Bearing Strength Determination on Bituminous Pavements by the Methods of Constant Rate of Loading or Deformation

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Bearing strength is defined as the maximum compressive load per unit area which a bituminous pavement can carry without undergoing failure. A method for its determination has been previously given (*1*) consists of subjecting a paving sample to a constant compressive load and measuring the resulting strain as a function of time. Under this condition the strain rate decreases with time, and the material hardens as the result of the work done on the system. When the strain rate approaches zero, another load increment is added, the process being continued over a suitable range of loads. At a certain stress the maximum amount of hardening is obtained; this stress represents the bearing strength. At greater stresses, the pavement becomes gradually weaker and finally fails. The bearing strength is, therefore, a measure of the true stability and is a material constant at a given temperature.

This method is rather tedious. It is the purpose of this paper to show, both theoretically and experimentally, that time-saving methods using constant rate of loading or deformation are suitable for the determination of bearing strength.

Under such conditions, the stress-strain relationship is first curvilinear and convex to the strain axis, then becomes linear, and finally curvilinear again but concave to the strain axis. The first part of the curve represents the region of hardening and terminates at a stress corresponding to the bearing strength. During the second part, the system approaches a state of equilibrium, and during the last part, the system is in the region of progressive failure.

Bearing strength measurements carried out in duplicate under a constant loading rate of 1 lb per second on sand-asphalt mixtures containing various amounts of two different asphalts showed good reproducibility of results. The bearing strength increased to an optimum with asphalt content.

For thermodynamic reasons, a deviation in the value of the bearing strength is expected at high rates of loading or deformation. For the determination of the rate limit, the sand-asphalt mixtures at optimum asphalt content were subjected to constant loading rates of 1, 2, 4, 8 and 16 lb per second. The bearing strengths obtained were reproducible at rates up to 4 lb per second, and were in good agreement with the values measured by static load tests at zero rate of loading. The effect of constant rates varies with the dimensions of the sample, the upper limit for constant rate of loading is 1.27 psi per second, and for constant rate of deformation the upper limit is of the order of 1.5 to 2.0 x 10^{-4} in. per inch thickness per second.

In contrast to this theory, it is generally assumed that the magnitude of the compactive effort affects the stability of bituminous pavements. To test this assumption, an asphaltic concrete mixture was compacted by the Mar-

shall method using 25, 50 and 75 blows, and the samples were subjected to a constant strain rate of 1.9×10^{-4} in. per inch per second. The bearing strength was the same in each case. However, increasing the degree of compaction reduced the work required to obtain the bearing strength.

The method of constant rate of loading or deformation also can be applied for the determination of the bearing strength of granular materials and soils. The knowledge of these strengths serves as a basis for the design of bituminous pavements.

● ROAD STRUCTURES are subjected to compressive loads. The maximum load per unit area which any part of the road structure can carry without undergoing initial failure, is defined as bearing strength.

A method already given (*1*) for measuring this strength consists of subjecting a sample to a constant compressive load and measuring the resulting strain as a function of time. Under this condition, the strain rate decreases with time and the material hardens as the result of the work done. When the strain rate approaches zero another load increment is added, the process being continued over a suitable range of loads. At a certain stress the maximum amount of hardening is obtained; this stress represents the bearing strength. At greater stresses the pavement becomes gradually weaker and finally fails.

This method is very time-consuming. A simple procedure for measuring the bearing strength would, therefore, be of great advantage. It is engineering practice in the mechanical testing of materials to establish their stress-strain relationships by subjecting them to the conditions of constant rate of loading, or constant rate of deformation. It is the purpose of this paper to show that these methods simplify the determination of the bearing strength.

DEFORMATION MECHANISM: CONSTANT RATE OF LOADING OR DEFORMATION

It has been shown (*1*) that the deformation of bituminous pavements is associated with the following types of strain:

1. An instantaneous elastic strain independent of time.
2. A retarded elastic strain which is a function of time.
3. A plastic strain whose rate decreases with time.

With both elastic deformations being of a transient nature, the plastic deformation is most important. This deformation contributes to the hardening effect within a certain stress region and causes an increase in strength of bituminous pavements. As a result of this effect, the moduli of elasticity associated with the instantaneous and retarded elastic strains increase within the hardening range with the stress.

The stress-strain relationships for the various types of deformation are as follows:

For the instantaneous elastic deformation

$$\sigma = A\ \varepsilon'^{a} \qquad (1)$$

in which σ is the compressive stress, ε' is the instantaneous elastic strain, A is a constant, and a is a dimensionless constant greater than 1.

The instantaneous elastic strain is followed by the retarded elastic strain, the latter consisting of an elastic strain and a plastic strain. Both strains are the same, but their corresponding stresses are additive; that is,

$$\sigma = \sigma' + \sigma'' \qquad (2)$$

in which σ' is the stress associated with the elastic strain and σ'' with the plastic strain. The stress-strain relationship for the retarded elastic deformation is

$$\sigma = B''\ [\dot{\varepsilon}''\ (t+1)]^{b} \qquad (3)$$

in which B is a constant, $\dot{\varepsilon}''$ $(=d\varepsilon''/dt)$ is the rate of the retarded elastic strain, and b is a constant usually larger than 1 or negative. The stress is a function of the product of strain rate and time, which has the dimension of strain.

The plastic deformation associated with hardening obeys a relationship similar to Eq. 3; that is,

$$\sigma = B'' \ [\dot{\varepsilon}''' \ (t+1)]^b \qquad (4)$$

in which $\dot{\varepsilon}'''$ is the plastic strain rate and b has the same magnitude as in Eq 3.

This plastic strain starts also at zero time and runs, therefore, concurrently with the retarded elastic deformation in such a manner that their strains are additive. Denoting their sum also by ε'', it follows from Eqs. 3 and 4 that

$$\sigma = B' \ [\dot{\varepsilon}'' \ (t+1)]^b \qquad (5)$$

in which $B' = B \ B''/(B^{1/b} + B''^{1/b})^b$. At constant stress, the retarded elastic strain has its maximum value at $t = t_o$. At t greater than t_o, the deformation is purely plastic and proceeds according to Eq. 4.

At a certain stress, σ_b, the elastic modulus associated with the instantaneous elastic deformation and the coefficient related to the retarded elastic and plastic deformations are maxima:

$$E_m = d\sigma/d\varepsilon' = a\sigma_b/\varepsilon'_b \qquad (6)$$

$$\phi_m = d\sigma/d[\dot{\varepsilon}'' \ (t+1)\ ] = b\sigma_b/[\dot{\varepsilon}'' \ (t+1)\ ]_b \qquad (7)$$

At stresses in excess of σ_b, the values of the modulus and coefficient decrease and the pavement is in a state of progressive weakening. Hence, σ_b is a constant characteristic of a bituminous pavement at constant temperature. It represents the bearing strength, or the maximum load per unit area a pavement can carry without causing initial failure.

For the condition of constant rate of loading or deformation, the stress varies continually and the strains associated with the various types of deformation are additive. In view of the continuous change in stress, the purely plastic deformation observed at constant stress in the region of hardening never comes into display. Hence, the strain for any given stress consists of the sum of the instantaneous elastic deformation and the strain of the combined retarded elastic and plastic deformation. The mathematical derivations for this condition are given in the Appendix. Since compression tests under constant rate of loading or deformation yield data in terms of load rather than stress, the various relationships are expressed in terms of load per original area. For simplification, this term is henceforth called stress.

The condition of constant rate of loading or deformation already contains the factor of time. This circumstance leads to simple equations which relate stress with strain only.

In the region of hardening, the stress-strain curve is curvilinear and convex to the strain axis. Within a certain stress range, the logarithm of stress is linear with the logarithm of the total strain, according to

$$S_b/S_l = (\varepsilon_b/\varepsilon_l)^n \qquad (8)$$

in which S_b is the load per original area expressing the bearing strength, ε_b is the corresponding strain, S_l and ε_l are the lower limits, and n is a constant greater than 1. The value of S_b is theoretically independent of the rate of loading or deformation and is, therefore, constant.

If the stress S_b is kept constant, the instantaneous elastic strain remains constant, but the combined retarded elastic and plastic strains increase with time from an initial to a final value in accordance with Eq. 7. This strain difference is also obtained, when the rate of loading or deformation is kept constant. However, in this case the stress must increase with the strain, and their relationship becomes linear according to

$$S_m - S_b = \phi_m(\varepsilon_m - \varepsilon_b)/b \qquad (9)$$

in which ε_m is the maximum strain in this region, S_m is the corresponding stress, and ϕ_m is the maximum coefficient defined by Eq. 7.

At stresses larger than S_m, the stress-strain relationship becomes curvilinear again, but is concave to the strain axis. In this region, the material fails progressively. The value of S_m depends on the onset of failure. Because the latter is a statistically disturbed event, S_m is not a constant.

Thus, the value of S_b is the most important factor for the evaluation of the mechanical stability of a bituminous pavement. A comparison of Eqs. 8 and 9 shows that there are two ways of determining S_b. In the first case, S_b is the upper stress limit of the linear relationship between log stress and log strain. In the other case, S_b is the lower stress limit of the linear stress-strain relationship.

EXPERIMENTAL

General Considerations

Theoretically, the value of the bearing strength should be constant and independent of the rate of loading or deformation. However, since the total deformation at the bearing strength is obtained in a shorter time interval under these conditions than under static loading, the following factors have to be taken into consideration:

1. The work done on the system is partly transformed into heat. If the experiment is carried out slowly, the heat will dissipate into the surroundings at about the same rate as it is formed. On the other hand, at high rates of loading or deformation the heat does not dissipate rapidly enough and causes a rise in temperature, thereby reducing the bearing strength.

2. The time-dependent strains are associated with a rotation of the mineral particles into new positions. Increasing rates of loading or deformation will decrease the time factor and retard the rotation. This effect will also influence the value of the bearing strength.

In view of these factors, the experiments were carried out at the lowest possible constant rate of loading of 1 lb per second, and the results compared with those obtained at various rates of loading, including zero rate of loading (that is, static loading).

Bearing Strength at Constant Rate of Loading

The tests were carried out on mixtures of asphalt, siliceous sand, and limestone filler. The mineral aggregate had the following grading:

U.S. Sieve No. Passing	U.S. Sieve No. Retained	Percent by Weight
10	20	24.5
20	40	30.5
40	80	33.0
80	200	1.8
200		10.2

The asphalts used had the following characteristics:

Asphalt	Soft. Point, °F (R and B)	Penetration 200 g/60 sec 32F	Penetration 100 g/5 sec 77F	Penetration 100 g/5 sec 100F	Ductility at 77F
A	111	16	86½	264	150+
B	125	18	80	224	150+

The test samples were prepared by mixing the sand and asphalt heated to 250 F and finally adding the filler heated to the same temperature. The mixtures were then compacted in a cylindrical mould by the double-plunger method, applying a load of 1,000 psi twice for a period of 1 min. The diameter of the compacted samples was 2 in. and the height approximately 1.85 in.

A Universal Baldwin testing machine was used for compression testing, which was carried out at 77 F. To reduce surface friction between the sample and the platens of the testing machine, the sample surfaces were coated with a paste prepared by vigorously stirring 10 parts of starch into 90 parts of a 1:1 mixture of water and glycerine heated to 200 F. This lubricant also helped to fill small cavities, thereby providing a more uniform distribution of the load over the surfaces of the samples.

The tests were carried out on sand-asphalt mixtures with an asphalt content varying between 6 and 10 parts of asphalt per 100 parts of mineral aggregate. The rate of loading was 1 lb per second, and the strains were expressed in terms of the natural strain, which is the product of 2.303 and the logarithm of the ratio of original height to height. All runs were made in duplicate.

The results were in agreement with the theory. Figure 1 is representative of the stress-strain curve obtained for a sand-asphalt mixture containing 8 percent of asphalt A. Between A and B the stress-strain curve is convex towards the strain axis and the material is in the region of hardening. Between B and C, the stress-strain relationship is linear, in agreement with Eq. 9. The curve then becomes concave towards the strain axis

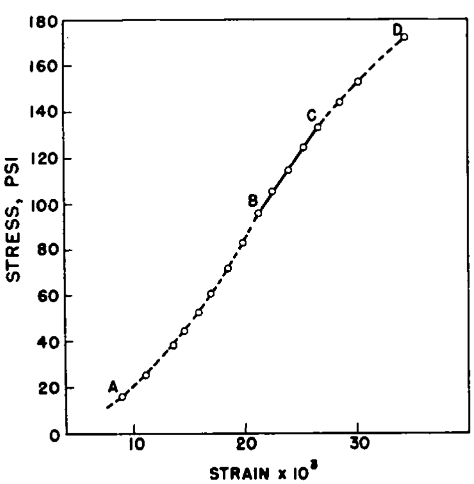

Figure 1. Stress-strain relationship for a sand-asphalt mixture prepared with asphalt A; loading rate, 1 lb per sec.

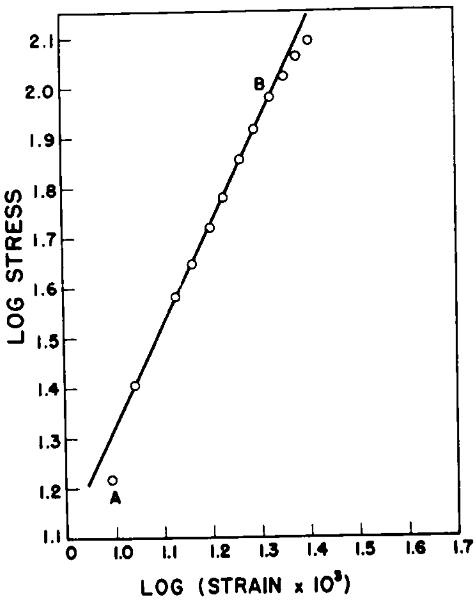

Figure 2. Relationship between stress and strain in the hardening range for mixture of sand and asphalt A.

(between C and D), indicative of progressive weakening. The stress at B represents the bearing strength S_b. It has been indicated that in the region of hardening the logarithm of stress should be a linear function of the logarithm of strain within a certain stress range, according to Eq. 8. Figure 2 shows this relationship to hold between the stress next to that at A and the bearing strength at B.

The results are given in Table 1, which contains data regarding the density and void content of the mixtures, the values of n, the power factor in Eq. 8, S_b the bearing strength, and S_m the upper stress limit of the linear part of the stress-strain curve. It will be seen that:

1. For a given asphalt content the densities and void contents of the two types of mixtures are practically the same. The increase in density and decrease in void content with increasing asphalt content is of the expected order.

2. The value of n is not reproducible for a given mixture. This constant is a function of the internal structure of the

TABLE 1
CHARACTERISTICS OF SAND-ASPHALT MIXTURES AT 77 F
(Rate of loading = 1 lb per second)

Item	Parts of Asphalt per 100 Parts of Aggregate				
	6	7	8	9	10
	(*a*) ASPHALT A				
Density	2.193, 2.193	2.234, 2.231	2.241, 2.239	2.250, 2.242	2.284, 2.286
Voids % vol.	10.7, 10.7	7.8, 7.9	6.27, 6.35	4.93, 5.12	0.21, 0.22
n	1.65, 2.15	2.18, 2.44	2.05, 2.28	1.73, 1.43	1.28, 1.44
S_b,psi	56.8, 56.7	78.7, 78.4	99.2, 97.6	69.4, 72.5	47.3, 47.3
S_m,psi	110.0, 119.0	144.0, 171.0	132.0, 160.0	135.0, 159.0	112.0, 105.6
	(*b*) ASPHALT B				
Density	2.189, 2.191	2.232, 2.230	2.243, 2.240	2.248, 2.250	2.283, 2.282
Voids, % vol.	10.7, 10.8	7.9, 8.0	6.22, 6.37	5.08, 4.98	0.22, 0.23
n	2.18, 2.21	1.85, 4.02	2.93, 4.11	2.05, 3.33	2.41, 2.21
S_b, psi	47.2, 48.7	66.2, 65.8	84.9, 84.5	62.6, 62.4	40.8, 40.9
S_m, psi	109.5, 159.0	144.5, 159.0	145.2, 156	103.0, 105.5	81.2, 84.4

material before testing, which cannot be controlled.

3. For a given mixture the value of S_b, the bearing strength, is reproducible with a satisfactory degree of precision.

4. The reproducibility of S_m is poor, as it represents the onset of failure, which is also not controllable.

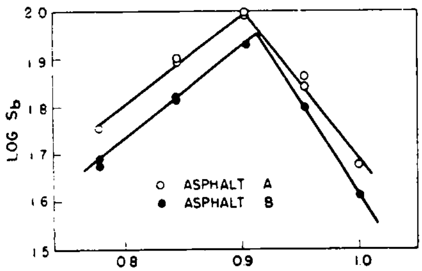

Figure 3. Relationship between bearing strength and asphalt content.

For both types of mixtures the bearing strength increases to a maximum at an asphalt content of 8 percent. This relationship is shown in Figure 3 as a plot of log bearing strength *vs* log asphalt content, and approaches linearity on either side of the inflection point. The optimum asphalt is 8 percent based on the aggregate for the mixture containing asphalt A and seems to be somewhat greater for the mixture with asphalt B.

The average bearing strengths at the optimum asphalt content are 98.4 psi for asphalt A and 84.7 psi for asphalt B. The bearing strengths obtained from static loading tests were 95.8 and 86.0 psi, respectively.

Bearing Strength as a Function of Loading Rate

In view of a possible variation of the bearing strength with the magnitude of the rate of loading, both sand-asphalt mixtures at the optimum asphalt content of 8 percent were subjected at 77 F to loading rates of 2, 4, 8, and 16 lb per second. The bearing strengths are given in Table 2 and Figure 4.

The results indicate that for both sand-asphalt mixtures the values of the bearing strength tally well within a rate of loading between 0 (static loading) and 4 lb per second. At greater rates, the agreement is fair for the mixture with asphalt A, but poor for the mixture with asphalt B. It appears, therefore, that a rate of loading of 4 lb per second represents the limit of reproducibility. On this basis, the bearing strength of mixture A is 98.6 ± 1.84 psi, and 85.0 ± 0.76 psi for mixture B.

Because the effect of constant rate of

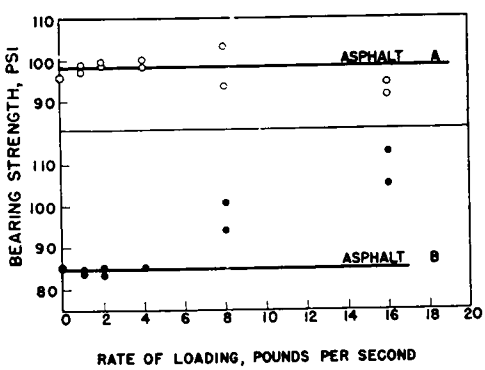

Figure 4. Bearing strength of a sand-asphalt mixture as a function of rate of loading.

TABLE 2
BEARING STRENGTH IN PSI FOR VARIOUS LOADING RATES

Asphalt	Loading Rate, lb per sec					
	0	1	2	4	8	16
A	95.8 —	99.2 97.6	100.0 98.7	100.3 98.8	103.5 93.9	91.6 94.6
B	86.0 —	84.9 84.5	83.7 85.5	85.4 85.3	101.0 94.2	105.0 113.0

loading varies with the dimensions of the sample, the upper limit of accuracy is 1.27 psi per second. Based on the linear stress-strain relationship at a rate of loading of 4 lb per second, the upper limit for constant rate of deformation becomes 1.5 to 2.0 x 10^{-4} in. per inch per second.

Bearing Strength in Relation to Compactive Effort

It is generally considered that the bearing strength or stability of a bituminous pavement increases with the compactive effort. This concept is not justified as will be seen from the following.

The bearing strength of a bituminous pavement is not an inherent property, but is acquired as a result of the work done to the system. Under stress the mineral particles rotate and shift to new positions. At a stress equal to the bearing strength, the particles are in positions of greatest stability, and the system behaves like a solid body under simple compression. The rearrangement of the particles is accompanied by a densification in the region of hardening. The work per unit volume done to the system at the point of maximum hardening is proportional to the product of bearing strength and its corresponding strain. This work is theoretically constant. Hence, if a bituminous mixture, compacted in an ordinary way, is first subjected to a special compactive effort and then to a load application, the total work is the sum of the work due to the separate manipulations. Therefore, the more work done by compaction, the less work will be required by loading. However, the value of the bearing strength will not be affected by the degree of compaction, as long as the load can still do work in the region of hardening.

To demonstrate this effect, a bituminous concrete mixture was compacted by the Marshall method (*2*) using 25, 50, and 75 blows on each face of the briquette. The mixture had the following composition:

Constituent	Parts by Weight
Trap rock	40
Sand	57
Filler	3
Asphalt, 85/100 pen.	5

The briquettes were 4 in. in diameter and about 4.2 in. high. They were subjected to a constant rate of deformation of 1.9 x 10^{-4} in. per inch per second at 77 F. The test on the sample compacted by 75 blows was not carried to the region of failure. It was subsequently unloaded and again strained.

The results are presented in Figure 5. It will be seen that in all these cases the stress-strain curve is convex to the strain axis to a value of 87 psi, then becomes linear. This stress is the bearing strength of the mixture. The area under these curves, which represents the work done per unit volume, decreases with increasing compaction in agreement with the theory. The sample compacted by 75 blows received its maximum hardening at the end of the test. At this stage the material behaved like a solid body, and reloading resulted in a linear stress-strain relationship outside the region of failure, except for a small deviation at low stresses, which is probably due to an adjustment of the surfaces of the sample and platens. A mixture in this state of hardening is not suitable for the de-

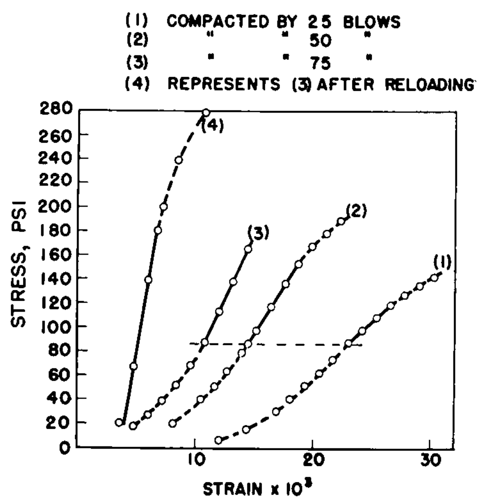

Figure 5. Stress-strain relationship for asphaltic concrete subjected to various compactive efforts.

termination of the bearing strength because of the absence of plastic deformation, and must be remolded.

Base Course Materials and Soils

A bituminous pavement rests on the base course and subsoil. The deformation mechanism of these materials is similar to that of bituminous pavements (*1*). Hence, the bearing strength of these materials can be determined by the method of constant rate of loading or deformation. In general, constant rate of deformation is the method preferred, because this rate is less subject to small fluctuations than is constant rate of loading.

Having no cohesion, the materials of the base course and subsoil are compacted in suitable containers and loaded centrally by means of a circular bearing plate or a piston, as with the California bearing test method. The size of the loaded area and depth of the layer must be adapted to the size of the largest stones present. The diameter of the loaded area and height of the layer should exceed four times the size of the largest stone.

For this condition, the loaded area remains constant, and the strain is proportional to the depression:

$$\varepsilon = 2z/\pi r \; (1 - \upsilon^2) \qquad (10)$$

in which z is the depression, r is the radius of the circular loaded area, and υ is Poisson's ratio. The load-depression curve has a similar S-shape as for bituminous mixtures shown in Figure 1. Such curves for soils have been published in the literature (*3*).

With knowledge of the bearing strength of the sublayers, the required thickness of pavement and base course can be calculated. The distribution of stresses in a road structure due to a load applied to the surface is not known at present. Its assessment is difficult, as the distribution of stresses varies with the physical state of the materials. However, because at a compressive stress corresponding to the bearing strength, the material behaves like a solid, the theory of elasticity can be applied at least to the distribution of vertical stresses.

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3. U. S. Corps of Engineers. Ad Interim Engineering Manual, Part XII, Fig. 7 (July 1946).

APPENDIX

STRESS-STRAIN RELATIONSHIPS OF BITUMINOUS MIXTURES UNDER CONSTANT RATE OF LOADING OR DEFORMATION

For the condition of constant rate of loading or deformation, the instantaneous elastic and combined retarded elastic and plastic deformations are of importance. They are represented by Eqs. 1 and 5.

Compressive tests are usually carried out on cylindrical specimens, which change the area under load with reduction in height. Because the experimental results refer to loads rather than stresses, it is necessary to express the latter in terms of loads.

The vertical strain under compression is

$$\varepsilon_1 = -\ln \; (h/h_0) = \ln \; (h_0/h) \qquad (11)$$

in which h and h_0 are the height and original height, and ln is the natural logarithm. The diameter of the specimen increases with decreasing height, and the resulting strains are

$$\varepsilon_2 = \varepsilon_3 = \ln \; (r/r_0) = \varepsilon_1 \upsilon \qquad (12)$$

in which r and r_0 are the radii and υ is Poisson's ratio (that is, the ratio of lateral to vertical strain).

The compressive stress is load per unit of area. Making use of Eq. 12, the ratio of area to original area is

$$\frac{\pi r^2}{\pi r_0^{\,2}} = \exp \; (2\upsilon\varepsilon_1) \qquad (13)$$

in which exp is the exponential. Hence, the stress is

$$\sigma = L/\pi r^2 = L/\pi r_0 \exp(2\nu\varepsilon_1) = S/\exp(2\nu\varepsilon_1) \qquad (14)$$

in which S is the load per unit of the original area.

Introducing Eq. 14 in Eq. 1 gives the following elastic strain as function of S:

$$\exp\left(\frac{2\nu\varepsilon'}{a}\right)\varepsilon' = \left(\frac{S}{A}\right)^{1/a} \qquad (15)$$

Expanding the exponential term into a series gives

$$\exp(2\nu\varepsilon'/a) = 1 + 2\nu\varepsilon'/a + (2\nu\varepsilon'/a)^2/2! + \ldots \qquad (16)$$

The value of Poisson's ratio is of the order of 0.25, the elastic strains are of the order of 10^{-3}, and a is larger than 1. Hence, all terms in the power of 2 and higher can be neglected and

$$(1 + 2\nu\varepsilon'/a)\varepsilon' \sim \varepsilon' = (S/A)^{1/a} \qquad (17)$$

Because this strain is purely elastic, it is independent of the rate of loading or deformation.

Rearranging Eq. 5 and introducing the value of σ from Eq. 14 gives

$$\exp\left(\frac{2\nu\varepsilon''}{b}\right)\dot{\varepsilon}'' = \exp\left(\frac{2\nu\varepsilon''}{b}\right)\frac{d\varepsilon''}{dt} = \left(\frac{S}{B'}\right)^{1/b}\frac{1}{(t+1)} \qquad (18)$$

which will be solved separately for the condition of constant loading and constant rate of deformation.

Because there is a retarded elastic strain at zero time, there is also a stress, and constant rate of loading is defined as

$$\frac{dS}{dt} = \dot{S} = \frac{S}{t+1} \qquad (19)$$

For this condition Eq. 18 becomes

$$\exp\left(\frac{2\nu\varepsilon''}{b}\right)d\varepsilon'' = \left(\frac{S}{B'}\right)^{1/b}\frac{dt}{t+1} = \left(\frac{S}{B'}\right)^{1/b}\frac{ddS}{S} \qquad (20a)$$

and integration gives

$$\frac{b}{2\nu}\left[\exp\left(\frac{2\nu\varepsilon''}{b}\right) - 1\right] \sim \varepsilon'' = b\left(\frac{S}{B'}\right)^{1/b} \qquad (20b)$$

Constant rate of deformation is defined as

$$-\frac{dh''}{dt} = \dot{h}'' = \frac{h_0 - h''}{(t+1)} \qquad (21)$$

Because

$$\frac{d\varepsilon''}{dt} = -\frac{1}{h''}\frac{dh''}{dt} = \frac{(h_0 - h'')}{h''(t+1)} \qquad (22)$$

Eq. 18 changes into

$$\exp\left(\frac{2\nu\varepsilon''}{b}\right)\cdot\frac{(h_0 - h'')}{h_0} = \left(\frac{S}{B'}\right)^{1/b} \qquad (23)$$

The term $(h_0 - h'')/h''$ is

$$\frac{h_0}{h''} - 1 = \exp(\varepsilon'') - 1 \qquad (24)$$

Introducing Eq. 24 in Eq. 23 gives

$$\exp\left(\frac{2\nu\varepsilon''}{b}\right)\left[\exp(\varepsilon'') - 1\right] = \exp\left(\frac{2\nu\varepsilon'' + b\varepsilon''}{b}\right) - \exp\frac{2\nu\varepsilon''}{b} \sim \frac{2\nu\varepsilon''}{b} + \varepsilon'' + 1 - \frac{2\nu\varepsilon''}{b} - 1 = \varepsilon'' \qquad (25)$$

and

$$\varepsilon'' = (S/B')^{1/b} \qquad (26)$$

It will be seen that Eqs. 20b and 26 are similar, and in both cases the strain is independent of the rate of loading or deformation and, therefore, independent of time. This peculiar result is due to the fact that the stress is a function of the product of strain-rate and time, which has the dimension of strain.

The instantaneous elastic strain and the combined retarded elastic and plastic strain are additive. The total strain is, therefore, the sum of the strains of Eqs. 17 and 20b or 26, and may be generalized as

$$\varepsilon = \varepsilon' + \varepsilon'' = \frac{1}{A^{1/a}} (S^{1/a} + q\, S^{1/b}) \quad (27)$$

in which $q = bA^{1/a}/B^{1/b}$ for the case of constant rate of loading, and $q = A^{1/a}/B^{1/b}$ for the case of constant rate of deformation.

For the elimination of A, Eq. 27 is differentiated and the result divided by the original equation, as follows:

$$\frac{d\varepsilon}{\varepsilon} = \left[\frac{1 + (q\, a/b)\, S^{(a-b)/ab}}{1 + q\, S^{(a-b)/ab}} \right] \frac{dS}{a\,S} \quad (28)$$

After a value of $S = S_1$ the term $S^{(a-b)/ab}$ becomes so large in comparison to 1, that the expression in brackets approaches a constant value, m. Hence,

$$\frac{d\varepsilon}{\varepsilon} = \frac{m}{a} \frac{dS}{S} = \frac{1}{n} \frac{dS}{S} \quad (29)$$

This equation represents the strain as a function of stress in the region of hardening. Integration between the limits S_1 and S_b, the load per original area corresponding to the bearing strength, and ε_1 and ε_b leads to

$$S_b/S_1 = (\varepsilon_b/\varepsilon_1)^n \quad (30)$$

In this region, n has a value larger than 1.

Differentiating Eq. 30 with respect to strain gives an apparent "modulus of elasticity" of the following form:

$$M = \frac{dS}{d\varepsilon} = n \frac{S}{\varepsilon} \quad (31)$$

This "modulus" increases with increasing values of S. At $S = S_b$, it has a maximum value and is a combination of the following two stress-strain ratios.

For the instantaneous elastic deformation, the following modulus of elasticity is obtained from Eq. 17:

$$E' = dS/d\varepsilon' = a\,S/\varepsilon' \quad (32)$$

Although this term is variable, it is a true modulus of elasticity, as the strain is independent of time. This modulus has a maximum at $S = S_b$:

$$E'_m = a\,S_b/\varepsilon'_b \quad (33)$$

The corresponding coefficient associated with the combined retarded elastic and plastic deformation is, according to Eq. 26,

$$\phi_m = dS/d\varepsilon'' = bS_b/\varepsilon''b \quad (34)$$

Rearranging these equations and adding the strains gives

$$\varepsilon_b = \varepsilon'_b + \varepsilon''_b = S_b\,[a/E'_m + b/\phi_m] \quad (35)$$

A comparison with Eq. 31 shows that the maximum apparent "modulus" is

$$M_m = nS_b/\varepsilon_b = nE'_m\phi_m/(a\phi_m + bE'_m) \quad (36)$$

Being independent of time, the elastic strain ε' reaches its maximum at $S = S_b$. However, the combined retarded elastic and plastic strain ε'' is not yet a maximum at this point and continues to increase with stress. This may be shown by comparing the behavior of the material at constant S_b with that at constant rate of loading or deformation.

Considering that, for small strains, the stress can be replaced by S, the maximum coefficient associated with the retarded elastic and plastic deformation is obtained from Eq. 5, as follows:

$$\phi_m = \frac{dS}{d\,[\dot\varepsilon''(t+1)]} = \frac{b\,S_b}{[\dot\varepsilon''(t+1)]_b} \quad (37)$$

The material is assumed to be loaded up to S_b, and then the latter is kept constant. Under this condition

$$\dot\varepsilon''(t+1) = \alpha = \text{constant} \quad (38)$$

and

$$\dot\varepsilon''dt + (t+1)d\dot\varepsilon'' = 0 \quad (39)$$

$$-\frac{d\dot\varepsilon''}{\dot\varepsilon''} = \frac{dt}{t+1} = \frac{\dot\varepsilon''dt}{\dot\varepsilon''(t+1)} = \frac{d\,\varepsilon''}{\alpha} \quad (40)$$

Integration between $\dot\varepsilon''_b$ and $\dot\varepsilon$, ε'' and ε''_b leads to

$$\dot\varepsilon''_b/\dot\varepsilon'' = \exp\,[\,(\varepsilon'' - \varepsilon''_b)/\alpha] \quad (41)$$

$$\dot\varepsilon''_b = \exp\,[\,(\varepsilon'' - \varepsilon''_b)/\alpha]\; d\varepsilon''/dt \quad (42)$$

A second integration between t_m and t_b, ε''_m and ε''_b gives

$$\dot{\varepsilon}''_b\ (t_m - t_b + 1) = \alpha \exp\left[(\varepsilon''_m - \varepsilon''_b)/\alpha\right] \quad (43)$$

It follows that at $t_m = t_b$, $\varepsilon''_m = \varepsilon''_b$, and $\dot{\varepsilon}''_b = \alpha$, hence

$$t_m - t_b + 1 = \exp\left[(\varepsilon''_m - \varepsilon''_b)/\alpha\right] \quad (44)$$

and

$$\varepsilon''_m - \varepsilon''_b = \alpha \ln (t_m - t_b + 1) \quad (45)$$

With $\alpha = \dot{\varepsilon}''(t+1)$, Eq. 37 may be written as follows for the condition S_b = constant:

$$\phi_m = \frac{b\ S_b \ln (t_m - t_b + 1)}{\varepsilon''_m - \varepsilon''_b} \quad (46)$$

For constant rate of loading or deformation, the term $S_b/(t+1)_b$ in Eq. 37 represents the rate of loading, and the coefficient ϕ_m is proportional to the ratio of loading rate to strain rate. Since ϕ_m is constant, both rates are constant. With $S_b/(t+1)_b = dS/dt$, and $\dot{\varepsilon}'' = d\varepsilon''/dt$, Eq. 37 becomes

$$\phi_m = b\ dS/d\varepsilon'' \quad (47)$$

which defines a linear relationship between stress and strain. Integration between the limits S_m and S_b, ε''_m and ε''_b yields

$$\phi_m = b\ (S_m - S_b)/(\varepsilon''_m - \varepsilon''_b) \quad (48)$$

Theoretically, the strain difference, $\varepsilon''_m - \varepsilon''_b$, should be the same in Eqs. 46 and 47, and S_m should be a multiple of S_b, as follows:

$$S_m = S_b\ [1 + \ln (t_m - t_b + 1)] \quad (49)$$

At values of S larger than S_m, the system is in the region of failure, and the elastic modulus and coefficient ϕ decrease with increasing stress. The stress-strain relationship in this region is similar in form to that in Eq. 30, but with a value of n smaller than 1.

The onset of failure is associated with the weakest part of the system and is, therefore, a statistically distributed event. For this reason, S_m, in contrast to S_b, is not a constant.