Prediction of Creep Behavior in Concrete from Sonic Properties

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● THE purpose of this paper is to present a method which will relate the creep behavior of concrete in axial compression with its viscoelastic properties obtained by vibrating the concrete specimens at relatively small amplitudes.

Since all available analyses of creep phenomena are entirely empirical in nature, the formulas obtained can predict the creep of concrete only when all variables are the same as those for which the formulas were developed. This has been a major obstacle to the practical application of these formulas.

To overcome these difficulties a more rational approach to the problem is needed. Such an approach is available if the concept that concrete is a viscoelastic material is accepted. Reiner (*1*) originally utilized this concept, which has not, to date, been widely used. In 1950, Flügge (*2*) used the concept of viscoelasticity to explain the creep behavior of concrete; he obtained creep constants by assuming the concrete specimens to be complicated linear mechanical or rheological models. Flügge found that for different types of concrete the creep constants were different.

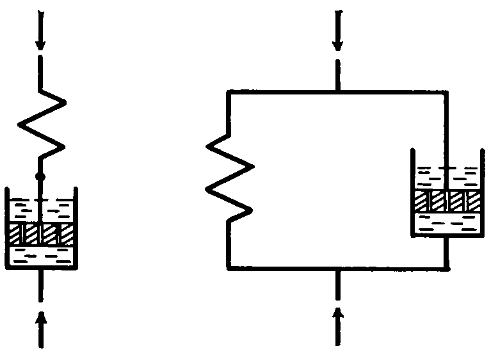

Figure 1. Maxwell's and Kelvin's rheological models.

If the concept that concrete is viscoelastic is correct, then a relationship should exist between the creep properties and the sonic properties, since fundamentally they are based on the same assumptions. This paper is a treatment of a semi-rational procedure to obtain the correlations of sonic and creep properties. The correlations are supported with experimental results.

THEORY OF RHEOLOGICAL MODELS

The theory of rheological models in short can be described as the theory of representing viscoelastic materials by means of a simple combination of finite or minute elastic and viscious elements which are idealized in nature to describe the actual rheological behavior of viscoelastic solids under load or deformation.

As early as 1868, in studying the dynamic theory of gases, Maxwell employed a simple rheological model, Figure 1*a*, built up by a series combination of idealized elastic and viscous elements to explain the relaxation of elastic stresses. Later, Kelvin employed a parallel combination of idealized elastic and viscous elements (Figure 1*b*) to explain the damping occurring in solids due to mechanical oscillations. Since then, many complicated models have been developed to describe the rheological behavior of various materials under sustained loading or free and forced vibrations at large or small amplitudes. The theory of rheological or mechanical models has two main uses: (a) to explain the creep and relaxation behaviors of liquids and solids under sustained loading; (b) to explain the phenomenon of resonance and hysteresis damping of vibrating liquids and solids.

Considering Kelvin's solid when subjected to a constant load (assuming small acceleration), the equation of motion is

$$c\,\frac{dx}{dt} + kx = p \tag{1}$$

The solution of this equation is

$$x = \frac{p}{k}\left(1 - e^{-k/c\cdot t}\right) \tag{2}$$

where: x = displacement
p = sustained load,
t = elapsed time,
k = spring constant, and
c = dashpot constant.

Equation 2 is the creep curve for the model when displacement is plotted against elapsed time. Therefore, in order to predict the creep behavior of a linear Kelvin solid, it is necessary only to determine the spring and dashpot constant of the model. These constants can be obtained by vibrating the model at different frequencies as will be discussed in the following paragraphs.

The differential equation governing the vibrating model, assuming the mass is concentrated at one end, can be obtained from the Newtonian equation of motion:

$$M\,\frac{d^2x}{dt^2} + c\,\frac{dx}{dt} + kx = p_1 \cos \omega t \tag{3}$$

where: M = mass of the model, and
$p_1 \cos \omega t$ = the force acting on the model.

This equation yields the solution of the displacement of the model in two terms as follows:

$$x = C_1 \cdot e^{-(c/2M\,t)} \cos\left[\sqrt{\frac{k}{M} - \frac{c^2}{4M^2}}\cdot t + C_2\right] + \frac{p_1}{\sqrt{(c\omega)^2 + (k - M\omega^2)^2}} \cos\left[\omega t - \tan^{-1}\left(\frac{c\omega}{k - M\omega^2}\right)\right] \tag{4}$$

The first term is identical to that for damped, free vibration and the second term is the steady-state solution. Considering only the second term,

$$x = \frac{p_1/k}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\frac{c}{c_c}\cdot\frac{\omega}{\omega_n}\right)^2}} \cos\left[\omega t - \tan\left(\frac{2\frac{c}{c_c}\cdot\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}\right)\right] \tag{5}$$

where: $\omega_n = \sqrt{k/M}$, undamped natural frequency of the model, and
$c_c = 2\sqrt{kM}$, the critical damping constant

This implies that the amplification factor, $|X/X_{st.}|$ is

$$\left|\frac{X}{X_{st}}\right| = \left|\frac{X}{\frac{p_1}{k}}\right| = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\frac{c}{c_c}\cdot\frac{\omega}{\omega_n}\right)^2}} \tag{6}$$

where: X = maximum displacement, and
X_{st} = displacement due to static force p_1.

If the amplification factor is plotted against the frequency ratio, ω/ω_n, then the familiar amplification curve results as shown in Figure 2.

The amplification factor becomes a maximum when the exciting frequency is slightly less than that of the natural frequency of the model vibrating freely without a dashpot.

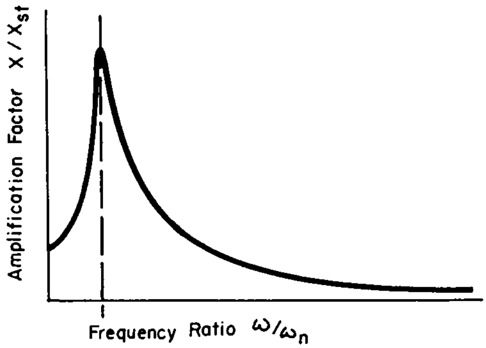

Figure 2. Amplification curve of forced, damped vibration of one degree of freedom.

For very small damping, the frequency for maximum amplitude can be assumed to be approximately the same as the natural frequency. The spring constant of the model can then be found from the following equation:

$$k = M\omega_n^2 = \frac{1}{4\pi^2} M \cdot f_0^2 \qquad (7)$$

where f_0 is the frequency corresponding to the maximum amplification in cycles per unit time.

This interesting phenomenon has been utilized extensively in material testing to obtain the modulus of elasticity of a material without actually obtaining a stress-strain diagram from the specimen.

Consider now the first term in Equation 4, which is the solution for damped free vibration. If the displacement is plotted against time, the solution is represented by an oscillating diminishing curve with a frequency of

$$q = \sqrt{\frac{k}{M} - \frac{c^2}{4M^2}} \qquad (8)$$

which is called the damped natural frequency. Such a curve is shown in Figure 3.

One interesting characteristic of the curve is that the natural logarithm of the ratio of the amplitude of any cycle of vibration to the amplitude of the immediately following cycle is constant. This constant is called the logarithmic decrement and can be found easily from its definition,

$$\delta = \log \frac{X_n}{X_{n+1}} = \log e\,(c\pi/qM) = \frac{c\pi}{qM} \qquad (9)$$

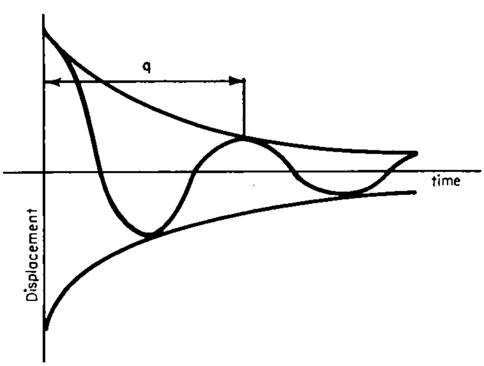

Figure 3. Decay curve of free, damped vibration of one degree of freedom.

where: δ = the logarithmic decrement,
X_n = amplitude for nth cycle.

The logarithmic decrement can be expressed in terms of the critical damping constant, c_c, directly as shown in the following equation:

$$\delta = \frac{c\pi}{qM} = \frac{2\pi \frac{c}{c_c}}{\sqrt{1 - \frac{c^2}{c_c^2}}} = 2\pi \frac{c}{c_c} \qquad (10)$$

$$(\text{for } c \ll c_c)$$

The logarithmic decrement is thus primarily related to the damping coefficient of the dashpot of Kelvin's solid. Furthermore, it can be shown that if the damping is small the logarithmic decrement can be approximated from the amplitude-frequency curve of the model under forced damped vibration, that is:

$$\delta = \pi \left(\frac{f_2 - f_1}{f_0} \right) \qquad (11)$$

f_0 = the frequency of maximum amplitude in cycles per unit time, and
f_1 and f_2 = frequencies on either side of resonance at which the amplitude of vibration is 0.707 of the maximum.

Thomson (*3*) determined the logarithmic decrement by using this equation.

It is therefore possible to obtain the elastic and viscous constants for an idealized Kelvin solid by subjecting it to forced vibration and obtaining the required frequencies. From these constants, it is not difficult to determine creep characteristics of the model. This is actually the foundation upon which many sonic investigations have been based. However, concrete can never be represented by a model as simple as that of the Kelvin type. Although concrete has a rather complicated viscoelastic representation, its behavior in most cases can generally be determined by subjecting it to forced vibrations and obtaining the elastic and viscous constants. The object of this paper is to determine the rheological coefficients of a model which can be considered as representative of concrete (these coefficients may be constant or varying, linear or non-linear) and then to utilize these coefficients to solve the differential equation of motion for the model under sustained loading.

A Kelvin solid with linear coefficients can-not describe the creep behavior of concrete adequately because the creep curve thus obtained has a limiting strain. This is due to the independence of the spring constant of the model with respect to the deformation of the model. If a model of Kelvin's type has a spring which softens as the strain increases, the limiting deformation of the spring becomes larger as the model deflects more. This is essentially identical to the characteristics of the creep curves for actual concrete specimens. Softening springs can be of many types. They may be time-softening, strain-softening, or both time- and strain-softening. Later analyses will show that the strain-softening type will be correct for concrete and that the spring stiffness can be assumed to be inversely proportional to the creep strain. It will also be shown that the dashpot coefficient is a power function of time and is entirely independent of the creep strain.

SONIC PROPERTIES OF CONCRETE CYLINDERS

In obtaining the elastic and viscous coefficients for actual concrete specimens by vibrating the specimens at relatively small amplitudes, the elastic and viscous constants can be assumed to be linear and equal to the initial values of the viscous and elastic coefficients of the non-linear model.

The dynamic modulus of elasticity of a cylinder is obtained by assuming the cylinder to be completely elastic. Timoshenko (*4*) used this assumption in developing the following equation.

$$\frac{E_d r_z^2}{\rho}\cdot\frac{\partial^4 v}{\partial x^4}+\frac{\partial^2 v}{\partial t^2}-r_z^2\left(1+\frac{E}{K'G_d}\right)\frac{\partial^4 v}{\partial x^2\partial t^2}+\frac{r_z^2}{K'G_d}\frac{\partial^4 v}{\partial t^4}=0 \qquad (12)$$

where: ρ = the density of the specimen,

r_z = the radius of gyration of the cross-section of the cylinder with respect to the centroidal z-axis which is perpendicular to the plane of vibration,

v = the displacement in the y-direction, which is perpendicular to the longitudinal axis and z-axis.

x = the coordinate in the longitudinal direction,

t = the time,

G_d = the dynamic modulus of rigidity,

E_d = the dynamic modulus of elasticity, and

K' = a constant introduced by Timoshenko to account for the effect of shear on the slope of the elastic line.

The solution for the first mode of flexural vibration has been given by Pickett (*5*) as:

$$E_d = C\cdot f_0^2\cdot W \qquad (13)$$

where: E_d = the dynamic modulus of elasticity,

f_0 = the natural frequency of the cylinder in cycles per unit time when vibrating in its first mode of flexural vibration,

W = the total weight of the specimen, and

C = a constant depending on the dimensions of the cylinder, the mode of vibration and Poisson's ratio. It is equal to 0.001168 for 6 by 12-inch cylinders when f_0 is in cycles per minute and W is in pounds.

This equation is almost identical to the theoretical equation for forced damped vibration of a linear Kelvin solid with the exception of the constant, C, owing to the difference of loading conditions and specimen sizes. The frequency at resonance obtained in sonic testing is determined by the criterion of maximum amplitude which, in fact, implies that the cylinder after all is not purely elastic. If the specimen were perfectly elastic, the amplitude at resonance under forced vibration becomes larger with time.

The damping in the system is found by assuming that the characteristics of the amplification curve, Figure 2, of the first mode of flexural vibration are approximately the same as those for linear forced vibration of a small particle with small damping. Equation 11 for the logarithmic decrement, δ, is therefore still applicable.

CREEP CURVE OF A NON-LINEAR KELVIN SOLID WITH STRAIN-SOFTENING SPRING AND TIME-THICKENING DASHPOT

It was found that the actual creep curve of a concrete cylinder could not be represented by the curve from a Kelvin solid with linear coefficients. Between small time intervals, however, the specimen can still be assumed to be linear. The variation of spring and dashpot-constants for the specimen can therefore be found by taking points on the actual creep curves and solving for the constants between points, i.e., using the method of finite differences to solve the non-linear differential equation in intervals. It was found that for every actual cylinder tested, the spring constant was essentially strain-softening, and the dashpot constant was a power function of time. The constants can be expressed in the following way:

$$\gamma = \frac{\gamma_1}{\epsilon_c},$$

and (14)

$$\eta = \eta_1 \cdot t_c^{n_\eta}$$

where: γ_1 = spring constant when t_c is unity,
ϵ_c = creep strain,
η_1 = dashpot constant when t_c is unity,
t_c = creep time, and
n_η = constant.

The differential equation governing the motion of such a non-linear model for small acceleration can be obtained by applying the Newtonian equation of motion to the model as follows:

$$\sigma = \eta_1 \cdot t_c^{n_\eta} \frac{d\epsilon_c}{dt_c} + \gamma_1 \tag{15}$$

where σ is the sustained compressive stress.

The solution of this equation for constant sustained stress is

$$\epsilon_c = \epsilon_{c_1} t_c^{n_{\epsilon_c}}, \tag{16}$$

where:

$$\epsilon_{c_1} = \left(\frac{\sigma - \gamma_1}{\eta_1}\right)\left(\frac{1}{1 - n_\eta}\right)$$

and is the creep strain when t_c is unity, and

$$n_{\epsilon_c} = 1 - n_\eta$$

Equation 16 indicates that the creep strain of a concrete cylinder under sustained axial compression is a power function of time. The creep curves should therefore be straight lines when time against strain is plotted on log-log paper. This was verified for all the cylinders tested, both for relatively short and for long durations of sustained loading. Furthermore, the coefficients of the creep functions can be obtained from the intersection of the creep curve and the strain axis when t_c is unity; the powers of the creep functions can be obtained from the slope of the creep curves plotted logarithmically.

SPECIMENS AND TEST PROCEDURES

The specimens tested were standard 6 by 12-inch cylinders, moist cured for seven days and then stored in a controlled environment at 76 F and 50 percent relative humidity. The specimens were aged in this environment from 21 to 351 days before being tested.

The aggregates used were from the Wabash River near Covington, Indiana. These aggregates are part of a glacial outwash, probably from the Wisconsin glaciation. The sand had an average fineness modulus of 3.0. The coarse aggregate was well graded and had a maximum size of 1 inch. The absorption of the aggregates was about one percent by weight of the surface dry aggregates. Both of these aggregates pass the usual specification tests. Type I portland cement was used. Different mixes were used and the strength at the time of test varied from 1100 to 5880 psi. The slump varied from 1 to 6½ inches.

The cylinders were placed in spring-loaded devices in pairs. Companion control cylinders were stored with the creep specimens, but not loaded. These control cylinders were used to correct the creep data for "shrinkage." Creep strain was measured over the center 6 inches of the cylinder.

Sonic tests were made immediately before the specimens were loaded for the creep tests. The sonic apparatus has been described in a previous paper (*6*).

CORRELATION BETWEEN SONIC PROPERTIES OF CONCRETE AND CREEP CONSTANTS

A close examination shows that the coefficient and the power that appear in Equation 16 are functions of the elastic properties, the viscous properties of the concrete, and the sustained stress in the concrete. The elastic and viscous properties of the cylinder can be assumed to be determined completely by the dynamic modulus of elasticity, the logarithmic decrement, and the ultimate strength of the concrete. This means that both the coefficient and the power that appear in Equation 16 can be considered to be functions of the dynamic modulus, the logarithmic decrement, the sustained compressive stress, and the ultimate strength of the concrete.

By dimensional analysis, the following can

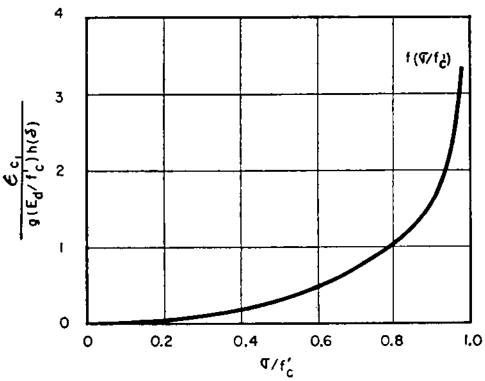

Figure 4. Dependence of ϵ_{c1} on the percentage of sustained stress.

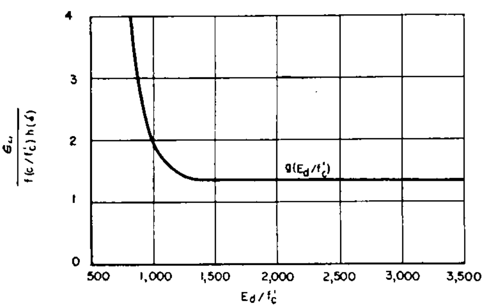

Figure 5. Dependence of ϵ_{c1} on the value of modulus-strength ratio.

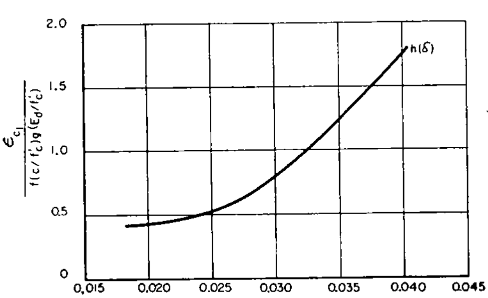

Figure 6. Dependence of ϵ_{c1} on the logarithmic decrement.

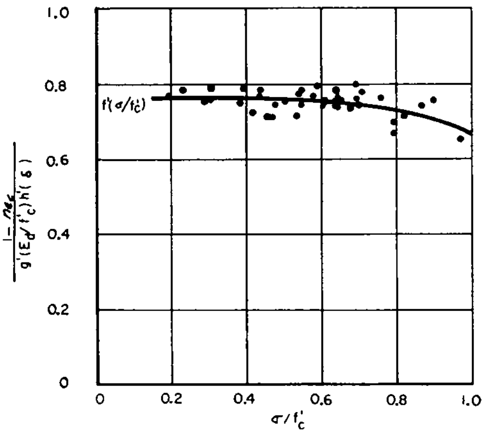

Figure 7. Dependence of $1 - n_{\epsilon c}$ on the percentage of sustained stress.

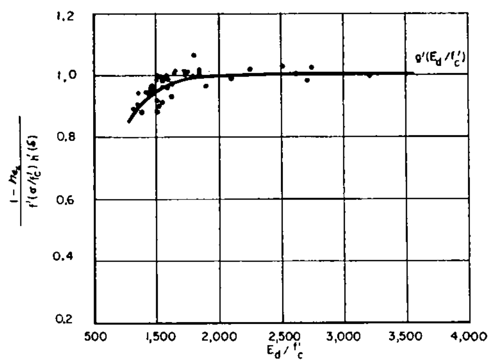

Figure 8. Dependence of $1 - n_{\epsilon c}$ on the modulus-strength ratio.

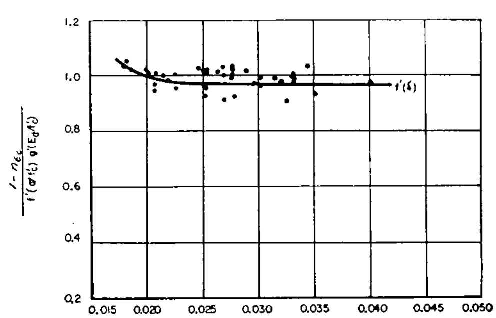

Figure 9. Dependence of $1 - n_{\epsilon c}$ on the logarithmic decrement.

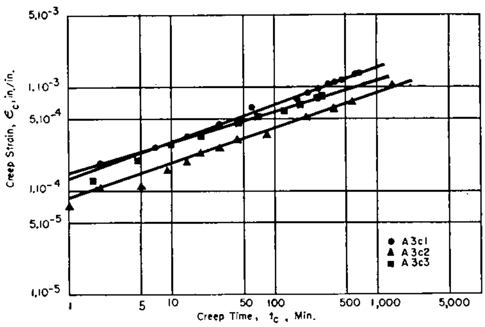

Figure 10. Theoretical creep curve and experimental creep data for short time tests.

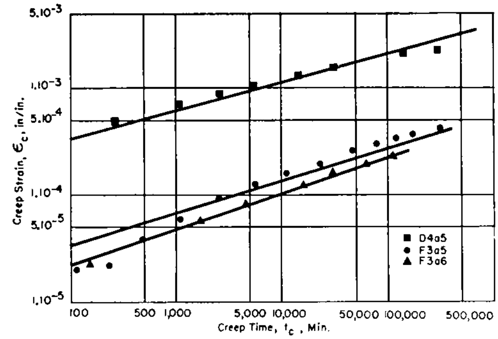

Figure 11. Theoretical creep curve and experimental creep data for long time creep tests.

TABLE 1
PROPERTIES OF CYLINDERS TESTED IN CREEP

Specimen No.	Slump	Age at Test	Strength	Dynamic Modulus	Logarithmic Decrement, δ	Creep Stress	σ/f_c'	Ed/f_c'	ϵ_{c_1} 10^{-4} in./in.	$1 - n_{\epsilon_c}$
	inches	*days*	*psi.*	*psi* $\times 10^6$		*psi.*				
A3c1	3½	53	5070	5.11	0.0268	4030	0.797	1008	1.303	0.645
A3c2	3½	58	5320	4.68	0.0207	3610	0.679	880	0.874	0.669
A3c3	3½	69	5880	5.07	0.0277	4070	0.693	863	1.585	0.697
C1a5	6½	58	1100	2.99.	0.0330	955	0.868	2720	1.000	0.742
F3a5	1	56	4900	5.00	0.0277	1240	0.253	1020	0.106	0.791
F3a6	1	61	5500	5.09	0.0253	1060	0.193	925	0.076	0.776

be obtained:

$$\begin{aligned} \epsilon_{c_1} &= F(\sigma/f_c', E_d/f_c', \delta) \\ n_{\epsilon_c} &= G(\sigma/f_c', E_d/f_c', \delta) \end{aligned} \tag{17}$$

where f_c' is the ultimate compressive strength of the concrete cylinder.

These functions can also be expressed by multiples of functions of each individual dimensionless products (7) as follows:

$$\begin{aligned} \epsilon_{c_1} &= f(\sigma/f_c')\cdot g(E_d/f_c')\cdot h(\delta) \\ n_{\epsilon_c} &= f'(\sigma/f_c')\cdot g'(E_d/f_c')\cdot h'(\delta) \end{aligned} \tag{18}$$

To obtain functions that appear in Equation 18, a total of 42 cylinders were sonic-tested and then creep-loaded. The results were then statistically analyzed. A method which combines step-by-step approximations and procedures of grossing errors was used. The resulting functions are shown graphically in Figures 4 through 9. Using the data in these figures, the sonic properties and the ultimate strength of a concrete cylinder, it is possible to predict the creep curve of that cylinder for a given sustained stress.

Theoretical creep curves obtained in this way were plotted on the log-log plots where actual creep data of the cylinders were also plotted. They were in very close agreement with the actual creep data. Some typical curves are shown in Figures 10 and 11. The data for the specimens for which results are shown in these figures are given in Table 1.

Similar excellent results have been obtained on cylinders which were made by others in which the mix, slump, etc. were unknown to the authors. These specimens were, however, also made from Type I portland cement and the same natural occurring aggregates as those for which results are presented.

SUMMARY

It may be seen from Figures 4 through 9 that the creep strain of concrete is not proportional to the percentage of sustained stress. However, for a sustained stress less than 50 per cent of the strength of the cylinder, the value of ϵ_{c_1} is approximately proportional to the percentage of stress and the value of n_{ϵ_1} is approximately constant. This indicates that for low percentages of sustained stress, the law of proportionality is approximately true.

The close agreement between the analysis presented and the experimental results indicates that a correlation exists between the sonic and creep properties of concrete. The ranges of variables covered in these tests are limited, however, and the analysis presented here should not be generally applied to other specimens, methods of curing, or other concretes. Further research is needed to determine the general applicability of this procedure.

ACKNOWLEDGMENTS

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REFERENCES

1. Reiner, M., *Twelve Lectures on Theoretical Rheology*. Interscience Publishers, Inc., New York, 1949, p. 102.
2. Flügge, W., "Mechanical Models in Plasticity and Their Use for Interpretation of Creep in Concrete," Technical Report No. 8, Division of Engineering Mechanics, Stanford University, 1950.
3. Thomson, W. T., "Measuring Changes in Physical Properties of Concrete by the Dynamic Methods," Proceedings, Am. Soc. Testing Mat., Vol. 40, 1940, p. 1113.
4. Timoshenko, S., *Vibration Problems in Engineering*. Van Nostrand, New York, 3rd edition, 1955, p. 331.
5. Pickett, G., "Equations for Computing Elastic Constants from Flexural and Torsional Resonant Frequencies of Vibration of Prisms and Cylinders," Proceedings, Am. Soc. Testing Mat., Vol. 45, 1945, p. 846.
6. Kesler, C., and Higuchi, Y., "Determination of Compressive Strength of Concrete by Using Its Sonic Properties," Proceedings, Am. Soc. Testing Mat., Vol. 53, 1953, p. 1044.
7. Chang, T. S., "Prediction of the Rheological Behavior of Concrete from Its Sonic Properties," Doctoral Thesis, University of Illinois, 1955, p. 109.