$\frac{rz \cos v + (r^2 - \rho^2) \sin v}{rz}$	$\frac{z}{R_2}$	1
b	$\frac{z^2 + r^2 + \rho^2}{R_1R_2}$	$\frac{z^2 + 2r^2}{zR_2}$	1
b	$\frac{z \cos v + r \sin v}{z}$	$\frac{z^2 + 2r^2}{zR_2}$	1

82851e

formula for the principal stresses is given by Timoshenko (Formula 108, page 186 of reference 9) and is a cubic equation in rectangular coordinates. It is convenient to translate this expression to cylindrical coordinates (ρ, θ, and z), and to observe that σ_θ acts on a plane of symmetry and is one of the three principal stresses.

The other two principal stresses are found from the expression:

$$S = \frac{\sigma_z + \sigma_\rho \pm \sqrt{(\sigma_z - \sigma_\rho)^2 + (2\tau_{\rho z})^2}}{2}$$

which is obtained by dividing Timoshenko's cubic equation by the factor $(S - \sigma_\theta)$ and

TABLE 5

$$E = \int_0^{\phi} \sqrt{1 - k^2 \sin^2\theta}\, d\theta \qquad F = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

ϕ	arcsin k	E	F
0	0	.0000	.0000
	90	.0000	.0000
1	0	.0175	.0175
	90	.0175	.0175
2	0	.0349	.0349
	90	.0349	.0349
3	0	.0524	.0524
	90	.0523	.0524
4	0	.0698	.0698
	90	.0698	.0699
5	0	.0873	.0873
	90	.0872	.0874
6	0	.1047	.1047
	90	.1045	.1049
7	0	.1222	.1222
	90	.1219	.1225
8	0	.1396	.1396
	90	.1392	.1401
9	0	.1571	.1571
	90	.1564	.1577
10	0	.1745	.1745
	90	.1736	.1754
11	0	.1920	.1920
	45	.1914	.1926
	90	.1908	.1932
12	0	.2269	.2269
	45	.2087	.2102
	90	.2079	.2110
13	0	.2269	.2269
	45	.2259	.2279
	90	.2250	.2289
14	0	.2443	.2443
	30	.2437	.2450
	45	.2431	.2456
	60	.2425	.2462
	90	.2419	.2468
15	0	.2618	.2618
	30	.2611	.2625
	45	.2603	.2633
	60	.2596	.2641
	90	.2588	.2648
16	0	.2793	.2793
	30	.2784	.2802
	45	.2775	.2811
	60	.2765	.2820
	90	.2756	.2830
17	0	.2967	.2967
	20	.2962	.2972
	30	.2956	.2978
	40	.2949	.2985
	45	.2946	.2989
	50	.2942	.2993
	60	.2935	.3000
	70	.2929	.3006
	90	.2924	.3012
18	0	.3142	.3142
	20	.3136	.3148
	30	.3129	.3154
	40	.3121	.3163
	45	.3116	.3167
	50	.3112	.3172
	60	.3103	.3181
	70	.3096	.3188
	90	.3090	.3195

ϕ	arcsin k	E	F
19	0	.3316	.3316
	20	.3309	.3323
	30	.3301	.3331
	40	.3291	.3341
	45	.3286	.3347
	50	.3281	.3352
	60	.3271	.3362
	70	.3263	.3371
	90	.3256	.3379
20	0	.3491	.3491
	20	.3483	.3499
	30	.3473	.3508
	35	.3468	.3514
	40	.3462	.3520
	45	.3456	.3526
	50	.3450	.3533
	55	.3444	.3539
	60	.3438	.3545
	70	.3429	.3555
	90	.3420	.3564
21	0	.3665	.3665
	20	.3656	.3675
	25	.3651	.3680
	30	.3645	.3685
	35	.3639	.3692
	40	.3632	.3699
	45	.3625	.3706
	50	.3618	.3714
	55	.3611	.3721
	60	.3604	.3728
	65	.3598	.3734
	70	.3593	.3740
	80	.3586	.3747
	90	.3584	.3750
22	0	.3840	.3840
	10	.3837	.3842
	20	.3829	.3851
	25	.3823	.3856
	30	.3817	.3863
	35	.3809	.3871
	40	.3802	.3879
	45	.3793	.3887
	50	.3785	.3896
	55	.3777	.3904
	60	.3770	.3912
	65	.3763	.3919
	70	.3757	.3926
	80	.3749	.3935
	90	.3746	.3938
23	0	.4014	.4014
	10	.4011	.4017
	20	.4002	.4027
	25	.3996	.4033
	30	.3988	.4041
	35	.3980	.4049
	40	.3971	.4059
	45	.3961	.4068
	50	.3952	.4078
	55	.3943	.4088
	60	.3935	.4097
	65	.3927	.4105
	70	.3920	.4113
	80	.3911	.4123
	90	.3907	.4127
24	0	.4189	.4189
	10	.4185	.4192
	15	.4181	.4197
	20	.4175	.4203
	25	.4168	.4210
	30	.4159	.4219
	35	.4150	.4229
	40	.4139	.4239
	45	.4129	.4250
	50	.4118	.4261
	55	.4108	.4272
	60	.4098	.4283
	65	.4090	.4292

ϕ	arcsin k	E	F
24	70	.4082	.4301
	75	.4076	.4308
	80	.4071	.4313
	90	.4067	.4317
25	0	.4363	.4363
	10	.4359	.4367
	15	.4354	.4372
	20	.4348	.4379
	25	.4339	.4387
	30	.4330	.4397
	35	.4319	.4408
	40	.4308	.4420
	45	.4296	.4433
	50	.4284	.4446
	55	.4272	.4458
	60	.4261	.4470
	65	.4251	.4481
	70	.4243	.4490
	75	.4236	.4498
	80	.4230	.4504
	90	.4226	.4509
26	0	.4538	.4538
	10	.4533	.4542
	15	.4528	.4548
	20	.4520	.4556
	25	.4511	.4565
	30	.4500	.4576
	35	.4488	.4588
	40	.4475	.4602
	45	.4462	.4616
	50	.4449	.4630
	55	.4436	.4645
	60	.4423	.4658
	65	.4412	.4670
	70	.4402	.4681
	75	.4394	.4690
	80	.4389	.4697
	90	.4384	.4702
27	0	.4712	.4712
	10	.4707	.4717
	15	.4701	.4724
	20	.4693	.4732
	25	.4682	.4743
	30	.4670	.4755
	35	.4657	.4769
	40	.4643	.4784
	45	.4628	.4800
	50	.4613	.4816
	55	.4598	.4832
	60	.4584	.4847
	65	.4572	.4861
	70	.4561	.4873
	75	.4552	.4884
	80	.4545	.4891
	90	.4540	.4897
28	0	.4887	.4887
	10	.4881	.4893
	15	.4874	.4899
	20	.4865	.4909
	25	.4854	.4921
	30	.4840	.4934
	35	.4825	.4950
	40	.4809	.4967
	45	.4793	.4985
	50	.4776	.5003
	55	.4760	.5021
	60	.4744	.5034
	65	.4730	.5053
	70	.4718	.5067
	75	.4708	.5079
	80	.4701	.5087
	90	.4695	.5094
29	0	.5061	.5061
	10	.5055	.5068
	15	.5048	.5075
	20	.5037	.5086
	25	.5025	.5099

ϕ	arcsin k	E	F
29	30	.5010	.5114
	35	.4993	.5132
	40	.4975	.5150
	45	.4957	.5170
	50	.4938	.5190
	55	.4920	.5210
	60	.4903	.5229
	65	.4887	.5247
	70	.4874	.5262
	75	.4863	.5275
	80	.4855	.5285
	90	.4848	.5293
30	0	.5236	.5236
	10	.5229	.5243
	15	.5221	.5251
	20	.5209	.5263
	25	.5195	.5277
	30	.5179	.5294
	35	.5161	.5313
	40	.5141	.5334
	45	.5120	.5356
	50	.5100	.5379
	55	.5080	.5401
	60	.5061	.5422
	65	.5044	.5442
	70	.5029	.5459
	75	.5016	.5474
	80	.5007	.5484
	90	.5000	.5493
31	0	.5411	.5411
	10	.5403	.5418
	15	.5394	.5427
	20	.5381	.5440
	25	.5366	.5456
	30	.5348	.5475
	35	.5327	.5496
	40	.5306	.5519
	45	.5283	.5543
	50	.5261	.5568
	55	.5239	.5593
	60	.5218	.5617
	65	.5199	.5639
	70	.5182	.5658
	75	.5169	.5674
	80	.5159	.5686
	90	.5150	.5696
32	0	.5585	.5585
	10	.5577	.5593
	15	.5567	.5603
	20	.5553	.5617
	25	.5536	.5635
	30	.5516	.5656
	35	.5494	.5679
	40	.5470	.5704
	45	.5446	.5731
	50	.5421	.5759
	55	.5396	.5786
	60	.5373	.5812
	65	.5352	.5837
	70	.5334	.5858
	75	.5319	.5876
	80	.5308	.5889
	90	.5299	.5900
33	0	.5760	.5760
	10	.5751	.5769
	15	.5740	.5780
	20	.5725	.5795
	25	.5706	.5814
	30	.5684	.5837
	35	.5660	.5862
	40	.5634	.5890
	45	.5607	.5920
	50	.5580	.5950
	55	.5553	.5980
	60	.5528	.6010
	65	.5505	.6037
	70	.5485	.6060
	75	.5468	.6080

TABLE 5—*Continued*

ϕ	arcsin k	E	F
33	80	.5456	.6095
	90	.5446	.6107
34	0	.5934	.5934
	10	.5924	.5944
	15	.5912	.5956
	20	.5896	.5973
	25	.5876	.5994
	30	.5852	.6018
	35	.5826	.6046
	40	.5797	.6077
	45	.5768	.6109
	50	.5738	.6143
	55	.5709	.6176
	60	.5681	.6208
	65	.5656	.6238
	70	.5634	.6265
	75	.5616	.6287
	80	.5603	.6303
	90	.5592	.6317
35	0	.6109	.6109
	10	.6098	.6119
	15	.6085	.6133
	20	.6067	.6151
	25	.6045	.6173
	30	.6019	.6200
	35	.5991	.6231
	40	.5960	.6264
	45	.5928	.6300
	50	.5895	.6336
	55	.5863	.6373
	60	.5833	.6408
	65	.5806	.6441
	70	.5782	.6471
	75	.5762	.6495
	80	.5748	.6513
	90	.5736	.6528
36	0	.6283	.6283
	10	.6272	.6295
	15	.6258	.6309
	20	.6238	.6329
	25	.6214	.6353
	30	.6186	.6383
	35	.6155	.6416
	40	.6122	.6452
	45	.6087	.6491
	50	.6051	.6531
	55	.6017	.6571
	60	.5984	.6610
	65	.5954	.6647
	70	.5928	.6679
	75	.5907	.6706
	80	.5891	.6726
	90	.5878	.6743
37	0	.6458	.6458
	10	.6445	.6470
	15	.6430	.6486
	20	.6409	.6507
	25	.6383	.6534
	30	.6353	.6565
	35	.6319	.6602
	40	6283	6641
	45	.6245	.6684
	50	.6207	.6727
	55	.6169	.6771
	60	.6134	.6814
	65	.6101	.6854
	70	.6073	.6890
	75	.6050	.6919
	80	.6032	.6941
	90	.6018	.6960
38	0	.6632	.6632
	10	.6619	.6646
	15	.6602	.6662
	20	.6580	.6685
	25	.6552	.6714
	30	.6519	.6749
	35	.6483	.6788

ϕ	arcsin k	E	F
38	40	.6444	.6831
	45	.6403	.6877
	50	.6361	.6925
	55	.6321	.6973
	60	.6282	.7019
	65	.6247	.7063
	70	.6216	.7102
	75	.6191	.7135
	80	.6172	.7159
	90	.6157	.7180
39	0	.6807	.6807
	10	.6792	.6821
	15	.6775	.6839
	20	.6750	.6864
	25	.6720	.6895
	30	.6685	.6932
	35	.6646	.6975
	40	.6604	.7021
	45	.6559	.7071
	50	.6515	.7123
	55	.6471	.7176
	60	.6429	.7227
	65	.6391	.7275
	70	.6357	7318
	75	6330	.7353
	80	.6310	.7380
	90	.6293	.7403
40	0	.6981	.6981
	10	.6966	.6997
	15	.6947	.7016
	20	.6921	.7043
	25	.6888	.7076
	30	.6851	.7116
	35	.6808	.7162
	40	.6763	.7213
	45	.6715	.7267
	50	.6667	.7323
	55	.6620	.7380
	60	.6575	.7436
	65	.6533	.7488
	70	.6497	.7535
	75	.6468	.7575
	80	.6446	.7604
	90	.6428	.7629
41	0	.7156	.7156
	10	.7139	.7173
	15	.7119	.7193
	20	.7091	.7222
	25	.7056	.7258
	30	.7016	.7301
	35	.6970	.7350
	40	.6921	.7405
	45	.6870	.7463
	50	.6818	.7524
	55	.6767	.7586
	60	.6719	.7647
	65	.6674	.7704
	70	.6636	.7756
	75	.6604	.7799
	80	.6580	.7831
	90	.6561	.7859
42	0	.7330	.7330
	10	.7313	.7348
	15	.7291	.7370
	20	.7261	.7401
	25	.7224	.7440
	30	.7180	.7486
	35	.7132	.7539
	40	.7079	.7598
	45	.7024	.7661
	50	.6969	.7727
	55	.6914	.7794
	60	.6862	.7860
	65	.6814	.7922
	70	.6772	.7979
	75	.6738	.8026
	80	.6712	.8062
	90	.6691	.8092

ϕ	arcsin k	E	F
43	0	.7505	.7505
	10	.7486	.7524
	15	.7463	.7548
	20	.7431	.7580
	25	.7391	.7622
	30	.7345	.7671
	35	.7293	.7728
	40	.7327	.7791
	45	.7178	.7859
	50	.7118	.7931
	55	.7059	.8004
	60	.7003	.8075
	65	.6952	.8143
	70	.6907	.8204
	75	.6870	.8256
	80	.6843	.8295
	90	.6820	.8328
44	0	.7679	.7679
	10	.7659	.7700
	15	.7634	.7725
	20	.7600	.7760
	25	.7558	.7804
	30	.7508	.7857
	35	.7453	.7918
	40	.7393	.7986
	45	.7330	.8059
	50	.7266	.8136
	55	.7204	.8215
	60	.7144	.8293
	65	.7088	.8367
	70	.7040	.8433
	75	.7000	.8490
	80	.6971	.8533
	90	.6947	.8569
45	0	.7854	.7854
	10	.7832	.7876
	15	.7806	.7903
	20	.7770	.7940
	25	.7725	.7988
	30	.7672	.8044
	35	.7613	.8109
	40	.7549	.8181
	45	.7482	.8260
	50	.7414	.8343
	55	.7346	.8428
	60	.7282	.8512
	65	.7223	.8592
	70	.7171	.8665
	75	.7129	.8727
	80	.7097	.8774
	90	.7071	.8814
46	0	.8029	.8029
	10	.8006	.8052
	15	.7977	.8080
	20	.7939	.8120
	25	.7891	.8170
	30	.7835	.8230
	35	.7772	.8300
	40	.7704	.8378
	45	.7633	.8462
	50	.7560	.8552
	55	.7488	.8643
	60	.7419	.8734
	65	.7356	.8821
	70	.7301	.8900
	75	.7255	.8968
	80	.7221	.9019
	90	.7193	.9063
47	0	.8203	.8203
	10	.8179	.8227
	15	.8149	.8258
	20	.8108	.8300
	25	.8057	.8353
	30	.7998	.8418
	35	.7931	.8492
	40	.7858	.8575
	45	.7782	.8666
	50	.7705	.8761

ϕ	arcsin k	E	F
47	55	.7628	.8860
	60	.7555	.8958
	65	.7488	.9053
	70	.7429	.9139
	75	.7380	.9212
	80	.7344	.9269
	90	.7314	.9316
48	0	.8378	.8378
	10	.8352	.8403
	15	.8320	.8436
	20	.8276	.8480
	25	.8223	.8537
	30	.8160	.8606
	35	.8089	.8685
	40	.8012	.8773
	45	.7931	.8870
	50	.7849	.8973
	55	.7768	.9079
	60	.7690	.9185
	65	.7618	.9287
	70	.7555	.9381
	75	.7502	.9461
	80	.7464	.9523
	90	.7431	.9575
49	0	.8552	.8552
	10	.8525	.8579
	15	.8491	.8614
	20	.8446	.8661
	25	.8389	.8721
	30	.8322	.8794
	35	.8247	.8878
	40	.8165	.8972
	45	.8079	.9076
	50	.7992	.9186
	55	.7905	.9300
	60	.7822	.9415
	65	.7746	.9525
	70	.7679	.9627
	75	.7623	.9714
	80	.7581	.9781
	90	.7547	.9838
50	0	.8727	.8727
	10	.8698	.8756
	15	.8663	.8792
	20	.8614	.8842
	25	.8554	.8905
	30	.8483	.8982
	35	.8404	.9072
	40	.8317	.9173
	45	.8227	.9283
	50	.8134	.9401
	55	.8042	.9523
	60	.7954	.9647
	65	.7872	.9766
	70	.7801	.9876
	75	.7741	.9971
	80	.7697	1.004
	90	.7660	1.011
51	0	.8901	.8901
	10	.8871	.8932
	15	.8834	.8970
	20	.8783	.9023
	25	.8719	.9090
	30	.8644	.9172
	35	.8560	.9267
	40	.8469	.9374
	45	.8373	.9491
	50	.8275	.9617
	55	.8177	9748
	60	8084	9881
	65	.7997	1.001
	70	.7921	1.013
	75	.7858	1.023
	80	.7811	1.031
	90	.7771	1.038
52	0	.9076	.9076
	10	.9044	.9108

TABLE 5—*Continued*

φ	arcsin k	E	F
52	15	.9004	.9148
	20	.8951	.9204
	25	.8884	.9275
	30	.8805	.9361
	35	.8716	.9462
	40	.8620	.9575
	45	.8518	.9701
	50	.8414	.9835
	55	.8311	.9976
	60	.8212	1.012
	65	.8120	1.026
	70	.8039	1.039
	75	.7972	1.050
	80	.7922	1.059
	90	.7880	1.066
53	0	.9250	.9250
	10	.9217	.9284
	15	.9175	.9326
	20	.9119	.9385
	25	.9048	.9460
	30	.8965	.9551
	35	.8872	.9658
	40	.8770	.9778
	45	.8663	.9912
	50	.8553	1.005
	55	.8444	1.021
	60	.8339	1.036
	65	.8241	1.051
	70	.8155	1.065
	75	.8084	1.077
	80	.8031	1.087
	90	.7986	1.095
54	0	.9425	.9425
	10	.9389	.9460
	15	.9346	.9505
	20	.9287	.9567
	25	.9212	.9646
	30	.9125	.9742
	35	.9026	.9855
	40	.8919	.9982
	45	.8806	1.012
	50	.8690	1.028
	55	.8575	1.044
	60	.8464	1.060
	65	.8361	1.076
	70	.8270	1.092
	75	.8194	1.105
	80	.8137	1.115
	90	.8090	1.124
55	0	.9599	.9599
	10	.9562	.9637
	15	.9517	.9683
	20	.9454	.9748
	25	.9376	.9832
	30	.9284	.9933
	35	.9181	1.005
	40	.9068	1.019
	45	.8949	1.034
	50	.8827	1.050
	55	.8705	1.067
	60	.8588	1.085
	65	.8479	1.102
	70	.8382	1.119
	75	.8302	1.133
	80	.8242	1.144
	90	.8192	1.154
56	0	.9774	.9774
	10	.9735	.9813
	15	.9687	.9862
	20	.9622	.9930
	25	.9540	1.002
	30	.9443	1.013
	35	.9335	1.025
	40	.9216	1.039
	45	.9091	1.055
	50	.8962	1.072
	55	.8834	1.091
	60	.8710	1.110
56	65	.8595	1.128
	70	.8493	1.146
	75	.8408	1.162
	80	.8344	1.174
	90	.8290	1.185
57	0	.9948	.9948
	10	.9908	.9989
	15	.9858	1.004
	20	.9789	1.011
	25	.9703	1.020
	30	.9602	1.032
	35	.9488	1.045
	40	.9363	1.060
	45	.9232	1.077
	50	.9096	1.095
	55	.8961	1.115
	60	.8831	1.135
	65	.8709	1.155
	70	.8601	1.174
	75	.8511	1.191
	80	.8443	1.205
	90	.8387	1.217
58	0	1.012	1.012
	10	1.008	1.017
	15	1.003	1.022
	20	.9956	1.029
	25	.9866	1.039
	30	.9760	1.051
	35	.9641	1.065
	40	.9510	1.081
	45	.9372	1.099
	50	.9230	1.118
	55	.9088	1.139
	60	.8950	1.160
	65	.8822	1.182
	70	.8707	1.203
	75	.8612	1.221
	80	.8540	1.236
	90	.8480	1.249
59	0	1.030	1.030
	10	1 025	1.034
	15	1.020	1.040
	20	1.012	1.048
	25	1.003	1.058
	30	.9918	1.070
	35	.9793	1.085
	40	.9656	1.102
	45	.9511	1.120
	50	.9362	1.141
	55	.9213	1.163
	60	.9068	1.186
	65	.8932	1.210
	70	.8812	1.232
	75	.8711	1.252
	80	.8635	1.268
	90	.8572	1.283
60	0	1.047	1.047
	10	1.043	1.052
	15	1.037	1.058
	20	1.029	1.066
	25	1.019	1.077
	30	1.008	1.090
	35	.9945	1.105
	40	.9081	1.123
	45	.9650	1.142
	50	.9493	1.164
	55	.9336	1.188
	60	.9184	1.213
	65	.9042	1.238
	70	.8194	1.262
	75	.8808	1.284
	80	.8728	1.301
	90	.8660	1.317
61	0	1.065	1.065
	10	1.060	1.070
	15	1.054	1.076
	20	1.046	1.084
61	25	1.035	1.095
	30	1.023	1.109
	35	1.010	1.125
	40	.9946	1.144
	45	.9787	1.165
	50	.9623	1.188
	55	.9459	1.213
	60	.9299	1.239
	65	.9149	1.266
	70	.9015	1.292
	75	.8903	1.316
	80	.8817	1.335
	90	.8746	1.352
62	0	1.082	1.082
	10	1.077	1.087
	15	1.071	1.094
	20	1.062	1 103
	25	1 052	1.014
	30	1.039	1.128
	35	1.025	1.145
	40	1.009	1.165
	45	.9924	1.187
	50	.9752	1.211
	55	.9580	1.238
	60	.9412	1.266
	65	.9254	1 295
	70	.9133	1.323
	75	.8995	1.349
	80	.8905	1.370
	90	.8829	1.389
63	0	1.100	1.100
	10	1.094	1.105
	15	1.088	1.112
	20	1.079	1.121
	25	1.068	1.133
	30	1.055	1.148
	35	1.040	1.166
	40	1.023	1.186
	45	1.006	1.209
	50	.9880	1.235
	55	.9700	1.263
	60	.9524	1.293
	65	.9358	1.324
	70	.9210	1.355
	75	.9085	1.383
	80	.8990	1.406
	90	.8910	1.427
64	0	1.117	1.117
	10	1.112	1.123
	15	1.105	1.129
	20	1.096	1.139
	25	1.084	1.152
	30	1.070	1.167
	35	1.055	1.186
	40	1.038	1.207
	45	1.019	1.232
	50	1.001	1.259
	55	.9818	1.289
	60	.9634	1.321
	65	.9460	1.354
	70	.9304	1.387
	75	.9173	1.418
	80	.9072	1.443
	90	.8988	1.466
65	0	1.134	1.134
	10	1.129	1.140
	15	1.122	1.147
	20	1.112	1.158
	25	1.100	1.171
	30	1.086	1.187
	35	1.070	1.206
	40	1.052	1.229
	45	1.033	1.254
	50	1.013	1.283
	55	.9936	1.315
	60	.9743	1.349
	65	.9561	1.384
	70	.9397	1.420
65	75	.9258	1.543
	80	.9152	1.481
	90	.9063	1.506
66	0	1.152	1.152
	10	1.146	1.158
	15	1.139	1.165
	20	1.129	1.176
	25	1.116	1.190
	30	1.101	1.207
	35	1.084	1.227
	40	1.066	1.250
	45	1.046	1.277
	50	1.026	1.308
	55	1.005	1.341
	60	.9850	1.377
	65	.9659	1.415
	70	.9487	1.454
	75	.9341	1.490
	80	.9230	1.520
	90	.9135	1.549
67	0	1.169	1.169
	10	1.163	1.176
	15	1.156	1.183
	20	1.145	1.194
	25	1.132	1.209
	30	1.117	1.226
	35	1.099	1.247
	40	1.080	1.272
	45	1.060	1.300
	50	1.038	1.332
	55	1.017	1.368
	60	.9956	1.406
	65	.9756	1.447
	70	.9576	1.488
	75	.9422	1.527
	80	.9305	1.561
	90	.9205	1.592
68	0	1.187	1.187
	10	1.180	1.193
	15	1.173	1.201
	20	1.162	1.213
	25	1.148	1.227
	30	1.132	1.246
	35	1.114	1.268
	40	1.094	1.294
	45	1.073	1.323
	50	1.051	1.357
	55	1.028	1.394
	60	1.006	1.435
	65	.9852	1.479
	70	.9662	1.523
	75	.9501	1.566
	80	.9377	1.603
	90	.9272	1.638
69	0	1.204	1.204
	10	1.198	1.211
	15	1.190	1.219
	20	1.178	1.231
	25	1.164	1.246
	30	1.148	1.266
	35	1.129	1.288
	40	1.108	1.315
	45	1.086	1.346
	50	1.063	1.382
	55	1.039	1.421
	60	1.016	1.465
	65	.9946	1.511
	70	.9747	1.559
	75	.9578	1.606
	80	.9447	1.647
	90	.9336	1.686
70	0	1.222	1.222
	10	1.215	1.229
	15	1.206	1.237
	20	1.195	1.250
	25	1.180	1.265
	30	1.163	1.285

TABLE 5—*Continued*

φ	arcsin k	E	F
70	35	1.144	1.309
	40	1.122	1.337
	45	1.099	1.370
	50	1.075	1.407
	55	1.051	1.448
	60	1.027	1.494
	65	1.004	1.544
	70	.9830	1.596
	75	.9652	1.647
	80	.9514	1.692
	90	.9397	1.735
71	0	1.239	1.239
	10	1.232	1.246
	15	1.223	1.255
	20	1.211	1.268
	25	1.196	1.285
	30	1.179	1.305
	35	1.158	1.330
	40	1.136	1.359
	45	1.112	1.393
	50	1.087	1.432
	55	1.062	1.476
	60	1.037	1.525
	65	1.013	1.578
	70	.9911	1.634
	75	.9724	1.689
	80	.9579	1.739
	90	.9455	1.788
72	0	1.257	1.257
	10	1.249	1.264
	15	1.240	1.273
	20	1.228	1.286
	25	1.212	1.304
	30	1.194	1.325
	35	1.173	1.351
	40	1.150	1.381
	45	1.125	1.417
	50	1.099	1.457
	55	1.073	1.504
	60	1.047	1.555
	65	1.022	1.612
	70	.9990	1.672
	75	.9794	1.733
	80	.9642	1.788
	90	.9511	1.843
73	0	1.274	1.274
	10	1.267	1.282
	15	1.257	1.291
	20	1.244	1.305
	25	1.228	1.323
	30	1.209	1.345
	35	1.188	1.372
	40	1.164	1.403
	45	1.138	1.440
	50	1.111	1.483
	55	1.084	1.532
	60	1.057	1.586
	65	1.031	1.647
	70	1.007	1.711
	75	.9862	1.777
	80	.9702	1.838
	90	.9563	1.901
74	0	1.292	1.292
	10	1.284	1.299
	15	1.274	1.309
	20	1.261	1.323
	25	1.244	1.342
	30	1.225	1.365
	35	1.202	1.392
	40	1.177	1.425
	45	1.151	1.464
	50	1.123	1.509
	55	1.094	1.560
	60	1.066	1.618
	65	1.039	1.682
	70	1.014	1.752
	75	.9928	1.824

φ	arcsin k	E	F
74	80	.9759	1.891
	90	.9613	1.962
75	0	1.309	1.309
	10	1.301	1.317
	15	1.291	1.327
	20	1.277	1.342
	25	1.260	1.361
	30	1.240	1.385
	35	1.217	1.413
	40	1.191	1.448
	45	1.163	1.488
	50	1.135	1.535
	55	1.105	1.588
	60	1.076	1.649
	65	1.048	1.718
	70	1.022	1.793
	75	.9992	1.871
	80	.9814	1.947
	90	.9659	2.028
76	0	1.326	1.326
	10	1.318	1.335
	15	1.308	1.345
	20	1.294	1.360
	25	1.276	1.380
	30	1.255	1.405
	35	1.231	1.434
	40	1.205	1.470
	45	1.176	1.512
	50	1.146	1.561
	55	1.116	1.617
	60	1.085	1.681
	65	1.056	1.754
	70	1.029	1.835
	75	1.005	1.921
	80	.9867	2.005
	90	.9703	2.097
77	0	1.344	1.344
	10	1.335	1.353
	15	1.325	1.363
	20	1.310	1.379
	25	1.292	1.399
	30	1.270	1.424
	35	1.246	1.455
	40	1.218	1.492
	45	1.189	1.536
	50	1.158	1.587
	55	1.126	1.646
	60	1.095	1.714
	65	1.064	1.791
	70	1.036	1.878
	75	1.011	1.972
	80	.9917	2.065
	90	.9744	2.172
78	0	1.361	1.361
	10	1.353	1.370
	15	1.342	1.381
	20	1.327	1.397
	25	1.308	1.418
	30	1.286	1.444
	35	1.260	1.476
	40	1.232	1.515
	45	1.201	1.560
	50	1.170	1.613
	55	1.137	1.675
	60	1.104	1.746
	65	1.072	1.828
	70	1.043	1.922
	75	1.017	2.024
	80	.9965	2.129
	90	.9781	2.253
79	0	1.379	1.379
	10	1.370	1.388
	15	1.359	1.399
	20	1.343	1.416
	25	1.324	1.437
	30	1.301	1.465

φ	arcsin k	E	F
79	35	1.275	1.498
	40	1.245	1.537
	45	1.214	1.584
	50	1.181	1.639
	55	1.147	1.704
	60	1.113	1.779
	65	1.080	1.866
	70	1.050	1.966
	75	1.023	2.078
	80	1.001	2.195
	90	.9816	2.340
80	0	1.396	1.396
	5	1.394	1.399
	10	1.387	1.406
	15	1.376	1.418
	20	1.360	1.434
	25	1.340	1.457
	30	1.316	1.485
	35	1.289	1.519
	40	1.259	1.560
	45	1.227	1.608
	50	1.193	1.666
	55	1.158	1.733
	60	1.122	1.813
	65	1.088	1.905
	70	1.056	2.012
	75	1.028	2.134
	80	1.055	2.265
	85	.9902	2.384
	90	.9848	2.436
81	0	1.414	1.414
	5	1.411	1.416
	10	1.404	1.423
	15	1.392	1.436
	20	1.376	1.453
	25	1.356	1.476
	30	1.331	1.505
	35	1.303	1.540
	40	1.272	1.582
	45	1.239	1.633
	50	1.204	1.693
	55	1.168	1.763
	60	1.132	1.846
	65	1.096	1.944
	70	1.063	2.058
	75	1.034	2.191
	80	1.010	2.339
	85	.9935	2.477
	90	.9877	2.542
82	0	1.431	1.431
	5	1.429	1.434
	10	1.421	1.441
	15	1.409	1.454
	20	1.393	1.472
	25	1.371	1.495
	30	1.346	1.525
	35	1.318	1.561
	40	1.286	1.605
	45	1.252	1.657
	50	1.215	1.719
	55	1.178	1.793
	60	1.141	1.880
	65	1.104	1.983
	70	1.069	2.106
	75	1.039	2.250
	80	1.014	2.416
	85	.9965	2.580
	90	.9903	2.660
83	0	1.449	1.449
	5	1.446	1.451
	10	1.439	1.459
	15	1.426	1.472
	20	1.409	1.490
	25	1.387	1.514
	30	1.362	1.545
	35	1.232	1.582
	40	1.299	1.628

φ	arcsin k	E	F
83	45	1.264	1.682
	50	1.227	1.746
	55	1.188	1.823
	60	1.150	1.914
	65	1.112	2.023
	70	1.076	2.154
	75	1.044	2.311
	80	1.017	2.496
	85	.9992	2.691
	90	.9925	2.794
84	0	1.466	1.466
	5	1.463	1.469
	10	1.456	1.477
	15	1.443	1.490
	20	1.425	1.509
	25	1.403	1.533
	30	1.377	1.565
	35	1.346	1.604
	40	1.313	1.650
	45	1.276	1.706
	50	1.238	1.773
	55	1.198	1.853
	60	1.158	1.948
	65	1.119	2.063
	70	1.082	2.202
	75	1.049	2.373
	80	1.021	2.581
	85	1.022	2.814
	90	.9945	2.949
85	0	1.484	1.484
	5	1.481	1.486
	10	1.473	1.494
	15	1.460	1.508
	20	1.442	1.527
	25	1.419	1.553
	30	1.392	1.585
	35	1.361	1.625
	40	1.326	1.673
	45	1.289	1.731
	50	1.249	1.800
	55	1.208	1.883
	60	1.167	1.983
	65	1.127	2.103
	70	1.088	2.252
	75	1.053	2.437
	80	1.024	2.669
	85	1.004	2.949
	90	.9962	3.131
86	0	1.501	1.501
	5	1.498	1.504
	10	1.490	1.512
	15	1.477	1.526
	20	1.458	1.546
	25	1.435	1.572
	30	1.407	1.605
	35	1.375	1.646
	40	1.340	1.696
	45	1.301	1.755
	50	1.261	1.827
	55	1.219	1.913
	60	1.176	2.017
	65	1.134	2.144
	70	1.094	2.302
	75	1.058	2.501
	80	1.028	2.761
	85	1.006	3.098
	90	.9976	3.355
87	0	1.518	1.518
	5	1.516	1.521
	10	1.507	1.530
	15	1.494	1.544
	20	1.475	1.564
	25	1.451	1.591
	30	1.422	1.625
	35	1.389	1.667
	40	1.353	1.718
	45	1.314	1.780

TABLE 5—*Concluded*

ϕ	arcsin k	E	F	ϕ	arcsin k	E	F	ϕ	arcsin k	E	F	ϕ	arcsin k	E	F
87	50	1.272	1.854	89	30	1.452	1.666	90	22	1.514	1.631	90	58	1.230	2.105
	55	1.229	1.944		35	1.418	1.710		23	1.509	1.637		59	1.221	2.130
	60	1.185	2.052		40	1.380	1.764		24	1.504	1.643		60	1.211	2.157
	65	1.142	2.185		45	1.338	1.829		25	1.498	1.649		61	1.202	2.184
	70	1.100	2.352		50	1.294	1.908		26	1.492	1.656		62	1.192	2.213
	75	1.063	2.567		55	1.249	2.004		27	1.486	1.663		63	1.183	2.244
	80	1.031	2.856		60	1.202	2.122		28	1.480	1.670		64	1.173	2.275
	85	1.008	3.262		65	1.156	2.267		29	1.474	1.678		65	1.164	2.309
	90	.9986	3.643		70	1.112	2.454		30	1.467	1.686		66	1.155	2.344
					75	1.072	2.701		31	1.461	1.694		67	1.145	2.381
88	0	1.536	1.536		80	1.037	3.053		32	1.454	1.703		68	1.136	2.420
	5	1.533	1.539		85	1.011	3.633		33	1.447	1.712		69	1.127	2.461
	10	1.525	1.547		90	.9998	4.741		34	1.440	1.721		70	1.118	2.505
	15	1.510	1.562						35	1.432	1.731		71	1.110	2.551
	20	1.491	1.583	90	0	1.571	1.571		36	1.425	1.741		72	1.101	2.600
	25	1.466	1.610		1	1.571	1.571		37	1.417	1.752		73	1.093	2.652
	30	1.437	1.645		2	1.570	1.571		38	1.409	1.763		74	1.084	2.708
	35	1.404	1.689		3	1.570	1.572		39	1.401	1.775		75	1.076	2.768
	40	1.366	1.741		4	1.569	1.573		40	1.393	1.787		76	1.069	2.833
	45	1.326	1.805		5	1.568	1.574		41	1.385	1.799		77	1.061	2.903
	50	1.283	1.881		6	1.566	1.575		42	1.377	1.812		78	1.054	2.979
	55	1.239	1.974		7	1.565	1.577		43	1.368	1.826		79	1.047	3.062
	60	1.194	2.087		8	1.563	1.578		44	1.359	1.840		80	1.040	3.153
	65	1.149	2.226		9	1.561	1.581		45	1.351	1.854		81	1.034	3.255
	70	1.106	2.403		10	1.559	1.583		46	1.342	1.869		82	1.028	3.370
	75	1.067	2.634		11	1.556	1.585		47	1.333	1.885		83	1.022	3.500
	80	1.034	2.954		12	1.554	1.588		48	1.324	1.901		84	1.017	3.652
	85	1.010	3.441		13	1.551	1.591		49	1.315	1.918		85	1.013	3.832
	90	.9994	4.048		14	1.548	1.595		50	1.306	1.936		86	1.009	4.053
					15	1.544	1.598		51	1.296	1.954		87	1.005	4.339
89	0	1.553	1.553		16	1.541	1.602		52	1.287	1.973		88	1.003	4.743
	5	1.550	1.556		17	1.537	1.606		53	1.278	1.993		89	1.001	5.435
	10	1.542	1.565		18	1.533	1.610		54	1.268	2.013		90	1.000	∞
	15	1.527	1.580		19	1.528	1.615		55	1.259	2.035				
	20	1.507	1.601		20	1.524	1.620		56	1.249	2.057				
	25	1.482	1.630		21	1.519	1.625		57	1.240	2.080				

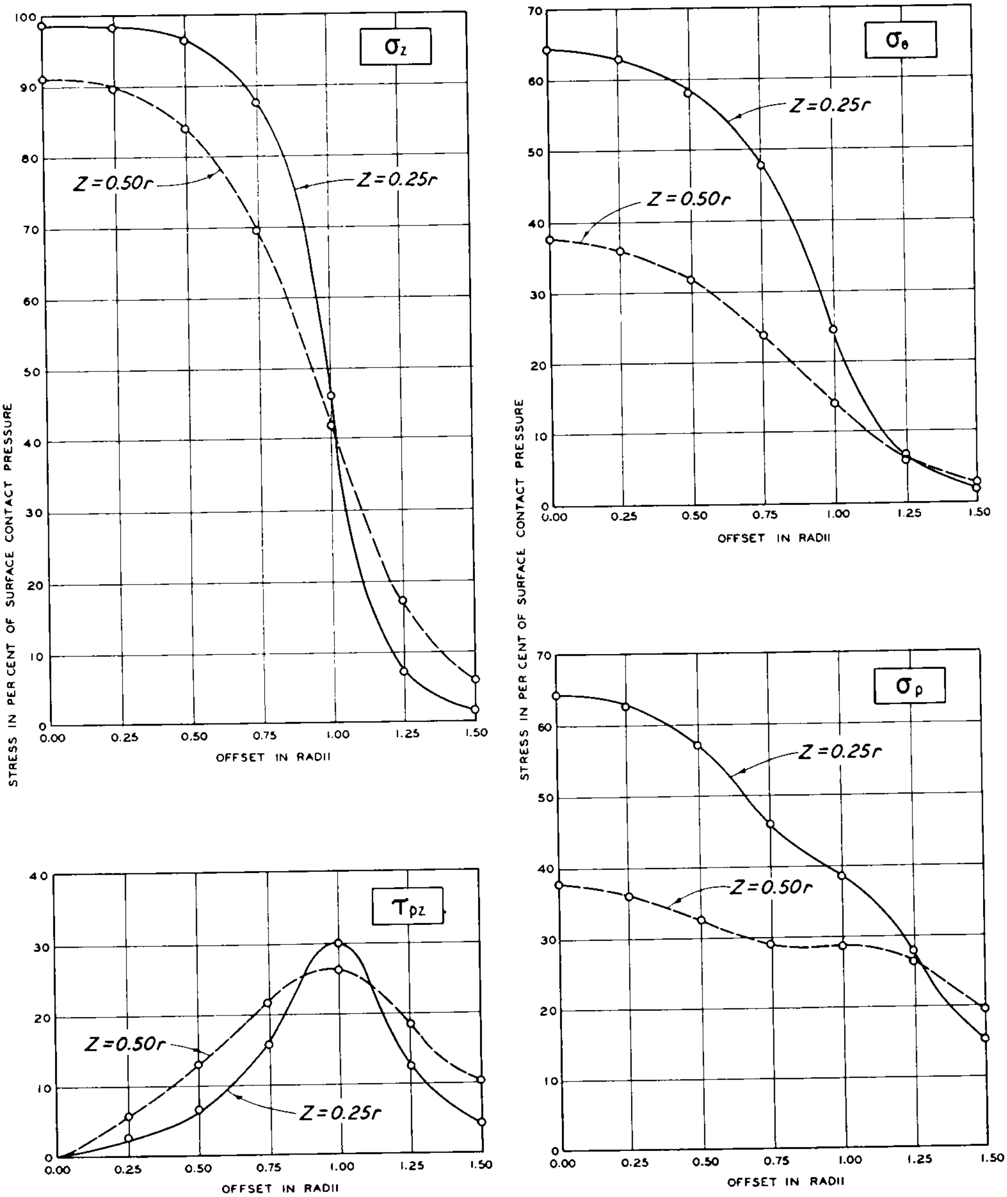
dropping out the terms containing the shearing stresses $\tau_{\rho\theta}$ and $\tau_{\theta Z}$ which are equal to zero because of the symmetry. Computations made with this equation show that, in the special case when Poisson's ratio is equal to 0.5, σ_θ is the intermediate principal stress σ_2 and that, therefore, the two roots of the equation are actually the major and minor principal stresses.

Expressions for the direct computation of these stresses and also for the maximum shearing stress which is equal to half their difference were obtained by substituting the terms shown on lines 24, 26, and 31 of table 3 for σ_z, σ_ρ, and $\tau_{\rho z}$. These expressions are shown on lines 28, 29, and 30 of table 3. The formulas also make it possible to derive an expression for the tangent of the angle of the major principal plane. This angle is designated as β and is the angle formed by the major principal plane with the horizontal. It is measured clockwise from the horizontal which puts the first quadrant on the lower right for all positive values of the offset distance ρ.

In the special case when Poisson's ratio is 0.5, the tangent of 2β can be computed from the expression shown on line 32 of table 3. It is to be noted that this expression does not contain any elliptic integrals. It can also be shown in this case that the angle β is equal to $180 - \frac{1}{2}(\theta - \gamma)$ and that the major principal plane is, therefore, perpendicular to the bisector of the angle v (see figures, table 2).

Summation of forces over an area. One of the problems encountered in the measurement of stresses within an earth mass is that of cell accuracy. To determine whether the cells measuring normal and shearing stresses are subject to any consistent over- or underregistration, a certain section of the earth was considered as a free body and the total forces acting over the body's several faces summed up. To check the accuracy of the values and the validity of the summation methods, the forces were also summed up theoretically by integration of the appropriate formulas. Three such integrations are shown on lines 40, 41, and 42 of table 3.

Numerical examples. To illustrate the use of the formulas, numerical examples are presented in which values for the stresses, strains, and deflections are computed at two points within the solid. The numerical values for the several elliptic integrals appearing in the



NOTE. LINES ARE COMPUTED STRESS.
POINTS REPRESENT STRESSES READ FROM NEWMARK'S CHARTS.

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Figure 1. Computed values of coordinate stresses.

formulas are obtained from tables compiled by Legendre (*8*) in 1826. These tables have been photographically reproduced and are appended to the report listed as reference 1.

The values of E, E', K, K', $E_{(k,\phi)}$, and $F_{(k,\phi)}$ are read from Legendre's tables by taking ϕ (shown in the left column of each page) as the upper limit of integration and the value of k or k' as the sine of the angle shown at the head of each column in parentheses after the letters E or F. The complete integrals E, E', K, and K' are thus read from the bottom line of every second page of Legendre's tables for the value of ϕ of 90° and in the column indicated for arc sine k or k'. E and E' are read from the "E" column and K and K' from the "F" column. Where k, k', or ϕ have values between those shown in the table, interpolations are made in the usual way.

For purposes of computation, it is convenient to note that arc sine k and arc sine k' are complementary angles and that where the angle v is greater than 90°, its cosine is negative. For those who do not need the accuracy of Legendre's tables, a brief abstract of the tables is appended hereto.

Values are computed at two points as follows:

Example 1

r	1.0
z	0.5
ρ	0.5
p	100 psi
E_m	10,000 psi
ν	0.5

$\tan\phi = \frac{z}{r-\rho}$	1.000
ϕ	45°
$\sin\phi$	0.7071
$R_1 = \frac{z}{\sin\phi}$	0.7071
$\tan\gamma = \frac{z}{r+\rho}$	0.3333
$\sin\gamma$	0.3162
$R_2 = \frac{z}{\sin\gamma}$	1.5811
R_1R_2	1.1180
$\cos v$	−0.4472
a	0.8945
b	1.3417
ab	1.2000
$k = \frac{R_1}{R_2}$	0.4472
arcsin k	26.56°
arcsin k'	63.44°
E'	1.1785
K'	2.2575
$K' - E'$	1.0790
$E_{(k,\phi)}$	0.7709
$F_{(k,\phi)}$	0.8004
J	0.8767
$B = 1 - \frac{J}{\pi}$	0.7210
$m = R_2E'$	1.8633
$n = R_1K'$	1.5963
$bm - n$	0.9037
$bn - m$	0.2785
$A = \frac{z}{2\pi\rho^2}$	0.3183

$\sigma_\rho =$ 32.51 psi
$\sigma_2 = \sigma_\theta =$ 31.64 psi
$\sigma_z =$ 83.95 psi
$\sigma_{vol} =$ 148.11 psi
$\sigma_1 =$ 87.00 psi
$\sigma_3 =$ 29.47 psi
$\tau_{max} =$ 28.76 psi
$\tau_{\rho z} =$ 12.86 psi

$\epsilon_\rho =$ −0.002529 in/in
$\epsilon_\theta =$ −0.002659 in/in
$\epsilon_z =$ 0.005188 in/in

$\omega_\rho =$ 0.001330 in
$\omega_\theta =$ 0.00 in
$\omega_z =$ 0.012305 in

Example 2

r	1.0
z	0.5
ρ	2.0
p	100 psi
E_m	10,000 psi
ν	0.5

$\tan\phi = \frac{z}{\rho - r}$	0.5000
ϕ	26.56°
$\sin\phi$	0.4472
$R_1 = \frac{z}{\sin\phi}$	1.1181
$\tan\gamma = \frac{z}{r+\rho}$	0.1667
$\sin\gamma$	0.1644
$R_2 = \frac{z}{\sin\gamma}$	3.0414
R_1R_2	3.4006
$\cos v$	0.9557
a	−0.8087
b	1.5438
ab	−1.2485
$k = \frac{R_1}{R_2}$	0.3676
arcsin k	21.57°
arcsin k_1	68.43°
E'	1.1324
K'	2.4375

$K' - E'$	1.3051
$E_{(k,\phi)}$	0.4614
$F_{(k,\phi)}$	0.4657
J	0.5169
$B = 1 - \frac{J}{\pi}$	0.1645
$m = R_2E'$	3.4441
$n = R_1K'$	2.7254
$bm - n$	2.5916
$bn - m$	0.7634
$A = \frac{z}{2\pi\rho^2}$	0.0199

$\sigma_\rho = 9.41$ psi
$\sigma_2 = \sigma_\theta = 0.65$ psi
$\sigma_z = 1.05$ psi
$\sigma_{vol} = 11.07$ psi
$\sigma_1 = 10.37$ psi
$\sigma_2 = 0.05$ psi
$\tau_{max} = 5.16$ psi
$\tau_{\rho z} = 3.03$ psi

$\epsilon_\rho = 0.000853$ in/in
$\epsilon_\theta = -0.000456$ in/in
$\epsilon_z = -0.000397$ in/in

$\omega_\rho = 0.000912$ in
$\omega_\theta = 0.00$ in
$\omega_z = 0.004007$ in

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REFERENCES

1. Investigations of Pressures and Deflections for Flexible Pavements: Report No. 3, Theoretical Stresses Induced by Uniform Circular Loads. Waterways Experiment Station, Corps of Engineers, TM No. 3-323 (September 1953).
2. Love, A. E. H., The Stress Produced in a Semi-Infinite Body by Pressure on Part of the Boundary, Philosophical Transactions of the Royal Society, series A, vol. 228 (1929).
3. Investigations of Pressures and Deflections for Flexible Pavements: Report No. 1, Homogeneous Clayey-Silt Test Section. Waterways Experiment Station, Corps of Engineers, TM No. 3-323 (March 1951).
4. Newmark, Nathan M., Influence Charts for Computation of Stresses in Elastic Foundations. University of Illinois Engineering Experiment Station Bulletin, No. 338 (November 10, 1942).
5. Newmark, Nathan M., Influence Charts for Computation of Vertical Displacement in Elastic Foundations. University of Illinois Engineering Experiment Station Bulletin, No. 367 (March 23, 1947).
6. Terazawa, Kwan-ichi, Journal of the College of Science. Imperial University, Tokyo, Japan (December 7, 1916).
7. Widder, David, V., Advanced Calculus. New York, Prentice-Hall, Inc., 1947.
8. Legendre, A. M., Traite des Fonctions Elliptiques. Paris, vol. II, 1826. Available at John Crerar Library, Chicago, Illinois.
9. Timoshenko, S., Theory of Elasticity. New York, McGraw-Hill Book Company, Inc., 1934.
10. Whittaker, E. T. and Watson, G. N., A Course of Modern Analysis. New York, The MacMillan Company, 1947